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ON THE MAGNETIZATION-BASED LAGRANGIAN METHODS FOR 2D AND 3D VISCOUS FLOWS. PART 1 – THEORETICAL BACKGROUND

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The paper presents the background of an alternative formulation of the Navier-Stokes equation using a variable called the magnetization. Several variants of governing equations, based on different choices of a particular gauge transform, are discussed. The remaining part of the paper is devoted to the formulation of a Lagrangian approach to 2D and 3D viscous flows. First, the carrier of the magnetization (the magneton) is defined and the corresponding induction law is derived. The instantaneous velocity field is constructed as a superposition of contributions from a large set of magnetons and a uniform stream. An essential feature of the method is a one-time-step operator splitting, consisting in the consecutive solution of three sub-problems: generation of the magnetization on solid boundaries, advection-diffusion of the magnetization and stretching.

Key words: viscous flow, Navier-Stokes equations, magnetization, gauge transform

1. Introduction

Viscous incompressible flows are usually described by the set of the Navier-Stokes and continuity equations

$$\begin{aligned} &\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla p + \nu\Delta\boldsymbol{v} \\ &\nabla\cdot\boldsymbol{v} = 0 \end{aligned} \tag{1.1}$$

The unknowns are the velocity field $\boldsymbol{v}$ and the pressure p. The fluid density is assumed here to be unary. It should be noted that the pressure is determined modulo an arbitrary function of time.

According to a well-known theorem, any vector field can be expressed as a sum of two components. One of these components is defined as a gradient of a scalar field, while the second one is a solenoid vector field. Thus, the divergence-free velocity $\boldsymbol{v}$ can be written as

$$\boldsymbol{v} = \boldsymbol{m} - \nabla\phi \tag{1.2}$$

The vector field $\boldsymbol{m}$ is called, after Buttke (Chorin, 1994), the "magnetization". The correspondence $\boldsymbol{m} \to (\boldsymbol{v}, \phi)$ is unique, providing that the normal component of the velocity $\boldsymbol{v}$ at the boundary is equal to zero, and appropriate regularity conditions are satisfied. On the other hand, the transform $\boldsymbol{v} \to (\boldsymbol{m}, \phi)$ obtained from (1.2) is not unique. This fact allows us to formulate the governing equations in alternative forms, employing various magnetization fields. In the paper we show a few of them, discuss their properties, formulate a Lagrangian numerical method for the magnetization and finally present results of some numerical experiments.

2. The gauge transform

Using velocity representation (1.2), the Navier-Stokes equation can be written as

$$\partial_t \boldsymbol{m} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \nu\Delta\boldsymbol{m} + \nabla(\partial_t\phi - \nu\Delta\phi - p) = 0 \tag{2.1}$$

Equation (2.1) can be viewed as a particular case of a more general equation

$$\partial_t \boldsymbol{m} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \nu\Delta\boldsymbol{m} + \nabla\lambda \tag{2.2}$$

where λ denotes some scalar field called a gauge field. We will refer to any particular selection of the gauge field λ as a gauge transform.

Let us define the subspace of solenoidal vector fields

$$V = \{\boldsymbol{v} \in L^2(\Omega) : \ \nabla\cdot\boldsymbol{v} = 0 \ \text{ in } \ \Omega, \ \ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \ \text{ at } \ \partial\Omega\}$$

Any square integrable vector field in Ω can be written as a sum of some element from the space V and the gradient of a certain scalar field. This decomposition is unique and the components are orthogonal with respect to the inner product in $L^2(\Omega)$. Thus, the orthogonal projection operator can be defined

$$\boldsymbol{m} \to \mathcal{P}\boldsymbol{m} \equiv \boldsymbol{v} \in V$$

and equation (2.2) can be formulated as follows

$$\partial_t \boldsymbol{m} + (\mathcal{P}\boldsymbol{m} \cdot \nabla)\mathcal{P}\boldsymbol{m} = \nu \Delta \boldsymbol{m} + \nabla\lambda \tag{2.3}$$

Assume that some gauge field λ has been chosen and the magnetization field obeys equation (2.3). Then the solenoidal vector field $\boldsymbol{v} = \mathcal{P}\boldsymbol{m}$ satisfies the governing equations of a viscous liquid motion. Indeed, the continuity equation is fulfilled since $\boldsymbol{v} \in V$. The magnetization can be expressed as $\boldsymbol{m} = \boldsymbol{v} + \nabla\phi$. Insertion of this form into (2.3) yields after some algebra the following equation

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \nu \Delta \boldsymbol{v} + \nabla(\lambda - \partial_t \phi + \nu \Delta \phi) \tag{2.4}$$

