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SHAKEDOWN- AND LIMIT ANALYSIS OF PERIODIC COMPOSITES

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A methodology for the assessment of periodic composites under variable repeated loads by means of the shakedown theory is presented and applied to metal-matrix-composites. The approach is based on the local shakedown analysis in a representative volume element of the composite and the use of averaging techniques to determine the domains of admissible stresses on the macro-level.

Key words: periodic composites, shakedown-analysis

Dedicated to Professor Czesław Woźniak on the occasion of his 70^{th} birthday

1. Introduction

Local failure due to fatigue can be considered as being caused by repeated, dissipative events occurring on the micro-structural level of materials. In this paper, the scale chosen for observation of these effects will be called "meso-scale", small compared to the scale for measuring macroscopic dimensions of a mechanical structural element and large compared to the atomistic scale. We suppose that on this meso-scopic level the laws and methods of the classical continuum mechanics are applicable, but that different constituents of the material can be recognised, forming a mechanical structure by itself on this level of observation. The interaction between these constituents determines the local response of the material, in particular the mechanisms leading to local failure and damage. Therefore, the study of the inelastic, dissipative behaviour on the meso-level of specific materials under repeated, variable loads can be helpful in better understanding of the failure mechanisms, and the methods developed for this purpose can be used in a constructive way for the design of materials. The appropriate choice of the scale and methodology of investigation, however, may be different from one material to another and for different types of loading programs.

A particularly interesting class of materials for this kind of studies are composites: here, the strong heterogeneity of the material causes in general large gradients of the mechanical field quantities such as stresses and strains, initiating local damage and overall fatigue failure by the interaction of different local effects, which depend upon the mechanical and geometrical properties of the individual components of the composite. Excluding non-mechanical effects like chemical reactions and effects of fluid-solid interactions in porous materials, one may quote brittle fracture of inclusions, local debonding between the matrix and fibrous reinforcements, localised plasticity and ductile damage due to different elastic-plastic properties of the individual components as examples for such initiation of the failure. To predict failure in such a case, it is important to understand and to model the mechanical processes on the meso-structural level and to link them to the characteristic macroscopic material properties. This can be facilitated if geometrical "patterns of periodicity" are formed by different components in the material. In this case, the averaging techniques such as Homogenisation Technique or Tolerance Averaging Methods (Suquet, 1982; Woźniak, 1999; Woźniak and Wierzbicki, 2000) can be used to bridge the gap between the local (mesoscopic) and global (macroscopic) properties of the considered composite.

In continuation of the preceding work (Weichert et al., 1999a,b) it is shown in this paper, how Direct Methods, in particular Shakedown Analysis (Limit Analysis being treated as particular case), can help to assess composites which exhibit plastic deformations on the meso-scale, and how these methods can be used in a constructive manner for the design of materials.

The classical field of application of shakedown theory is the assessment of mechanical structures or structural elements exposed to variable thermo- and/or mechanical loads. It addresses basically failure (non-shakedown) caused by unlimited growth of plastic dissipation during the loading process, leading to incremental collapse or alternating plasticity. Limit analysis covers the particular case of the instantaneous collapse under a monotonous loading. The foundations of shakedown theory have been laid by Melan (1938) and Koiter (1960), who derived sufficient criteria for shakedown and non-shakedown, respectively, for elastic-perfectly plastic structures in the framework of geometrically linearised continuum mechanics. Due to the evident practical importance, their classical theorems have been extended to larger classes of problems and widely applied to structural analysis. Reviews and overviews of such studies can be found e.g. in Gokhfeld and Cherniavsky (1980), König (1987), Mróz et al. (1995), Weichert and Maier (2000).

Here, this theory is applied to the study of periodic composites, in which at least one component exhibits ductile properties. This type of behaviour can be found, e.g. in metal matrix composites (MMC's) (Ponter and Leckie, 1998; Weichert et al., 1999a,b; Carvelli et al., 1999). To relax the classical assumptions of shakedown theory, in the theoretical part of this paper, plastic damage is taken into account for the ductile components of the composite, as well as brittle failure of eventual reinforcing components and their debonding from the matrix material of the composite.

