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INTERACTIONS BETWEEN VIBRATIONS OF FLEXIBLE LINKS AND BASE MOTION OF MANIPULATORS

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The paper presents an application of the Rigid Finite Element Method to modelling of flexible links of a manipulator. The equations of motion are derived assuming that the base motion is not known and thus it depends on generalised co-ordinates. Numerical calculations are carried out in order to show not only the influence of the base motion on vibrations of the flexible links but also the reverse connection.

Key words: dynamic analysis, flexibility of links, changing configuration

1. Introduction

Dynamics of flexible systems has been widely investigated over decades. The possibility of changing configuration is the feature of flexible systems causing most problems. Large base motion causes additional centrifugal forces and vibrations of the flexible links. Most popular approaches to solve the problem are based on the Finite Element Method (FEM) (Gao et al., 1989; Bircout et al., 1990; Du et al., 1996). In order to reduce the number of degrees of freedom and thus the dimension of the non-linear differential equations the modal method is used by Kane et al. (1987), Du et al. (1992).

A different approach used for discretisation of flexible links is based on the Rigid Finite Element Method (RFEM) described by Kruszewski et al. (1975). The method has been successfully applied to dynamic analysis of both planar (Wojciech, 1984; Adamiec and Wojciech, 1993) and spatial systems with changing configuration (Wojciech, 1990; Adamiec, 1992; Wittbrodt and Wojciech, 1995). Not only the flexibility of links but also friction in joints (Wojciech, 1995) as well as the flexibility of connecting elements (Wojciech,

1996) have been investigated. The results obtained using the RFEM have been compared with those obtained using the FEM (Wojciech, 1996; Plosa and Wojciech, 2000) as well as with experiments (Adamiec, 1992; Plosa and Wojciech, 2000). This comparison shows that the RFEM gives reliable results despite its simplicity and can be used to simulate the dynamics of systems with changing configuration.

In the above papers the base motion is assumed to be known. However, in such systems not only the base motion influences the vibrations of flexible links but there is also an opposite effect – the base motion can be influenced by the vibrations of the flexible links. In the present paper the RFEM is applied in order to derive the equations of motion for flexible systems with changing configurations assuming that the base motion is not known. In this way it will be possible to investigate the influence of flexible vibrations on the base motion. Since the link after discretisation is treated as a system of rigid bodies connected by joints, the conventional kinematic description widely employed for rigid manipulators can be used in a straightforward way for flexible mechanisms (Wojciech, 1996). The RFEM enables us to obtain the equations of motion of a system with changing configuration by merely a slight modification of the equations of motion of a system with all rigid links. Moreover, the equations of motion of a rigid system can be obtained as a special case of generalised equations of motion. Thus, using the same equations of motion, both vibrations of the flexible link caused by known motion of the base and the influence of vibrations on the base motion in the case of force input will be analysed in the paper.

2. Rigid finite element method and description of the model

The RFEM is based upon discretisation of continuous systems into undeformable bodies called rigid finite elements. These elements are connected with spring-damping elements, the characteristics of which are linear. In order to describe the motion of the system a local co-ordinate system is connected with each *rfe* (rigid finite element). The beginning of the system is placed in the mass centre and the axes coincide with the principal inertial axes before deformation of the link.

The co-ordinate system $\{r\}$ is connected with the base (Fig.1) and the vibrations of the system of *rfe*'s caused by the base motion are taken into consideration.

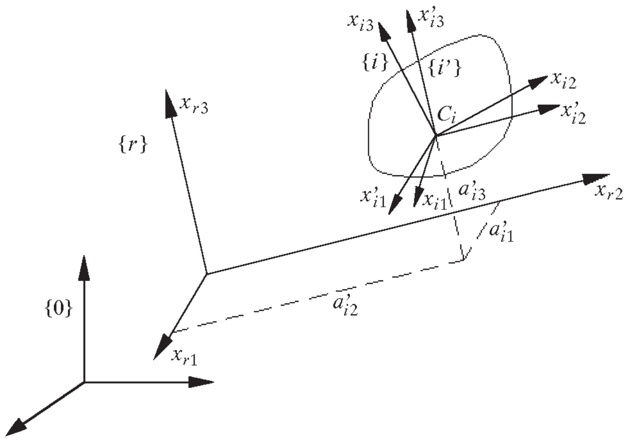

Fig. 1. Co-ordinate systems: $\{0\}$ – inertial system, $\{r\}$ – system connected with the moving base, $\{i'\}$ – local system connected with the rfe_i before deformation of the link

It is assumed that the base motion is not known and it depends on generalised co-ordinates defined by the vector $\boldsymbol{q}_b = [q_1^{(b)}, ..., q_n^{(b)}]^\top$. Then the base motion can be described using a transformation matrix dependent on the vector $\boldsymbol{q}_b$ in the following form

$$\mathbf{B}_0(\boldsymbol{q}_b) = {}_r^0\mathbf{T}(\boldsymbol{q}_b) = \mathbf{B}_1^{(b)}\mathbf{B}_2^{(b)}...\mathbf{B}_m^{(b)} \tag{2.1}$$

The matrices $\mathbf{B}_i^{(b)}$ are the transformation matrices which are used in the description of rigid systems by Craig (1988).