Thus, the vector field $\boldsymbol{v}$ satisfies the Navier-Stokes equation, and the corresponding pressure is given as

$$p = \partial_t \phi + \nu \Delta \phi - \lambda + f(t) \tag{2.5}$$

It can be shown that the gauge field λ can be chosen arbitrarily. Assume there are two different magnetization fields $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ corresponding to the gauge fields λ_1 and λ_2, respectively. The field $\boldsymbol{m}_1$ can be expressed as a sum $\boldsymbol{m}_1 = \boldsymbol{v} + \nabla\phi_1$. It is immediate to verify that equation (2.3) for $\lambda = \lambda_2$ admits the solution $\boldsymbol{m}_2$ in the following form

$$\boldsymbol{m}_2 = \boldsymbol{v} + \nabla\phi_2$$

Thus, the solution $\boldsymbol{m}_2$ defines the same velocity field and can be also verified that

$$\nabla(\partial_t \phi_1 - \nu \Delta \phi_1 - \lambda_1) = \nabla(\partial_t \phi_2 - \nu \Delta \phi_2 - \lambda_2)$$

thus the resulting pressure fields differ only by a (time-dependent) constant.

If an initial/boundary-value problem formulated for equation (2.3) permits a unique solution, then one concludes that the magnetization fields computed for different gauges will correspond to the equivalent velocity and pressure fields. Thus, the choice of a particular gauge is arbitrary. Note that the nonlinear term in equation (2.3) is a bilinear one. Looking for convenient variants of equation (2.3), we assume the following form of the gauge field λ

$$\lambda = \frac{1}{2}\alpha \boldsymbol{v} \cdot \boldsymbol{v} + \beta \boldsymbol{v} \cdot \boldsymbol{m} + \frac{1}{2}\gamma \boldsymbol{m} \cdot \boldsymbol{m} \tag{2.6}$$

After elementary calculations, we find the gradient of the above expression

$$\begin{aligned} \nabla\lambda \;&=\; \alpha[(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \boldsymbol{v} \times \operatorname{rot} \boldsymbol{v}] + \gamma[(\boldsymbol{m} \cdot \nabla)\boldsymbol{m} + \boldsymbol{m} \times \operatorname{rot} \boldsymbol{m}] \\ &+\; \beta[(\boldsymbol{v} \cdot \nabla)\boldsymbol{m} + (\boldsymbol{m} \cdot \nabla)\boldsymbol{v} + \boldsymbol{v} \times \operatorname{rot} \boldsymbol{m} + \boldsymbol{m} \times \operatorname{rot} \boldsymbol{v}] \end{aligned} \tag{2.7}$$

In (2.7) α, β, γ are arbitrary constants. Any particular choice of these constants defines some gauge transform, which, in turn, determines a governing equation for the magnetization field. Having it solved one is able to find the potential ϕ, and then the pressure field can be determined with the use of (2.5).

Here is a list of some particular, possibly interesting variants of the gauge transform and the corresponding form of the governing equation:

Case 1: $\alpha = \gamma = 1,\ \beta = -1$

$$\begin{aligned} &\partial_t \boldsymbol{m} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{m} = \nu\Delta\boldsymbol{m} + (\boldsymbol{m}\cdot\nabla)\nabla\phi \\ &p + \frac{1}{2}v^2 = f(t) + \partial_t\phi - \nu\Delta\phi - \frac{1}{2}\boldsymbol{m}\cdot\boldsymbol{m} + \boldsymbol{m}\cdot\boldsymbol{v} \end{aligned} \tag{2.8}$$

Case 2: $\alpha = 1,\ \beta = -1,\ \gamma = 0$

$$\begin{aligned} &\partial_t \boldsymbol{m} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{m} = \nu\Delta\boldsymbol{m} - (\nabla\boldsymbol{v})^\top\boldsymbol{m} \\ &p + \frac{1}{2}v^2 = f(t) + \partial_t\phi - \nu\Delta\phi + \boldsymbol{m}\cdot\boldsymbol{v} \end{aligned} \tag{2.9}$$

Case 3: $\alpha = 1,\ \beta = 0,\ \gamma = -1$

$$\begin{aligned} &\partial_t \boldsymbol{m} + (\boldsymbol{m}\cdot\nabla)\boldsymbol{m} = \nu\Delta\boldsymbol{m} - \nabla\phi\times\operatorname{rot}\boldsymbol{m} \\ &p + \frac{1}{2}v^2 = f(t) + \partial_t\phi - \nu\Delta\phi + \frac{1}{2}\boldsymbol{m}\cdot\boldsymbol{m} \end{aligned} \tag{2.10}$$

Case 4: $\alpha = 1,\ \beta = \gamma = 0$

$$\begin{aligned} &\partial_t \boldsymbol{m} - \boldsymbol{v}\times\operatorname{rot}\boldsymbol{m} = \nu\Delta\boldsymbol{m} \\ &p + \frac{1}{2}v^2 = f(t) + \partial_t\phi - \nu\Delta\phi \end{aligned} \tag{2.11}$$

Case 5: $\alpha = \beta = \gamma = 0,$

$$\begin{aligned} &\partial_t \boldsymbol{m} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \nu\Delta\boldsymbol{m} \\ &p = f(t) + \partial_t\phi - \nu\Delta\phi \end{aligned} \tag{2.12}$$