Plastic material damage is taken into account by using the concept of effective stress (Kachanov, 1958; Lemaitre, 1985), combined with specific models of damage evolution. To model brittle failure of inclusions and fibrous reinforcements as well as debonding between the reinforcements and matrix material following the theory by Needleman (1987), subsidiary conditions on stresses on the mesoscopic level are introduced.

The simulation of material behaviour by means of structural mechanics is nowadays well established. Nevertheless, the use of shakedown analysis to assess and to design composites is rather new: first attempts had been undertaken in a pioneering work by Tarn et al. (1975) for the determination of safe loading domains of unidirectional composites under an axisymmetric loading. Ponter and Leckie (1998) investigated the shakedown behaviour of an aluminium/alumina system under fluctuating temperatures by means of the homogenisation technique, focusing on the application of the upper bound theorem. Making use of the finite-element analysis, Carvelli et al. (1999) applied the upper bound theorem of the shakedown theory to two-dimensional problems. The authors of the present paper calculated the admissible loading domains for composites by using the lower bound theorem (Weichert et al., 1999a,b). It should be noted, that the three last quoted groups of researchers followed similar lines of thinking: *The two principal theoretical ingredients being averaging techniques, in particular the homogenisation technique combined with shakedown analysis for a "representative volume element" (RVE) or "unit cell" V on the meso-level.* This allows under the assumption of periodicity of the composite, to link the results of shakedown analysis on the meso-level to the overall material properties on the macro-level. From this point of view, the presented methodology can be regarded as an extension of that given by Suquet (1983) for the limit analysis of heterogeneous media.

The scenario of failure of metal-matrix-composites under variable loads, assumed in the present study, is as follows: during loading, on the meso-scale level unlimited accumulation of inelastic deformations occurs in some areas. These accumulated plastic deformations lead to material damage in the ductile matrix of the composite, which causes the initiation of micro-cracks. These may in the sequel of the loading process propagate and initiate failure of the considered structural element. However, the crack initiation and propagation are not specifically addressed by the presented analysis: *If in some part of the composite the unlimited accumulation of plastic deformations is detected, we say that the material fails.* Similarly, brittle failure and debonding of reinforcements are assumed to initiate failure due to fatigue and are not admissible for safe states of the material.

2. Definitions and general assumptions

We consider a material which is composed of two or more constituents, one of which identified as the matrix material, exhibiting ductile properties. The other components are usually reinforcements, in form of particles or fibres. It is supposed that the reinforcing components are embedded in the matrix material according to a regular pattern. Each component is assumed to be homogeneous, occupying the volume fraction V_i $(\sum V_i = 1,\ i = 1, ...n)$, where n denotes the number of components of the composite. The macroscopic behaviour of this heterogeneous material is observed on the scale $\boldsymbol{x}$ and the mesoscopic behaviour on the scale $\boldsymbol{y}$ (Fig. 1). For reasons of simplicity, effects of geometrical changes occurring during deformation are neglected. We note that the concept developed in this paper applies also to materials with regularly distributed voids or perforations, which are considered as material components with zero resistance.

Material damage of elastic-plastic material constituents is modelled in the context of continuum mechanics with the help of the concept of the effective stresses (Kachanov, 1958). This means that the behaviour of the damaged material can be represented by the constitutive equations of the virgin material where the usual stresses $\boldsymbol{\sigma}$ on the meso-level are replaced by effective stresses $\widetilde{\boldsymbol{\sigma}}$. If we restrict our considerations to isotropic damage, they are defined by