The flexible system is divided into $n+1$ *rfe*s, and the rfe_0 is the last link of the base (Fig.2).

The vector of generalised co-ordinates of each *rfe* takes the following form

$$\boldsymbol{q}_i^{(f)} = [x_{i1}, x_{i2}, x_{i3}, \varphi_{i1}, \varphi_{i2}, \varphi_{i3}]^\top \tag{2.2}$$

where

x_{i1}, x_{i2}, x_{i3} – co-ordinates of the mass centre of the rfe_i

$\varphi_{i1}, \varphi_{i2}, \varphi_{i3}$ – generalised co-ordinates describing bending and torsion of the rfe_i.

Thus, the generalised co-ordinates of the whole flexible system form the vector

$$\boldsymbol{q}^{(f)} = [\boldsymbol{q}_1^{(f)\top}, \boldsymbol{q}_2^{(f)\top}, ..., \boldsymbol{q}_n^{(f)\top}]^\top \tag{2.3}$$

The position of any *rfe* before deformation can be defined if the matrices of transformation from the system connected with rfe_i to the system of the

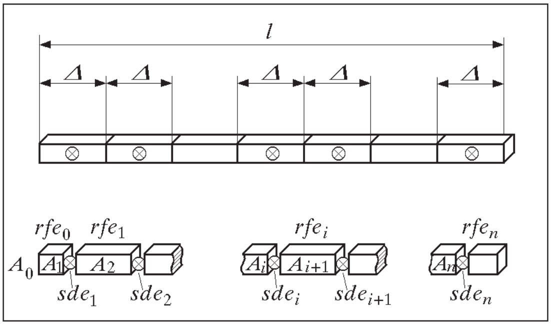

Fig. 2. System of *rfe*s and *sde*s

rfe$_0$ ${}^{r}_{i'}\mathbf{T} = \text{const}$, $i = 1,2,...,n$ are known. The matrix corresponding to the transformation from the co-ordinate system after deformation to that before the deformation can be written as follows

$$
{}^{i'}_{i}\mathbf{T} = \begin{bmatrix} c_{i2}c_{i3} & -c_{i2}s_{i3} & s_{i2} & x_{i1} \\ s_{i1}s_{i2}c_{i3} + c_{i1}s_{i3} & -s_{i1}s_{i2}s_{i3} + c_{i1}c_{i3} & -s_{i1}c_{i2} & x_{i2} \\ -c_{i1}s_{i2}c_{i3} + s_{i1}s_{i3} & c_{i1}s_{i2}s_{i3} + s_{i1}c_{i3} & c_{i1}c_{i2} & x_{i3} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.4}
$$

where: $x_{i1}, x_{i2}, x_{i3}, \varphi_{i1}, \varphi_{i2}, \varphi_{i3}$ are the generalised co-ordinates of the *rfe*$_i$, and $s_{ij} = \sin\varphi_{ij}$, $c_{ij} = \cos\varphi_{ij}$ for $i = 1,...,n$; $j = 1,2,3$.

The matrix of transformation from the system connected with the *rfe*$_i$ to the inertial system can be determined from the formula

$$
\mathbf{B}_i^{(f)} = {}^{0}_{r}\mathbf{T}\;{}^{r}_{i'}\mathbf{T}\;{}^{i'}_{i}\mathbf{T} = \mathbf{B}_0(\boldsymbol{q}_b){}^{r}_{i'}\mathbf{T}\;{}^{i'}_{i}\mathbf{T}(\boldsymbol{q}_i^{(f)}) = \mathbf{B}_0(\boldsymbol{q}_b)\mathbf{B}_i^{\prime(f)}(\boldsymbol{q}_i^{(f)}) \tag{2.5}
$$

where $\mathbf{B}_0$ is the matrix of transformation from the *rfe*$_0$ system to the inertial system defined by the formula (2.1), and

$$
\mathbf{B}_i^{\prime(f)}(\boldsymbol{q}_i^{(f)}) = {}^{r}_{i'}\mathbf{T}\;{}^{i'}_{i}\mathbf{T}(\boldsymbol{q}_i^{(f)})
$$

3. Equations of motion

In order to derive the equations of motion the Lagrange equations of the second order are used

$$
\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_l} - \frac{\partial T}{\partial q_l} + \frac{\partial V}{\partial q_l} = Q_l \qquad l = 1,...,m+6n \tag{3.1}
$$

where

T, V	–	kinetic and potential energy of the system, respectively
Q_l	–	generalised forces,
q_l	–	generalised coordinate which is a component of the following vector

$$\boldsymbol{q}_s = [q_1^{(b)},...,q_m^{(b)},x_{11},x_{12},x_{13},\varphi_{11},\varphi_{12},\varphi_{13},...,x_{n1},x_{n2},x_{n3},\varphi_{n1},\varphi_{n2},\varphi_{n3}]^\top \tag{3.2}$$

The equations of motion require a definition of the kinetic and potential energies of the system.