Case 2 has been introduced by Buttke (Chorin, 1994), Russo and Smerka (1999) analysed cases 2, 4 and 5. Summers and Chorin (1996) made use of the equations in case 2 and formulated the Lagrangian method for particles carrying magnetization. Generally, the Buttke gauge seems to be most popular. Russo and Smerka remarked that the stretching term in the first equation in (2.9), i.e. $(\nabla \boldsymbol{v})^{\top} \boldsymbol{m}$ is similar to the vortex stretching term in the Helmholtz equation for the vorticity. This term describes the mechanism of the deformation of vortex structures in the flow, consisting in local stretching in one space direction and compression in the remaining two. This behavior is related to the existence of a large positive (or negative) eigenvalue of the tensor $\nabla \boldsymbol{v}$. The dominating eigenvalue of $\nabla \boldsymbol{v}$ renders the magnetization (or vorticity) tend to be concentrated into the form of thin filaments oriented locally along the corresponding eigendirection.

Changing the gauge one may be able to reduce this phenomenon. It seems that the gauge defined in case 1 is a better choice. The stretching term in this case has the form

$$(\boldsymbol{m} \cdot \nabla) \nabla \phi = \boldsymbol{m} H(\phi)$$

where $H(\phi)$ is the Hessian, i.e. $H(\phi) = \{\partial_{x_i x_j} \phi\}$. The trace of this matrix, equal to the sum of the eigenvalues, is not restricted by the incompressibility condition. This fact can reduce the tendency for high concentration and instability.

In this study, the formulation with inhomogeneous terms with respect to the magnetization will not be analysed. Any inhomogeneous term in the governing equation acts as a source of the magnetization, i.e. the magnetization is created at points where it was previously equal to zero. It seems to be unreasonable to formulate a Lagrangian (or particle) approach based on such formulation. In addition, the case 4 is not convenient for numerical computations. According to Russo and Smerka (1999) this formulation suffers from essential instability which may be difficult to overcome.

3. The carrier of the magnetization

In order to construct a numerical method based on the Lagrangian approach one should define an elemental "source" (or particle) of the magnetization field. It is convenient (and natural) to assume spherical symmetry of the spatial distribution of the magnetization insidesuch a particle. The radius of the particle can be finite – in such a case the magnetization differs from zero inside

the spherical "core" and vanishes identically outside the core. If the radius is infinite, the magnetization fills the whole space. The spatial distribution of the magnetization should be, however, concentrated in a relatively small neighborhood of the particle center. The effect of the "localization" can be achieved by applying, for instance, a "slim" Gaussian distribution.

In the remaining part of this work, we will refer to the magnetization particles (independently on the details of their construction) as the magnetons.

The magnetization field carried by a single magneton is defined as follows

$$\boldsymbol{m} = \boldsymbol{m}_0(t)g(r) \tag{3.1}$$

where r denotes the distance from the magneton center and g is a regular core function.

The vector "charge" of the magneton is, in general, time-dependent and it is characterized by the scaling factor $\boldsymbol{m}_0(t)$.

It should be clear that the potential term in decomposition (1.2) is defined as a solution to the following Poisson equation

$$\Delta\phi = \operatorname{div}\boldsymbol{m} = (\boldsymbol{m}_0 \cdot \nabla)g(r) \tag{3.2}$$

The function ϕ can be sought in the form of $(\boldsymbol{m}_0 \cdot \nabla)\Phi(r)$, where Φ denotes a new unknown function. It is described by another Poisson's equation. Indeed, insertion of the assumed form of ϕ in (3.2) gives

$$0 = \Delta\phi - (\boldsymbol{m}_0 \cdot \nabla)g = \Delta(\boldsymbol{m}_0 \cdot \nabla)\Phi - (\boldsymbol{m}_0 \cdot \nabla)g = (\boldsymbol{m}_0 \cdot \nabla)(\Delta\Phi - g)$$

Since $\boldsymbol{m}_0$ is non-zero we conclude that:

— 2D case, angular symmetry

$$\Delta\Phi = g(r) \quad \Rightarrow \quad \frac{1}{r}\frac{d}{dr}\Big(r\frac{d\Phi(r)}{dr}\Big) = g(r) \tag{3.3}$$

— 3D case, spherical symmetry

$$\Delta\Phi = g(r) \quad \Rightarrow \quad \frac{1}{r^2}\frac{d}{dr}\Big(r^2\frac{d\Phi(r)}{dr}\Big) = g(r) \tag{3.4}$$

Then, the derivative of Φ is given as

$$\frac{d\Phi}{dr} = \begin{cases} \dfrac{1}{r}\displaystyle\int_0^r \xi g(\xi)\, d\xi & \text{for 2D case} \\[2ex] \dfrac{1}{r^2}\displaystyle\int_0^r \xi^2 g(\xi)\, d\xi & \text{for 3D case} \end{cases} \tag{3.5}$$