$$\widetilde{\boldsymbol{\sigma}} = \frac{\boldsymbol{\sigma}}{1-D} \tag{2.1}$$

The scalar $D = 0$ corresponds to the undamaged state, $D \in (0, D_c)$ corre-

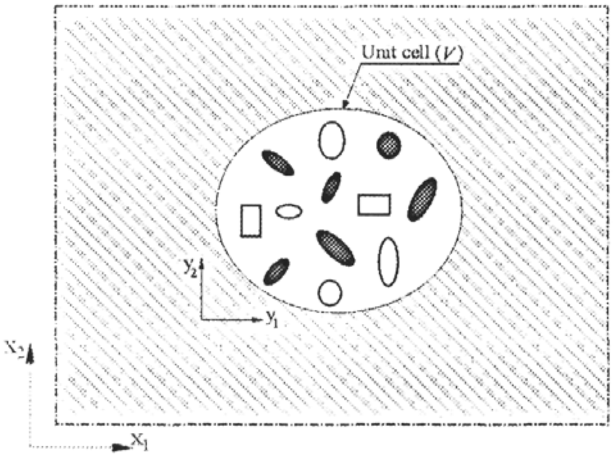

Fig. 1. Heterogeneous material

sponds to a partly damaged state and $D = D_c$ defines complete local rupture ($D_c \in [0,1]$). A simple model of isotropic ductile plastic damage is given by Lemaitre (1985). This model is linear with the equivalent plastic strain ε_{eq} and depends upon three material constants, damage threshold strains ε_D, strain at fracture ε_R and the critical value of damage parameter at fracture D_c for damage properties and Poisson's ratio ν

$$D = \frac{D_c}{\varepsilon_R - \varepsilon_D}\langle R_\nu \varepsilon_{eq} - \varepsilon_D\rangle \tag{2.2}$$

with R_ν as the triaxiality ratio given by

$$R_\nu = \frac{2}{3}(1+\nu) + 3(1-2\nu)\Big(\frac{\sigma_H}{\sigma_{eq}}\Big)^2 \tag{2.3}$$

where σ_{eq} denotes the von Mises equivalent stress, $\sigma_H = \sigma_{ii}/3$ denotes the hydrostatic stress and $\langle\cdot\rangle$ the Macauley operator, i.e. $\langle x\rangle = (x+|x|)/2$.

In the sequel, the superposed tilde indicates quantities related to the damaged state of the material. Extensions of the shakedown theory to the case of non-isotropic damage can be found in Druyanov and Roman (2000).

According to the restriction to the geometrically linear theory, the total strains $\boldsymbol{\varepsilon}(\boldsymbol{y})$ can be split into purely elastic $\boldsymbol{\varepsilon}^e$ and purely plastic $\boldsymbol{\varepsilon}^p$ ones, respectively

$$\boldsymbol{\varepsilon}(\boldsymbol{y}) = \boldsymbol{\varepsilon}^e(\boldsymbol{y}) + \boldsymbol{\varepsilon}^p(\boldsymbol{y}) \tag{2.4}$$

For the considered unit cell V, we adopt the usual homogenisation assumption for the local displacement field $\boldsymbol{u}$ at the position $\boldsymbol{y}$

$$\boldsymbol{u} = \mathbf{E}\cdot\boldsymbol{y} + \boldsymbol{u}^{per} \tag{2.5}$$

where E is the macroscopic strain tensor and $\boldsymbol{u}^{per}$ is the displacement field satisfying the periodicity conditions. Then, the Hill relationship (Hill, 1963) holds

$$\boldsymbol{\Sigma} : \mathsf{E} = \langle \boldsymbol{\sigma} : \boldsymbol{\varepsilon} \rangle = \frac{1}{V} \int\limits_{(V)} \boldsymbol{\sigma} : \boldsymbol{\varepsilon} \, dV \tag{2.6}$$

with

$$\begin{aligned} \boldsymbol{\Sigma}(\boldsymbol{x}) &= \langle \boldsymbol{\sigma}(\boldsymbol{y}) \rangle = \frac{1}{V} \int\limits_{(V)} \boldsymbol{\sigma}(\boldsymbol{y}) \, dV \\ \mathsf{E}(\boldsymbol{x}) &= \langle \boldsymbol{\varepsilon}(\boldsymbol{y}) \rangle = \frac{1}{V} \int\limits_{(V)} \boldsymbol{\varepsilon}(\boldsymbol{y}) \, dV \end{aligned} \tag{2.7}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are mesoscopic stresses and strains also satisfying the periodicity condition. Within the unit cell, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ fulfil compatibility and equilibrium conditions, respectively.