The kinetic energy includes the energies of both parts, the base and the flexible part

$$T = T_b + T_f \tag{3.3}$$

Having used the transformation matrices the following vectors describing the position of any point of the system can be defined

$$\begin{aligned}
&{}^0_i\boldsymbol{r} = \boldsymbol{r}_i = \mathbf{B}_1^{(b)}\mathbf{B}_2^{(b)}...\mathbf{B}_i^{(b)}\ {}^i\boldsymbol{r}_b && \text{for any point of the base}\\
&{}^0_i\boldsymbol{r} = \boldsymbol{r}_i = \mathbf{B}_0(\boldsymbol{q}_b)\ {}^r_{i'}\mathbf{T}\ {}^{i'}_i\mathbf{T}(\boldsymbol{q}_i^{(f)})\ {}^i\boldsymbol{r}_i = \mathbf{B}_0\mathbf{B}_i^{'(f)}\ {}^i\boldsymbol{r}_i && \text{for any point of the } rfe_i
\end{aligned} \tag{3.4}$$

Thus, the kinetic energy of the base is equal to

$$T_b = \sum_{i=1}^m T_{bi} = \frac{1}{2}\sum_{i=1}^m \mathrm{tr}\,(\dot{\mathbf{B}}_i^{(b)}\mathbf{H}_{bi}\dot{\mathbf{B}}_i^{(b)\top}) \tag{3.5}$$

where $\mathbf{B}_i^{(b)}$ and $\mathbf{H}_{bi}$ are the transformation and inertial matrices, respectively, defined according to the rules for rigid systems.

The kinetic energy of the flexible part is a sum of the energies of all *rfe*s, and the energy of each *rfe* can be calculated as follows

$$\begin{aligned}
T_{fk} &= \frac{1}{2}\int\limits_{m_k} \dot{\boldsymbol{r}}_k^\top\dot{\boldsymbol{r}}_k\ dm_k = \frac{1}{2}\int\limits_{m_k} \mathrm{tr}\,(\dot{\boldsymbol{r}}_k\dot{\boldsymbol{r}}_k^\top)\ dm_k =\\
&= \frac{1}{2}\int\limits_{m_k} \mathrm{tr}\,(\dot{\mathbf{B}}_k^{(f)}\ {}^k\boldsymbol{r}_k\ {}^k\boldsymbol{r}_k^\top\ \dot{\mathbf{B}}_k^{(f)\top})\ dm_k =\\
&= \frac{1}{2}\,\mathrm{tr}\Big(\dot{\mathbf{B}}_k^{(f)}\int\limits_{m_k} {}^k\boldsymbol{r}_k\ {}^k\boldsymbol{r}_k^\top\ dm_k\ \dot{\mathbf{B}}_k^{(f)\top}\Big) = \frac{1}{2}\,\mathrm{tr}\,(\dot{\mathbf{B}}_k^{(f)}\mathbf{H}_{fk}\dot{\mathbf{B}}_k^{(f)\top})
\end{aligned} \tag{3.6}$$

where

$$\mathbf{H}_{fk} = \int\limits_{m_k} [{}^kx_{k1}, {}^kx_{k2}, {}^kx_{k3}, 1]^\top [{}^kx_{k1}, {}^kx_{k2}, {}^kx_{k3}, 1]\ dm_k$$

If the local system is a system of inertial principal axes, then

$$\mathbf{H}_{jk} = \operatorname{diag}[h_{kj}] \tag{3.7}$$

where

$$h_{kj} = \int\limits_{m_k} {}^k x_{kj}^2 \, dm_k \qquad j = 1,2,3 \qquad h_{k4} = \int\limits_{m_k} dm_k = m_k$$

Eventually, the kinetic energy of the whole system is calculated as follows

$$\begin{aligned} T &= \frac{1}{2}\sum_{k=1}^{m} \operatorname{tr}(\dot{\mathbf{B}}_k^{(b)}\mathbf{H}_{bk}\dot{\mathbf{B}}_k^{(b)\top}) + \frac{1}{2}\sum_{k=1}^{n} \operatorname{tr}(\dot{\mathbf{B}}_k^{(f)}\mathbf{H}_{fk}\dot{\mathbf{B}}_k^{(f)\top}) \\ \dot{\mathbf{B}}_k^{(b)} &= \frac{d\mathbf{B}_k^{(b)}}{dt} = \sum_{i=1}^{m} \frac{\partial \mathbf{B}_k^{(b)}}{\partial q_i^{(b)}}\dot{q}_i^{(b)} = \sum_{i=1}^{m} \mathbf{B}_{ki}^{(b)}\dot{q}_i^{(b)} \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \mathbf{B}_{ki}^{(b)} &= \frac{\partial \mathbf{B}_k^{(b)}}{\partial q_i^{(b)}} \\ \dot{\mathbf{B}}_k^{(f)} &= \frac{d\mathbf{B}_k^{(f)}}{dt} = \frac{d(\mathbf{B}_0\mathbf{B}_k^{'(f)})}{dt} = \frac{d\mathbf{B}_0}{dt}\mathbf{B}_k^{'(f)} + \mathbf{B}_0\frac{d\mathbf{B}_k^{'(f)})}{dt} = \\ &= \sum_{i=1}^{m} \mathbf{B}_{0i}^{(b)}\mathbf{B}_k^{'(f)}\dot{q}_i^{(b)} + \mathbf{B}_0\sum_{i=1}^{6}\frac{\partial \mathbf{B}_k^{'(f)}}{\partial q_{ki}^{(f)}}\dot{q}_{ki}^{(f)} = \sum_{i=1}^{m} \mathbf{B}_{0i}^{(b)}\mathbf{B}_k^{'(f)}\dot{q}_i^{(b)} + \mathbf{B}_0\sum_{i=1}^{6}\mathbf{B}_{ki}^{'(f)}\dot{q}_{ki}^{(f)} \end{aligned} \tag{3.9}$$