The gradient of the function ϕ is expressed as follows

$$\nabla\phi = \nabla[(\boldsymbol{m}_0 \cdot \nabla)\Phi] = \nabla\Big(\frac{\boldsymbol{m}_0 \cdot \boldsymbol{r}}{r}\frac{d\Phi}{dr}\Big) \tag{3.6}$$

Finally, the velocity field induced by the single magneton with the center at the origin can be derived as

$$\boldsymbol{u} = \begin{cases} \boldsymbol{m}_0\Big[g(r) - \dfrac{1}{r^3}\displaystyle\int_0^r \xi^2 g(\xi)d\xi\Big] - \dfrac{(\boldsymbol{m}_0 \cdot \boldsymbol{r})\boldsymbol{r}}{r^2}\Big[g(r) - \dfrac{3}{r^3}\displaystyle\int_0^r \xi^2 g(\xi)d\xi\Big] & \text{for 3D} \\ \boldsymbol{m}_0\Big[g(r) - \dfrac{1}{r^2}\displaystyle\int_0^r \xi g(\xi)d\xi\Big] - \dfrac{(\boldsymbol{m}_0 \cdot \boldsymbol{r})\boldsymbol{r}}{r^2}\Big[g(r) - \dfrac{2}{r^2}\displaystyle\int_0^r \xi g(\xi)d\xi\Big] & \text{for 2D} \end{cases} \tag{3.7}$$

In particular, the velocity induced at the center of the magneton is equal to

$$\boldsymbol{u}\Big|_{r=0} = \begin{cases} \dfrac{2}{3}g(0)\boldsymbol{m}_0 & \text{for 3D case} \\ \dfrac{1}{2}g(0)\boldsymbol{m}_0 & \text{for 2D case} \end{cases} \tag{3.8}$$

The reader may notice that, in the case of a finite core, the velocity induced outside the core is potential. The magneton charge can be defined as

$$Q = \begin{cases} \displaystyle\int_0^a \xi^2 g(\xi)\, d\xi & \text{for 3D case} \\ \displaystyle\int_0^a \xi g(\xi)\, d\xi & \text{for 2D case} \end{cases} \tag{3.9}$$

The symbol a denotes here the radius of the core. One can easily conclude from (3.7) that for $r \geqslant a$ the induced velocity is given by the formulas

$$\boldsymbol{u} = \begin{cases} -\nabla\Big(Q\dfrac{\boldsymbol{m}_0 \cdot \boldsymbol{r}}{r^3}\Big) & \text{for 3D case} \\ -\nabla\Big(Q\dfrac{\boldsymbol{m}_0 \cdot \boldsymbol{r}}{r^2}\Big) & \text{for 2D case} \end{cases} \tag{3.10}$$

The boundary of the magneton core (the radius $r = a$) moves due to the self-induction with the velocity

$$\boldsymbol{U} = \begin{cases} \dfrac{2Q}{a^3}\boldsymbol{m}_0 & \text{for 3D case} \\ \dfrac{Q}{a^2}\boldsymbol{m}_0 & \text{for 2D case} \end{cases} \tag{3.11}$$

The spherical shape of the core is preserved during the self-induced motion. Note that in the case when $Q = 0$ there is no self-induction.

In the case of an unbounded core, one must reasonably assume that the self-movement is equal to zero. This is equivalent to

$$\begin{aligned} \lim_{r\to\infty} \frac{1}{r^3} \int_0^r \xi^2 g(\xi)\, d\xi = 0 \quad &\text{for 3D case} \\ \lim_{r\to\infty} \frac{1}{r^2} \int_0^r \xi g(\xi)\, d\xi = 0 \quad &\text{for 2D case} \end{aligned} \tag{3.12}$$

Consider now the superposition of a number of magnetons with characteristic vectors $\boldsymbol{m}_{0k}$ and with the centers located at points $\boldsymbol{r}_k$. The total velocity field is a sum of the contributions from each magneton

$$\boldsymbol{v} = \sum_k \boldsymbol{m}_{0k}(t)\mathfrak{U}(|\boldsymbol{r} - \boldsymbol{r}_k|) \tag{3.13}$$

In the above $\mathfrak{U}$ denotes the matrix operator defined by induction formulas (3.7).

In the following sections of this part of the paper, we focuse on the 3D case. The two dimensional variant of the method can be derived analogously.

4. Lagrangian decomposition

In the rest of this paper, we will focus on those variants of the magnetization equation, which can be cast in the following form

$$\partial_t \boldsymbol{m} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{m} = \nu \Delta \boldsymbol{m} + \boldsymbol{S}(\boldsymbol{m}) \tag{4.1}$$

Equation (4.1) can be re-written using the material derivative

$$\frac{D\boldsymbol{m}}{Dt} = \nu \Delta \boldsymbol{m} + \boldsymbol{S}(\boldsymbol{m}) \tag{4.2}$$

The above form describes the rate of change in time of the magnetization field seen by an observer moving with a fluid element. This form of the governing equation is an appropriate starting point for the formulation of a Lagrangian method.