For the plastic part of the material behaviour we assume the validity of the normality rule for the plastic flow in a sub-differential form, such that

$$\dot{\boldsymbol{\varepsilon}}^p \in \delta\varphi(\boldsymbol{\sigma}) \tag{2.8}$$

where $\delta\varphi(\boldsymbol{\sigma})$ denotes the sub-gradient of the plastic potential $\varphi(\boldsymbol{\sigma})$ which is the indicator function of a convex elastic domain $P(\boldsymbol{y})$ of all plastically admissible stress states

$$\boldsymbol{\sigma}(\boldsymbol{y}) \in P(\boldsymbol{y}) \qquad \forall \boldsymbol{y} \in V \tag{2.9}$$

$P(\boldsymbol{y})$ is defined by means of a yield function $\mathcal{F}(\widetilde{\boldsymbol{\sigma}}, \boldsymbol{y})$

$$P(\boldsymbol{y}) = \Big\{ \boldsymbol{\sigma} \;\Big|\; \mathcal{F}(\widetilde{\boldsymbol{\sigma}}, \boldsymbol{y}) \leqslant 0, \ \ \forall \boldsymbol{y} \in V \Big\} \tag{2.10}$$

The convexity of $\mathcal{F}(\widetilde{\boldsymbol{\sigma}}, \boldsymbol{y})$ and the validity of the normality rule can be expressed by the maximum plastic work inequality

$$(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{(s)}) : \dot{\boldsymbol{\varepsilon}}^p \geqslant 0 \qquad \forall \boldsymbol{\sigma}^{(s)}(\boldsymbol{y}) \in \overline{P}(\boldsymbol{y}) \tag{2.11}$$

where $\boldsymbol{\sigma}^{(s)}$ is any safe state of stresses defined by

$$\overline{P}(\boldsymbol{y}) = \Big\{ \boldsymbol{\sigma}^{(s)} \;\Big|\; \mathcal{F}(\widetilde{\boldsymbol{\sigma}}^{(s)}, \boldsymbol{y}) < 0, \ \ \forall \boldsymbol{y} \in V \Big\} \tag{2.12}$$

The smooth von Mises yield function, defined by

$$\mathcal{F}(\widetilde{\boldsymbol{\sigma}}, \boldsymbol{y}) = \sqrt{\frac{3}{2}\Big(\frac{\boldsymbol{\sigma}^D}{1-D}\Big) : \Big(\frac{\boldsymbol{\sigma}^D}{1-D}\Big)} - \sigma_Y(\boldsymbol{y}) \tag{2.13}$$

where $\sigma_Y(\boldsymbol{y})$ denotes the yield stress and $\boldsymbol{\sigma}^D$ the deviatoric part of $\boldsymbol{\sigma}$, is a particular case and will be used in the numerical calculations presented in Section 5.

3. Interface model

A cohesive zone model of Needleman (1987) is used for describing the separation of two phases along a predefined process zone by defining an interface potential specifying the dependence of the tractions in the interface T (force by unit reference area) consisting of normal and tangential components T_n, T_t and T_b upon the corresponding discontinuity in the displacement field across the interface

$$[\![\boldsymbol{u}]\!] = \boldsymbol{u}^+ - \boldsymbol{u}^- \tag{3.1}$$

where $\boldsymbol{u}^+$ and $\boldsymbol{u}^-$ are displacement vectors at the interior and the exterior borders of the interface zone. At each point of the interface, we define