Having calculated the necessary derivative and after necessary transformations, the Lagrange equation component dependent on the kinetic energy for the rigid part can be written in the following form (for $j = 1,...,m$)

$$\begin{aligned} \varepsilon_j^{(b)} &= \frac{d}{dt}\frac{\partial T^{(b)}}{\partial \dot{q}_j^{(b)}} - \frac{\partial T^{(b)}}{\partial q_j^{(b)}} = \sum_{i=1}^{m} a_{ji}^{(b)}\ddot{q}_k^{(b)} + \sum_{k=1}^{n}\sum_{i=1}^{6} a_{jki}^{(f)}\ddot{q}_{ki}^{(f)} + \\ &+ \sum_{i=1}^{m}\sum_{l=1}^{m} b_{jil}^{(b)}\dot{q}_i^{(b)}\dot{q}_l^{(b)} + 2\sum_{i=1}^{m}\sum_{k=1}^{n}\sum_{l=1}^{6} b_{jikl}^{(bf)}\dot{q}_i^{(b)}\dot{q}_{kl}^{(f)} \end{aligned} \tag{3.10}$$

where

$$a_{ji}^{(b)} = \sum_{k=1}^{m} \operatorname{tr}(\mathbf{B}_{kj}^{(b)}\mathbf{H}_{bk}\mathbf{B}_{ki}^{(b)\top}) + \sum_{k=1}^{n} \operatorname{tr}\Big(\mathbf{B}_{0j}\mathbf{B}_k^{'(f)}\mathbf{H}_{fk}(\mathbf{B}_{0i}\mathbf{B}_k^{'(f)})^\top\Big)$$

$$a^{(f)}_{jki} = \operatorname{tr}(\mathbf{B}_{0j}\mathbf{B}^{'(f)}_{k}\mathbf{H}_{fk}\mathbf{B}^{(f)\top}_{ki})$$

$$b^{(b)}_{jil} = \sum_{k=1}^{m} \operatorname{tr}(\mathbf{B}^{(b)}_{kj}\mathbf{H}_{bk}\mathbf{B}^{(b)\top}_{kil}) + \sum_{k=1}^{n} \operatorname{tr}\Big(\mathbf{B}_{0j}\mathbf{B}^{'(f)}_{k}\mathbf{H}_{fk}(\mathbf{B}_{0il}\mathbf{B}^{'(f)}_{k})^{\top}\Big)$$

$$b^{(bf)}_{jikl} = \operatorname{tr}\Big(\mathbf{B}_{0j}\mathbf{B}^{'(f)}_{k}\mathbf{H}_{fk}(\mathbf{B}_{0i}\mathbf{B}^{'(f)}_{kl})^{\top}\Big)$$

And the component for the flexible part equals (for $k = 1, ..., n$; $j = 1, ..., 6$)

$$\begin{aligned}\varepsilon^{(f)}_{kj} &= \frac{d}{dt}\frac{\partial T^{(f)}}{\partial \dot{q}^{(f)}_{kj}} - \frac{\partial T^{(f)}}{\partial q^{(f)}_{kj}} = \sum_{i=1}^{m} a^{(b)}_{kji}\ddot{q}^{(b)}_{i} + \sum_{i=1}^{6} a^{(f)}_{kji}\ddot{q}^{(f)}_{ki} + \\ &+ \sum_{i=1}^{m}\sum_{l=1}^{m} b^{(b)}_{kjil}\dot{q}^{(b)}_{i}\dot{q}^{(b)}_{l} + 2\sum_{i=1}^{m}\sum_{l=1}^{6} b^{(bf)}_{kjil}\dot{q}^{(b)}_{i}\dot{q}^{(f)}_{kl}\end{aligned} \tag{3.11}$$

where

$$a^{(b)}_{kji} = \operatorname{tr}\Big(\mathbf{B}^{(f)}_{kj}\mathbf{H}_{fk}(\mathbf{B}'_{0i}\mathbf{B}^{(f)}_{k})^{\top}\Big)$$