Among different variants of the governing equations discussed in Section 2, only first two cases lead to form (4.2). They differ with the shape of the stretching terms, namely

$$\boldsymbol{S}(\boldsymbol{m}) = \begin{cases} (\boldsymbol{m}\cdot\nabla)\nabla\phi & \text{in case 1} \\ (\boldsymbol{m}\cdot\nabla)\boldsymbol{v} & \text{in case 2} \end{cases} \tag{4.3}$$

In the remaining part of this Section, we explain the basic elements of the proposed numerical method.

We will use the Lagrangian decomposition of the magnetization field. First, we write the magnetization as the sum of magnetons currently present in the flow domain

$$\boldsymbol{m} = \sum_k \boldsymbol{m}_{0k} g(|\boldsymbol{r} - \boldsymbol{r}_k|) \tag{4.4}$$

The magnetons move accordingly to the fluid velocity and self-induced velocity expressed by (3.11). The viscous effects can be simulated by random walks.

Indeed, taking the set of the stochastic differential equations

$$dx_k^i = v_k^i\, dt + \sqrt{2\nu}\, dW_k^i \qquad\qquad x_k^i\Big|_{t=0} = x_{0k}^i \tag{4.5}$$

where dW_k^i denotes infinitesimal increments of the Wiener processes, we obtain the Fokker-Planck-Kolmogorov equation for the density of the transition probability (Gardiner, 1990)

$$\partial_t p + \partial_{x_i}(v^i p) = \nu \Delta p \tag{4.6}$$

The initial condition formulated for the transition probability is following

$$\lim_{t\to t_0} p(t, \boldsymbol{r}|t_0, \boldsymbol{r}_0) = \delta(\boldsymbol{r} - \boldsymbol{r}_0)$$

The solution to the homogeneous advection-diffusion problem can be expressed as follows (Szumbarski and Styczek, 1997)

$$\boldsymbol{m} = \int_\Omega p(t, \boldsymbol{r}|t_0, \boldsymbol{r}_0)\boldsymbol{m}_0(\boldsymbol{r}_0)\, d_3 r_0 \approx \sum_k p(t, \boldsymbol{r}|t_0, \boldsymbol{r}_{0k}) 4\pi \boldsymbol{m}_{0k} \tag{4.7}$$

Following Chorin and Marsden (1997), we introduce the following fractional-step approach. The time advancing the magnetization field at each interval $[t_n, t_{n+1}]$ is carried out in three consecutive sub-steps:

Step 1. Creation of new magnetons at the boundary

At the beginning of a new time step the boundary distribution of the magnetization should be modified in order to satisfy the assumed boundary condition for the velocity field. Thus, we can write

$$\boldsymbol{m}' = \boldsymbol{m}^{(n)} + \boldsymbol{m}_{\partial\Omega}^{(n+1)} \tag{4.8}$$

where $\boldsymbol{m}_{\partial\Omega}^{(n+1)}$ denotes the new magnetization created in close vicinity of the boundary at the beginning of the $(n+1)$th time step. This new magnetization is such that the velocity "induced" by the magnetization $\boldsymbol{m}'$ defined as $\boldsymbol{v}' = \mathcal{P}\boldsymbol{m}'$ satisfies the boundary condition.

The magnetization field obtained in this sub-step can be expressed by introducing the operator B as follows

$$\boldsymbol{m}' = B\boldsymbol{m}^{(n)} \equiv \boldsymbol{m}^{(n)} + \boldsymbol{m}_{\partial\Omega}^{(n+1)} \tag{4.9}$$

In the Lagrangian method, the magnetization $\boldsymbol{m}_{\partial\Omega}^{(n+1)}$ is approximated by a certain number of new magnetons injected into the flow domain near the boundary of an immersed body.

Detailed description of this procedure is given in the next Section.

Step 2. Advection-diffusion

In this sub-step, the magnetization is modified due to the transport by advection and diffusion. It means that all magnetons are moved to their new locations accordingly to the Itô equations

$$d\boldsymbol{r}_{i'} = \boldsymbol{v}'_{i'}\, dt + \sqrt{2\nu}\, d\boldsymbol{W}_{i'} \tag{4.10}$$

The lower subscript i' refers to all magnetons, including new magnetons injected near the boundary. Stochastic equations (4.10) are to be integrated over the time interval $[t_n, t_{n+1}]$ using some numerical integration scheme. The length of the interval is $\tau = t_{n+1} - t_n$. Consequently, we obtain modified magnetization field. One can summarize this operation as follows

$$\boldsymbol{m}'(\boldsymbol{r}) = \sum_{i'\in I'} \boldsymbol{m}_{0i'} g(|\boldsymbol{r} - \boldsymbol{r}_{i'}|) \Rightarrow \boldsymbol{m}''(\boldsymbol{r}) = \sum_{i'\in I''} \boldsymbol{m}_{0i'} g(|\boldsymbol{r} - \boldsymbol{r}_{i'} - \Delta\boldsymbol{r}_{i'}|) \tag{4.11}$$

In expression (4.11), the vector function $\boldsymbol{m}_{i'}$ describes the magnetization distribution induced by the i'th magneton. It should be remarked that the range I'' of the second summation might be different than the range I' of the first

summation. This is so because a certain number of the magnetons always "diffuses" across the boundary beyond the computational domain.