$$u_n = \boldsymbol{n} \cdot [\![\boldsymbol{u}]\!] \qquad u_t = \boldsymbol{t} \cdot [\![\boldsymbol{u}]\!] \qquad u_b = \boldsymbol{b} \cdot [\![\boldsymbol{u}]\!] \tag{3.2}$$

and

$$T_n = \boldsymbol{n} \cdot \boldsymbol{T} \qquad T_t = \boldsymbol{t} \cdot \boldsymbol{T} \qquad T_b = \boldsymbol{b} \cdot \boldsymbol{T} \tag{3.3}$$

where $\boldsymbol{n}$, $\boldsymbol{t}$ and $\boldsymbol{b}$ form a right-hand coordinate system chosen so that a positive u_n corresponds to increasing interfacial separation and a negative u_n corresponds to decreasing interfacial separation.

The mechanical response of the interface is described by a constitutive relation that gives the dependence of the tractions T_n, T_t and T_b on u_n, u_t and u_b. Here, this response is specified in terms of the function ϕ (Lissenden and Herakovich, 1995; Ismar et al., 2000)

$$\phi(\lambda) = \frac{27}{4}\sigma_{\max}(1-\lambda)^2 \tag{3.4}$$

with λ as a dimensionless parameter defined by

$$\lambda = \sqrt{\Big(\frac{\langle u_n \rangle}{\delta_n}\Big)^2 + \Big(\frac{u_t}{\delta_t}\Big)^2 + \Big(\frac{u_b}{\delta_t}\Big)^2} \tag{3.5}$$

where the Macauley operator $\langle\cdot\rangle$ specifies that under a compressive loading ($u_n \leqslant 0$) the debonding occurs only in the tangential direction. The characteristic length δ_n and δ_t are material parameters which correspond to the work of separation in connection with the maximum normal traction $\sigma_{\max}$. In (3.4), $\lambda = 0$ corresponds to the perfect bonding, whereas for values $\lambda \geqslant 1$ no more cohesive stresses can be supported.

The non-linear relations between tractions and displacement difference depend upon the maximum value of λ in the course of the precedent loading history, $\lambda_{\max}$, in order to prevent healing of the interface with decreasing values of λ. For $\lambda_{\max} < 1$, the interfacial tractions are defined by (Ismar et al., 2000)

$$
\begin{gathered}
T_n = \begin{cases} \dfrac{27}{4}\sigma_{\max}\dfrac{u_n}{\delta_n} & \text{for} \quad u_n \leqslant 0 \\ \phi(\lambda_{\max})\dfrac{u_n}{\delta_n} & \text{for} \quad u_n > 0 \end{cases} \\
T_t = \beta\phi(\lambda_{\max})\frac{u_t}{\delta_t} \qquad\qquad T_b = \beta\phi(\lambda_{\max})\frac{u_b}{\delta_t}
\end{gathered} \tag{3.6}
$$

where the maximum tangential traction is denoted by $\tau_{\max} = \beta\sigma_{\max}$.

4. Formulation of shakedown theorems

We define the admissible domain $P^m(\boldsymbol{x})$ as the set of macroscopic states of stress $\boldsymbol{\Sigma}(\boldsymbol{x})$ for all mesoscopic states of plastically admissible stress $\boldsymbol{\sigma}(\boldsymbol{y})$ (Suquet, 1983)

$$
P^m(\boldsymbol{x}) = \Big\{ \boldsymbol{\Sigma} \,\Big|\, \exists \boldsymbol{\sigma},\ \boldsymbol{\sigma}(\boldsymbol{y}) \in P(\boldsymbol{y}),\ \forall \boldsymbol{y} \in V \Big\} \tag{4.1}
$$

To determine $P^m(\boldsymbol{x})$, shakedown analysis is carried out on the meso-level. For this, we introduce the notion of a "*reference representative volume element* (RVE$^{(c)}$)" differing from the actual one only by the fact that the material is supposed to behave purely elastically without damage. All quantities related to this reference representative volume element are indicated by the superscript "(c)".