$$a^{(f)}_{kji} = \operatorname{tr}\Big(\mathbf{B}^{(f)}_{kj}\mathbf{H}_{fk}(\mathbf{B}_{0}\mathbf{B}^{'(f)}_{ki})^{\top}\Big)$$

$$b^{(b)}_{kjil} = \operatorname{tr}\Big(\mathbf{B}^{(f)}_{kj}\mathbf{H}_{fk}(\mathbf{B}_{0il}\mathbf{B}^{'(f)}_{k})^{\top}\Big)$$

$$b^{(bf)}_{kjil} = \operatorname{tr}\Big(\mathbf{B}^{(f)}_{kj}\mathbf{H}_{fk}(\mathbf{B}_{0i}\mathbf{B}^{'(f)}_{kl})^{\top}\Big)$$

The potential energy of the system consists of the energy of spring deformation of the sdes and the energy of gravity forces. When the sde_e connects the elements l and p (Fig. 3), then the elastic energy of this element can be formulated as follows

$$V_{se} = \frac{1}{2}\sum_{j=1}^{3} c^{e}_{j}(x^{e}_{pj} - x^{e}_{lj})^2 + \frac{1}{2}\sum_{j=1}^{3} k^{e}_{j}(\varphi^{e}_{pj} - \varphi^{e}_{lj})^2 \tag{3.12}$$

where

c^{e}_{j} – coefficients of longitudinal stiffness (Kruszewski et al., 1975)
k^{e}_{j} – coefficients of rotational stiffness (Kruszewski et al., 1975)

and x^{e}_{lj}, x^{e}_{pj}, φ^{e}_{lj}, φ^{e}_{pj} are the co-ordinates with respect to the base system.

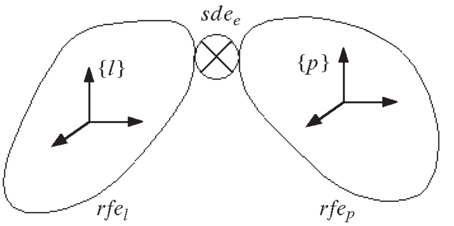

Fig. 3. *sde* element connecting two *rfe*s

The potential energy of the whole system is a sum of the elastic strain energies of all *sde*s

$$V_s = \sum_{e=1}^{n_{est}} V_{se} \tag{3.13}$$

In order to differentiate the expression for spring deformation energy it is assumed that the co-ordinates of the sde_e with respect to the co-ordinate systems connected with rfe_l and rfe_p are defined by the following vectors, respectively

$$\begin{aligned} {}^l\boldsymbol{r}_e &= [\eta_{l1}, \eta_{l2}, \eta_{l3}]^\top \qquad \text{in the } \{l\} \text{ system} \\ {}^p\boldsymbol{r}_e &= [\eta_{p1}, \eta_{p2}, \eta_{p3}]^\top \qquad \text{in the } \{p\} \text{ system} \end{aligned} \tag{3.14}$$

The elastic energy of the sde_e is defined with respect to the co-ordinate system $\{r\}$ connected with the moving base. Thus, the following holds

$$ {}^r\boldsymbol{r}_{el} = {}^r_{l'}\mathbf{T}\, {}^{l'}_{l}\mathbf{T}\, {}^l\boldsymbol{r}_e \qquad\qquad {}^r\boldsymbol{r}_{ep} = {}^r_{p'}\mathbf{T}\, {}^{p'}_{p}\mathbf{T}\, {}^p\boldsymbol{r}_e \tag{3.15}$$

where

${}^r_{l'}\mathbf{T}$, ${}^r_{p'}\mathbf{T}$	–	matrices with constant coefficients
${}^{l'}_{l}\mathbf{T}$, ${}^{p'}_{p}\mathbf{T}$	–	matrices with coefficients dependent on the generalised co-ordinates of rfe_l and rfe_p, respectively.

The matrices ${}^{i'}_{i}\mathbf{T}$ for $i \in \{l, p\}$ are defined in (2.4).

After necessary calculations we obtain the following formulae for the co-ordinates

$$\begin{aligned} x^e_{i1} &= c_{i2}c_{i3}\eta_{i1} - c_{i2}s_{i3}\eta_{i2} + s_{i2}\eta_{i3} + x_{i1} + u_{i1} \\ x^e_{i2} &= (s_{i1}s_{i2}c_{i3} + c_{i1}s_{i3})\eta_{i1} + (c_{i1}c_{i3} - s_{i1}s_{i2}s_{i3})\eta_{i2} - s_{i1}c_{i2}\eta_{i3} + x_{i2} + u_{i2} \\ x^e_{i3} &= (s_{i1}s_{i3} - c_{i1}s_{i2}c_{i3})\eta_{i1} + (c_{i1}s_{i2}s_{i3} + s_{i1}c_{i3})\eta_{i2} + c_{i1}c_{i2}\eta_{i3} + x_{i3} + u_{i3} \end{aligned} \tag{3.16}$$

where $\boldsymbol{u}_i = [u_{i1}, u_{i2}, u_{i3}]^\top$ is the position vector of the co-ordinate system connected with the ith *rfe* with respect to the co-ordinate system $\{r\}$.