The instantaneous magnetization field $\boldsymbol{m}''$ resulting from the advection-diffusion sub-step can be expressed by introducing another operator

$$\boldsymbol{m}'' = \Xi_\tau^1 \boldsymbol{m}' \tag{4.12}$$

Step 3. Stretching

In the last sub-step, one should account for the stretching term in magnetization equation (4.1). Hence, the following (deterministic) differential equation

$$\partial_t \boldsymbol{m} = \boldsymbol{S}(\boldsymbol{m}) \tag{4.13}$$

should be integrated over the time interval $[t_n, t_{n+1}]$ with the initial condition $\boldsymbol{m}(t_n) = \boldsymbol{m}''$. In effect, one obtains the final distribution of the magnetization at the time instant t_{n+1}. Again, the operator notation can be used

$$\boldsymbol{m}^{(n+1)} = \Xi_\tau^2 \, \boldsymbol{m}'' \tag{4.14}$$

The composition of the three operators introduced above defines completely the evolution of the magnetization field during one time step, namely

$$\boldsymbol{m}^{(n+1)} = \Xi_\tau^2 \, \Xi_\tau^1 B \, \boldsymbol{m}^{(n)} \tag{4.15}$$

The above formula can be used to obtain an approximate solution at the time instant $t = t_0 + T$ (here t_0 denotes the initial time). Indeed, fixing the number of time steps n, we can set $\tau = T/n$ and using recursively (4.15), we arrive at

$$\boldsymbol{m}\Big|_{t_0+T} \approx (\Xi_\tau^2 \, \Xi_\tau^1 \, B)^n \boldsymbol{m}\Big|_{t_0} \tag{4.16}$$

The convergence of the splitting approach with is guaranteed by the Lie-Trotter formula (see Chorin and Marsden, 1997), which in our case can be written as follows

$$\boldsymbol{m}\Big|_{t_0+T} = (\Xi_{T/n}^2 \, \Xi_{T/n}^1 B)^n \boldsymbol{m}\Big|_{t_0} \tag{4.17}$$

being actually the limit form of (4.16).

5. Velocity decomposition and boundary conditions

Consider a flow past a rigid body. The flow domain is unbounded and the stream at infinity is assumed homogeneous

$$\boldsymbol{v}\Big|_{\infty} = \boldsymbol{U}_{\infty} \tag{5.1}$$

On the surface of the body, the velocity field vanishes, i.e.

$$\boldsymbol{v}\Big|_{A} = 0 \tag{5.2}$$

We express the velocity field in the flow domain as a sum of the contributions induced by two ensembles of the magnetons. The first ensemble contains the magnetons created at earlier time steps and remaining in the flow domain at the considered time instant – these magnetons will be referred to as "old" magnetons. The second ensemble is formed with the magnetons created in the flow domain at the current instant of time, i.e. with "new" magnetons. Thus, the complete velocity field can be written as follows

$$\boldsymbol{v} = \boldsymbol{U}_{\infty} + \sum_{k} \boldsymbol{m}_{0k}^{0} \mathfrak{U}(|\boldsymbol{r} - \boldsymbol{r}_{k}^{0}|) + \sum_{k=1}^{M} \boldsymbol{m}_{0k}^{n} \mathfrak{U}(|\boldsymbol{r} - \boldsymbol{r}_{k}^{n}|) \tag{5.3}$$

It is assumed that all parameters $\boldsymbol{m}_{0k}^{0}$ and $\boldsymbol{r}_{k}^{0}$ characterising the current "charge" and position of all old magnetons are known. On the other hand, the parameters $\boldsymbol{m}_{0k}^{n}$ and $\boldsymbol{r}_{k}^{n}$ of the new magnetons are unknown and have to be determined at each time step of the flow simulation. We assume that the new magnetons will be generated near the body surface A.

The surface A is divided into M small parts $A_1, A_2, .., A_M$, and the kth new magneton is placed over A_k, with its center located at a small distance above the surface. The positions $\boldsymbol{r}_{k}^{n}$, $k = 1, .., M$ of the "generation points" are then defined and the only remaining unknowns are the characteristic vectors $\boldsymbol{m}_{0k}^{n}$, $k = 1, .., M$.