The statical shakedown theorem states that: if there exists a safety factor $\alpha > 1$, a time-independent field of the periodic residual stresses $\overline{\boldsymbol{\sigma}}^r$ and a

Sanctuary of Elasticity (Nayroles and Weichert, 1993)

$$\overline{P}^m(\boldsymbol{x}) \subset P^m(\boldsymbol{x}) \tag{4.2}$$

with

$$\overline{P}^m(\boldsymbol{x}) = \left\{ \boldsymbol{\Sigma} \;\middle|\; \exists \boldsymbol{\sigma}^{(s)},\ \boldsymbol{\sigma}^{(s)}(\boldsymbol{y}) \in \overline{P}(\boldsymbol{y}),\ \forall \boldsymbol{y} \in V \right\} \tag{4.3}$$

then the periodic composite material shakes down. Here, the safe state of stresses $\boldsymbol{\sigma}^{(s)}$ is defined as usual

$$\boldsymbol{\sigma}^{(s)} = \alpha \boldsymbol{\sigma}^{(c)} + \overline{\boldsymbol{\sigma}}^{(r)} \tag{4.4}$$

where $\boldsymbol{\sigma}^{(c)}$ is the stress field which would occur in the RVE$^{(c)}$ under the same boundary conditions as the actual RVE such that the following relations hold

$$\begin{array}{lll} \operatorname{Div} \boldsymbol{\sigma}^{(c)} = \mathbf{0} & \text{in} & V \\ \boldsymbol{u}(\boldsymbol{y}) - \mathbf{E} \cdot \boldsymbol{y} \ \text{ periodic} & \text{on} & \partial V \\ \boldsymbol{\sigma}^{(c)} \cdot \boldsymbol{n} \ \text{ anti-periodic} & \text{on} & \partial V \\ \boldsymbol{\sigma}^{(c)} = \mathbf{L} : [\boldsymbol{\varepsilon}(\boldsymbol{u}^{per}) + \mathbf{E}] & \text{in} & V \end{array} \tag{4.5}$$

However, the field of the residual stresses $\overline{\boldsymbol{\sigma}}^r$ satisfies

$$\begin{array}{lll} \operatorname{Div} \overline{\boldsymbol{\sigma}}^r = \mathbf{0} & \text{in} & V \\ \overline{\boldsymbol{\sigma}}^r \cdot \boldsymbol{n} \ \text{ anti-periodic} & \text{on} & \partial V \\ \langle \overline{\boldsymbol{\sigma}}^r \rangle = 0 & \text{in} & V \end{array} \tag{4.6}$$

and such that

$$\langle \boldsymbol{\sigma}^{(s)}(\boldsymbol{y}) \rangle = \boldsymbol{\Sigma}(\boldsymbol{x}) \tag{4.7}$$

where $\boldsymbol{n}$ is the outward vector normal to ∂V and Div the divergence operator.

4.1. Particular case

If one considers that the boundary conditions on the edges of the representative volume element are the uniform constraints $\boldsymbol{\Sigma}$, as suggested by Suquet (1982) (see also Marigo et al., 1987), then we have to consider the following domains of macroscopic stresses

$$\overline{P}^m_{\boldsymbol{\Sigma}}(\boldsymbol{x}) = \left\{ \boldsymbol{\Sigma} \;\middle|\; \exists \boldsymbol{\sigma}^{(s)},\ \boldsymbol{\sigma}^{(s)}(\boldsymbol{y}) \in \overline{P}(\boldsymbol{y}),\ \forall \boldsymbol{y} \in V \right\} \tag{4.8}$$

with the following conditions

$$\begin{aligned} &\operatorname{Div} \boldsymbol{\sigma}^{(c)} = \mathbf{0} && \text{in} \quad V \\ &\boldsymbol{\sigma}^{(c)} \cdot \boldsymbol{n} = \boldsymbol{\Sigma} \cdot \boldsymbol{n} && \text{on} \quad \partial V \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} &\operatorname{Div} \overline{\boldsymbol{\sigma}}^{r} = \mathbf{0} && \text{in} \quad V \\ &\overline{\boldsymbol{\sigma}}^{r} \cdot \boldsymbol{n} = 0 && \text{on} \quad \partial V \end{aligned} \tag{4.10}$$