Having used (3.12) the derivatives of the elastic energy with respect to the generalised co-ordinates can be written as follows

$$
\begin{aligned}
\frac{\partial V_e}{\partial x_{pj}} &= c_j^e(x_{pj}^e - x_{lj}^e) \\
\frac{\partial V_e}{\partial \varphi_{pj}} &= k_j^e(\varphi_{pj}^e - \varphi_{lj}^e) + \sum_{k=1}^{3} c_k^e(x_{pj}^e - x_{lj}^e)\frac{\partial x_{pk}^e}{\partial \varphi_{pj}^e} \\
\frac{\partial V_e}{\partial x_{lj}} &= -c_j^e(x_{pj}^e - x_{lj}^e) \\
\frac{\partial V_e}{\partial \varphi_{lj}} &= -k_j^e(\varphi_{pj}^e - \varphi_{lj}^e) - \sum_{k=1}^{3} c_k^e(x_{pj}^e - x_{lj}^e)\frac{\partial x_{lk}^e}{\partial \varphi_{lj}^e}
\end{aligned}
\tag{3.17}
$$

The potential energy of the gravity forces can be calculated as follows

$$
V_p = g\boldsymbol{\Theta}_3^\top\Big(\sum_{i=1}^{m} m_i^{(b)}\mathbf{B}_i^{(b)}\boldsymbol{r}_{bci} + \sum_{i=1}^{n} m_{fi}\mathbf{B}_i\boldsymbol{\Theta}_4\Big) \tag{3.18}
$$

where

m_{bi}	–	mass of the ith link of the base system
m_{fi}	–	mass of the ith *rfe* of the flexible link

and

$$
\boldsymbol{\Theta}_4 = [0,0,0,1]^\top \qquad\qquad \boldsymbol{\Theta}_3^\top = [0,0,1,0]
$$

The respective derivatives are calculated as follows

$$
\begin{aligned}
\frac{\partial V_p}{\partial q_j^{(b)}} &= g\boldsymbol{\Theta}_3^\top\Big(\sum_{i=1}^{m} m_{bi}\mathbf{B}_{ij}^{(b)}\boldsymbol{r}_{bci} + \sum_{i=1}^{n} m_{fi}\mathbf{B}_{0j}\mathbf{B}_i'\boldsymbol{\Theta}_4\Big) && j = 1,...,m \\
\frac{\partial V_p}{\partial q_{kj}^{(f)}} &= g\boldsymbol{\Theta}_3^\top m_{fk}\mathbf{B}_{kj}^{(f)}\boldsymbol{\Theta}_4 && k = 1,...,n \\
& && j = 1,...,6
\end{aligned}
\tag{3.19}
$$

Substituting equations (3.11), (3.17) and (3.19) into the Lagrange equations (3.1), the equations of motion, which form a set of $m + 6n$ non-linear differential equations of the second order, are obtained. Various methods can be used for integrating those equations.

4. Example of calculations

The derived equations of motion can be very easily adapted for dynamic analysis of a system when the base motion is known. Numerical calculations have been carried out for a manipulator described by Kane et al. (1987) (Fig. 4).

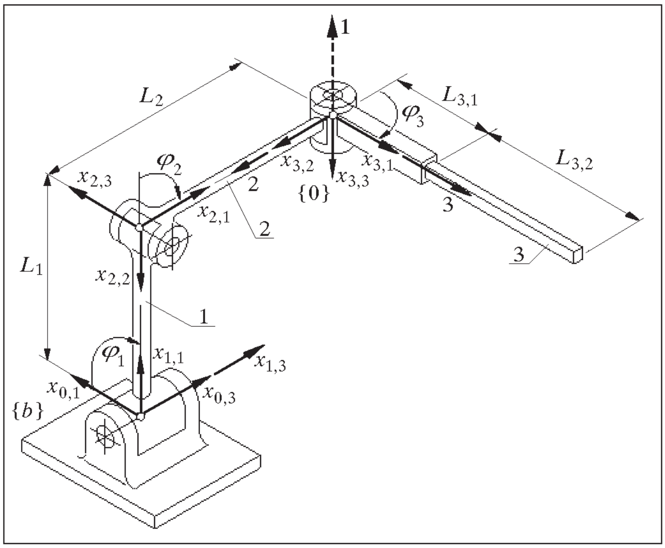

Fig. 4. Model of the manipulator

Kane et al. (1987) considered an example of base motion called a deployment process, which is characterised by a smooth change of the angles

$$\psi_i(t) = \begin{cases} \pi - \dfrac{\pi}{2}\alpha_i\Big(t - \dfrac{T}{2\pi}\sin\dfrac{2\pi t}{T}\Big) & \text{for} \quad 0 \leqslant t \leqslant T \\ \dfrac{\pi}{2} & \text{for} \quad t > T \end{cases} \tag{4.1}$$

where $i = 1,2,3$, $\alpha_1 = 1/2$, $\alpha_2 = 3/4$, $\alpha_3 = 1$ and $T = 15$.