In can be concluded from (3.10) that $\boldsymbol{m}_0 \mathfrak{U}(r) \to 0$ as $r \to \infty$, and thus, asymptotic condition (5.1) is satisfied. Using boundary condition (5.2) one writes for $\boldsymbol{r} \in A$

$$\sum_{k=1}^{M} \boldsymbol{m}_{0k}^{n} \mathfrak{U}(|\boldsymbol{r} - \boldsymbol{r}_{k}^{n}|) = -\boldsymbol{U}_{\infty} - \sum_{k} \boldsymbol{m}_{0k}^{0} \mathfrak{U}(|\boldsymbol{r} - \boldsymbol{r}_{k}^{0}|) \tag{5.4}$$

Let $\boldsymbol{e}_{\alpha}\Big|_{A_i}$, $\alpha = 1, 2, 3$ denote the triple of the versors at a certain collocation point within the surface segment A_i. A possible choice is the triple consisting

of the normal and two different tangent versors at each collocation point, or simply the three versors of the global Cartesian reference frame. Computing the scalar product of the last equation with three versors at each collocation point, we obtain $(\alpha = 1, 2, 3)$

$$\sum_{k=1}^{M} \boldsymbol{m}_{0k}^{n} \mathfrak{U}(|\boldsymbol{r}_{A_i} - \boldsymbol{r}_k^n|) \boldsymbol{e}_\alpha \Big|_{A_i} = -\boldsymbol{U}_\infty \boldsymbol{e}_\alpha \Big|_{A_i} - \sum_k \boldsymbol{m}_{0k}^{0} \mathfrak{U}(|\boldsymbol{r} - \boldsymbol{r}_k^0|) \boldsymbol{e}_\alpha \Big|_{A_i} \tag{5.5}$$

where $\boldsymbol{r}_{A_i}$ denotes the position of a collocation point belonging to the segment A_i. Consequently, we have a linear algebraic system of 3M equations. If the shape of the surface and other geometrical parameters are fixed in time, the matrix of the system is fixed as well and it can be evaluated (and possibly LU-factorized) once and forever.

Solution to the linear system yields complete information about new generation of the magnetons, and thus about the complete velocity and vorticity fields at a given instant of time. The velocity field satisfies the boundary condition at the chosen array of the collocation points.

It should be remarked that alternative approaches to the boundary conditions are also possible. One can consider an integral-type rather than pointwise enforcement of the boundary condition on the surface. Such an approach would consist in integration of the tangent velocity along two different (possibly perpendicular) line segments located within the surface element A_i, and demanding these integrals to be equal to zero. The normal component of the velocity is set to zero at the collocation point, as previously.

6. Advection-diffusion and stretching

The advection-diffusion operator working over one time step has been already mentioned. It describes a one-time-step advancing of the magnetization field. Let us summarize the essential steps defining this operator. Having complete knowledge of the magnetons created up to a given instant, we construct a "magnetization layer", which is adjacent to the boundary and consists of the newly born magnetons. These new magnetons are created at fixed locations but their vector "charges" are initially unknown. They are chosen so that the boundary condition for the velocity field is satisfied. At this point one has to solve the linear system of $3M$ equation, where M denotes the number of the newly generated magnetons. After that, the magnetization field is completely defined and all magnetons are moved to their new locations by

performing one-time-step integration of stochastic differential equations (4.2). The instantaneous velocity of the jth magneton is expressed as

$$\boldsymbol{v}_j = \boldsymbol{U}_\infty + \frac{2Q}{a_j^3}\boldsymbol{m}_{0j} + \sum_{k\neq j} \boldsymbol{m}_{0k}\mathfrak{U}(|\boldsymbol{r}_j - \boldsymbol{r}_k|) \tag{6.1}$$

i.e. as the sum of the free-stream velocity, self-induced velocity (3.11) and the velocity induced by other magnetons in the flow domain. The change of the positional vector $\boldsymbol{r}_j$ of the magneton during one time step is determined by numerical integration of Itô differential equations (4.10). As an example, one can consider the Euler integration scheme leading to the following formula

$$\Delta\boldsymbol{r}_j = \boldsymbol{v}_j\Delta t + \sqrt{2\nu\tau}\,\boldsymbol{\mathcal{N}}(0,1)_j \tag{6.2}$$

In the above, τ denotes the length of the time step and $\boldsymbol{\mathcal{N}}(0,1)_j$ denotes the random vector, the components of which are independent random variables with the standard Gaussian distribution. More sophisticated, higher-order integration schemes are also available. The reader should refer to Kloeden and Platen (1999) for more details.

It should be noted that due to the presence of the random component in equation (6.2), the magnetons can penetrate the surface and jump into the body interior. This is a manifestation (on the "kinetic" level) of the diffusion of the magnetization through the boundaries. The magnetons beyond the flow domain can be removed from the simulation or they can be reflected back. Kinetic equations (6.2), together with the removal or reflection approach, complete the description of the advection-diffusion operator Ξ_τ^1.

The last operator is defined by a stretching equation. We re-write it in the following form

$$\partial_t m^\alpha = -m^\beta \partial_{x_\alpha} v^\beta \tag{6.3}$$

in case 2, or as

$$\partial_t m^\alpha = m^\beta \partial_{x_\alpha, x_\beta}\phi \tag{6.4}$$

in case 1.

Both matrix operators appearing in the right-hand sides of (6.3) and (6.4) are known. We substitute here the Lagrangian decomposition and solve equation (6.3) or (6.4) for each magneton. This approach is justified by the following fact. If for a given matrix operator $B_{\alpha\beta}$ and vectors $\boldsymbol{m}_k$, $k = 1, 2, .., M$ one has

$$\partial_t m_k^\alpha = B_{\alpha\beta} m_k^\beta \tag{6.5}$$

then, as a result of the formal linearity with respect to $\boldsymbol{m}$, the basic equation holds. Inserting the expression for m_k^α we write

$$g(|\boldsymbol{r}-\boldsymbol{r}_k|)\frac{dm_{0k}^\alpha}{dt} = \begin{cases} -m_{0k}^\beta g(|\boldsymbol{r}-\boldsymbol{r}_k|)\partial_{x_\alpha} v^\beta \\ m_{0k}^\beta g(|\boldsymbol{r}-\boldsymbol{r}_k|)\partial_{x_\alpha,x_\beta}\phi \end{cases} \tag{6.6}$$

Integrating the above equation over the support of the core function g we get

$$4\pi\frac{dm_{0k}^\alpha}{dt} = B_{\alpha\beta}^0 m_{0k}^\beta \tag{6.7}$$

where the matrix operator depends on $\boldsymbol{r}_k$ only and has the following form

$$B_{\alpha\beta}^0 = \int\limits_{\rho\leqslant a} \left\{ \begin{array}{c} v^\beta(\boldsymbol{r}_k+\boldsymbol{\rho}) \\ -\partial_{x_\beta}\phi(\boldsymbol{r}_k+\boldsymbol{\rho}) \end{array} \right\} \frac{dg(\rho)}{d\rho}\frac{x_\alpha}{\rho}\, d\Omega_\rho \tag{6.8}$$

In the derivation of (6.8) the equality g(a)=0 and the integration by parts have been employed.

As we see, the stretching is described by the system of 3N ordinary differential equations with constant coefficients. Its dimension is equal to the tripled number of all magnetons present in the flow domain at a given instant of time. Thus, its dimension is large and can be excessively large for long flow simulations.

7. Final remarks

The Lagrangian approach to the velocity-vorticity formulation known as the Vortex Blobs Method (see Protas (2000) for exhaustive list of references) allows solving many interesting problems. In the vortex method, the formulation involves only kinematic quantities: the velocity and vorticity. Once these fields are determined, the pressure can be recovered (protas et al., 2000). As a physical quantity, the pressure must be a univalued function of time and space variables. This fact brings a fundamental condition constraining the total charge of the vorticity, being nontrivial for flows in multi-connected domains. It is essential that the implementation of this condition gives an effect of stabilization of a numerical process (see Błażewicz and Styczek, 1993; Szumbarski and Styczek, 1999; Protas, 2000), which otherwise tends to generate physically absurd "solutions".

The existence of an additional condition of global nature is one of the main differences between the vortex and the magnetization methods. The pressure recovery does not require any further restriction of the magnetization because it is based on explicit formula (2.5) (in the vortex method it is based on a formula for the pressure gradient).

There exists, however, a numerical evidence, which seems to indicate that some kind of (yet unknown) stabilizing condition is indeed necessary. Russo and Smerka (1999) and Summers and Chorin (1996) presented a sample of numerical computations. They encountered unstable behavior rendering impossible sensible long-time simulations of flow evolution. These observations are in agreement with the results obtained by the authors, discussed in Section 2 of this paper.

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Lagrangeowska metoda magnetyzacji dla dwu i trójwymiarowych ruchów płynu lepkiego. Część I – Podstawowe sformułowania

Streszczenie

W artykule przedstawione jest sformułowanie problemu granicznego dla równań Naviera-Stokesa z użyciem tzw. pola magnetyzacji. Sformułowanie nie jest jednoznaczne, lecz wiąże się z przyjętą transformacją cechowania. Rozważane są różne postacie tej transformacji i dokonuje się wyboru odpowiednich wariantów. Pole magnetyzacji przedstawione jest w formie lagrangeowskiej. Wprowadza się cząstki będące źródłami tego pola i określa się związane z ich zbiorem pole prędkości. Cząstki magnetyzacji (zwane magnetonami) poruszają się w indukowanym polu prędkości, wykonują ruch losowy odpowiadający dyfuzji i podlegają przekształceniu w sposób opisany członem źródłowym (tzw. stretching). Warunek brzegowy formułowany na opływanym ciele jest realizowany przez tworzenie w każdej chwili nowych cząstek ulokowanych w bliskim otoczeniu powierzchni ciała.

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