Wittbrodt and Wojciech (1995) as well as Płosa and Wojciech (2000) presented a comparison of results obtained using the RFEM with those obtained using the FEM and presented by Kane et al. (1987) and Du et al. (1992). A very good correspondence of the results has been achieved when the base motion was known.

For the purpose of this paper it is assumed that the base motion is not known and the case when each *rfe* has only three degrees of freedom in relative motion ($x_{i1} = x_{i2} = x_{i3} = 0$) is considered. The motion of the system is realised by application of external forces, and the following procedure has been applied:

1. Mass parameters are attributed to links 1 and 2 of the manipulator. It is assumed that $E = 6.895 \cdot 10^{10}\,\mathrm{Nm}^{-2}$, $G = 2.6519 \cdot 10^{10}\,\mathrm{Nm}^{-2}$, $\rho = 2766.67\,\mathrm{kg\,m}^{-3}$ and both links have a square box cross-section where $h_z = 0.05\,\mathrm{m}$ and $h_w = 0.0475\,\mathrm{m}$ are external and internal dimensions, respectively. The third link has the same parameters as in the above mentioned papers and consists of two segments. The first segment B_1 is 2.667 m long and has a symmetric box cross section, while the second segment B_2 is a 5.333 m long channel. Both segments are made of a material for which $E = 6.895 \cdot 10^{10}\,\mathrm{Nm}^{-2}$, $G = 2.6519 \cdot 10^{10}\,\mathrm{Nm}^{-2}$, $\rho = 2766.67\,\mathrm{kg\,m}^{-3}$. The section parameters for B_1 are: cross-section area $A = 3.84 \cdot 10^{-4}\,\mathrm{m}^2$, cross-sectional moments of inertia $I_{yy} = I_{zz} = 1.5 \cdot 10^{-7}\,\mathrm{m}^4$, the torsional one about the axis $J = 2.2 \cdot 10^{-11}\,\mathrm{m}^4$, sectional mass centre offset $x_{gc} = 0$, while the corresponding parameters of B_2 are: $A = 7.3 \cdot 10^{-5}\,\mathrm{m}^2$, $I_{yy} = 8.2181 \cdot 10^{-9}\,\mathrm{m}^4$, $I_{zz} = 4.8746 \cdot 10^{-9}\,\mathrm{m}^4$, $J = 2.433 \cdot 10^{-7}\,\mathrm{m}^4$, $x_{gc} = 0.01875\,\mathrm{m}$.

2. Torques necessary for realisation of the motion described by (4.1) are calculated assuming that all three links are rigid. The equations of motion of the rigid system can be obtained as a particular case of the general equations of motion assuming that the number of *rfe*s=0.

3. The torques calculated in 2 have been applied to the joints assuming that the third link is flexible and is divided into 6 *rfe*s. Having integrated the equations of motion the angles ψ_i, $i = 1, 2, 3$, their velocities and accelerations as well as deflections and rotation of the tip of the flexible link have been calculated.

In the rest of the paper the kinematic input means the case of motion realised by a change in the angles described by (4.1), and the force input means the torques, calculated according to the above procedure, applied to the joints.

Figure 5 shows a comparison of the deflections and rotation of the tip of the flexible link obtained for the kinematic and force input. The denotations by Kane et al. (1987) are used and the following holds

$$u_1 = -q^{(f)}_{n,1} - \frac{1}{2}q^{(f)}_{n,5}l_n \qquad u_2 = -q^{(f)}_{n,2} - \frac{1}{2}q^{(f)}_{n,4}l_n \qquad \theta = q^{(f)}_{n,6}$$

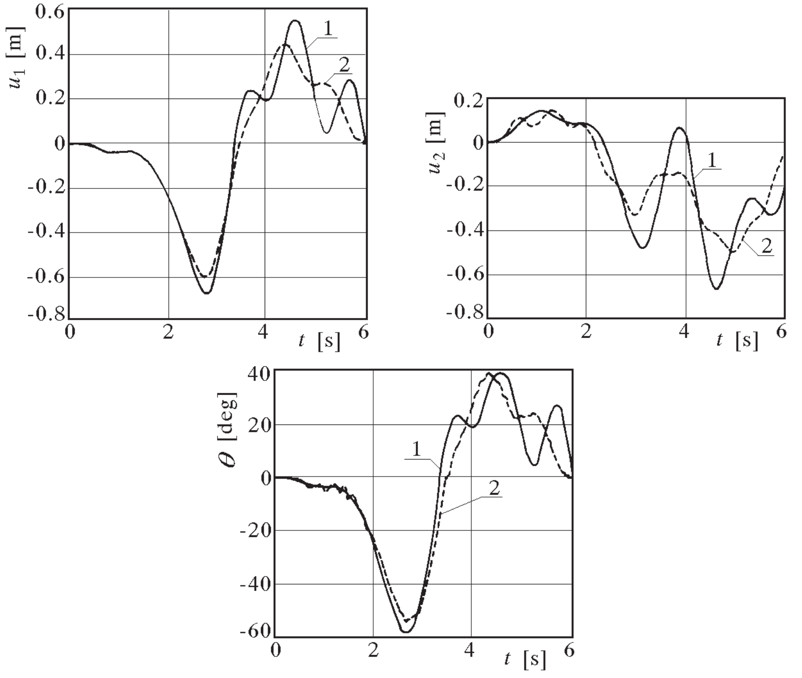

Fig. 5. Deflections u_1, u_2 and rotation θ of the end of the third link; 1 – kinematic input, 2 – force input

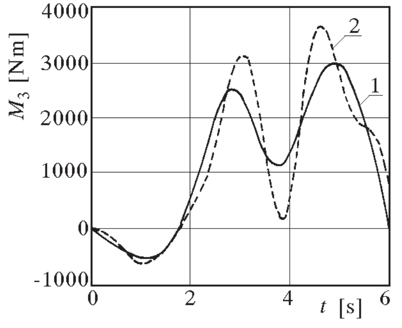

Fig. 6. Torque in the third joint required for realisation of motion; 1 – rigid link, 2 – flexible link

where $q^{(f)}_{n,i}$, $i = 1, ..., 6$ are the generalised co-ordinates of the rfe_n, and l_n is its length.

The torque which should be applied to joint 3 in order to ensure the motion as in (4.1) is presented in Fig. 6. One course is calculated assuming that the third link is rigid and the other one when the link is flexible.

It can be seen that the difference is quite large, which means that the torque calculated for a rigid system can be applied to a flexible system only when small deflections are considered. However, the procedure presented enables us to estimate whether the influence of the flexible vibrations on the base motion is significant.

Flexibility also affects the velocities and accelerations of the joint variables. The influence of vibrations of the third link on the first and second joints is not as large as on the third one.

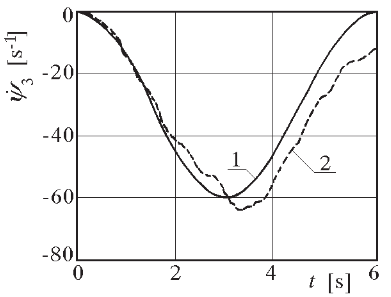

Fig. 7. Velocity in the third joint; 1 – kinematic input, 2 – force input

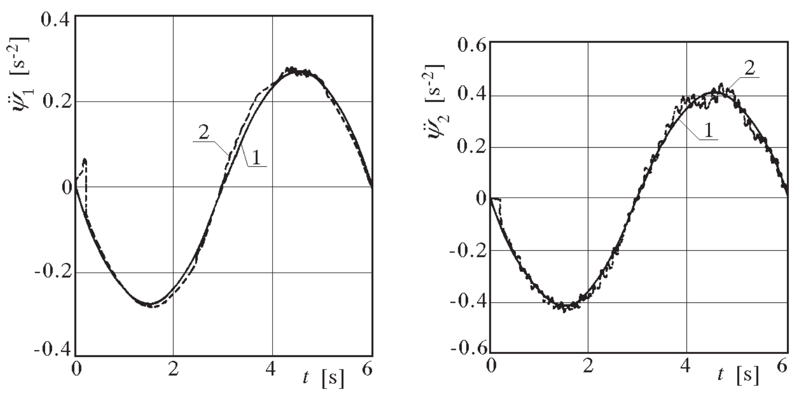

Fig. 8. Accelerations in the first and second joint, respectively; 1 – kinematic input, 2 – force input

Figure 7 presents the velocity in the third joint for both kinematic and force inputs, while Fig. 8 shows the accelerations in the first and second joints for the same inputs.

5. Final remarks

The paper presents the rigid finite element method applied to dynamic analysis of systems with flexible links when the base motion is not known. An essential feature of the method is that the equations of motion are derived only once, and they allow us to analyse both rigid and flexible systems, the base motion is known or unknown. This means that not only the influence of the base motion on the vibrations of a flexible link can be analysed but also the effect which flexibility has on the base motion.

The method can be easily generalised to consider more complex problems when the flexible links are found between chains of rigid links.

The aim of the paper is to show that in certain cases it is necessary to take into account the influence of flexible vibrations on the base motion. This influence depends on the parameters of links realising the base motion as well as on the drive input.

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Wzajemne oddziaływanie ruchu bazowego i drgań członów podatnych manipulatorów

Streszczenie

W artykule przedstawiono zastosowanie metody Sztywnych Elementów Skończonych do modelowania członów podatnych manipulatora. Równania ruchu wyprowadzono zakładając, że ruch bazowy nie jest znany, czyli, że zależy od pewnych współrzędnych uogólnionych. Przeprowadzone obliczenia numeryczne pokazują wzajemne oddziaływanie drgań członu podatnego oraz ruchu bazowego.

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