If, on the contrary, one assumes that the boundary conditions on the edges of the representative volume element are the uniform strains E, then the following domains of macroscopic stresses have to be considered

$$\overline{P}^{m}_{\mathsf{E}}(\boldsymbol{x}) = \left\{ \boldsymbol{\Sigma} \;\middle|\; \exists \boldsymbol{\sigma}^{(s)},\ \boldsymbol{\sigma}^{(s)}(\boldsymbol{y}) \in \overline{P}(\boldsymbol{y}),\ \forall \boldsymbol{y} \in V \right\} \tag{4.11}$$

with the following conditions

$$\begin{aligned} &\operatorname{Div} \boldsymbol{\sigma}^{(c)} = \mathbf{0} && \text{in} \quad V \\ &\boldsymbol{u}^{(c)} = \mathsf{E} \cdot \boldsymbol{y} && \text{on} \quad \partial V \\ &\boldsymbol{\varepsilon}^{(c)} = \nabla_s(\boldsymbol{u}^{(c)}) && \text{in} \quad V \\ &\boldsymbol{\sigma}^{(c)} = \mathsf{L} : \boldsymbol{\varepsilon}^{(c)} && \text{in} \quad V \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} &\operatorname{Div} \overline{\boldsymbol{\sigma}}^{r} = \mathbf{0} && \text{in} \quad V \\ &\langle \overline{\boldsymbol{\sigma}}^{r} \rangle = 0 && \text{in} \quad V \\ &\langle \boldsymbol{\sigma}^{(s)} \rangle = \boldsymbol{\Sigma}(\boldsymbol{x}) && \end{aligned} \tag{4.13}$$

For the imposed uniform constraints on the edges of the RVE, the domains of macroscopic stresses are underestimated and for the imposed uniform strains these domains are overestimated (cf. Suquet, 1982; Marigo et al., 1985)

$$\overline{P}^{m}_{\Sigma} \subset \overline{P}^{m}(\boldsymbol{x}) \subset \overline{P}^{m}_{\mathsf{E}} \tag{4.14}$$

The condition for shakedown according to the static theorem for the determination of the macroscopic admissible domain $\overline{P}^{m}$ against failure due to inadmissible damage or unlimited accumulation of plastic deformations can be expressed by the following optimisation problem:

Find

$$\alpha_{SD} = \max_{\lambda_{\max}, D, \overline{\boldsymbol{\sigma}}^{r}} \alpha \tag{4.15}$$

subjected to (4.6) and

$$\begin{aligned} &\lambda_{\max} < 1 \\ &D(\boldsymbol{y}) < D_c && \forall \boldsymbol{y} \in V \\ &\boldsymbol{\sigma}^{(s)}(\boldsymbol{y}) \in \overline{P}(\boldsymbol{y}) && \forall \boldsymbol{y} \in V \\ &\langle \boldsymbol{\sigma}^{(s)} \rangle = \boldsymbol{\Sigma} \end{aligned} \tag{4.16}$$

such that

$$\boldsymbol{\sigma}^{(s)} = \alpha \boldsymbol{\sigma}^{(c)} + \overline{\boldsymbol{\sigma}}^{r} \tag{4.17}$$

Condition $(4.16)_1$ assures that there is no complete debonding at the interface, and condition $(4.16)_2$ assures structural safety against the failure in the matrix due to plastic damage (see, e.g. Hachemi and Weichert, 1997). Condition $(4.16)_3$ assures that the safe state of stresses is plastically admissible.

5. Illustrative examples

To illustrate the method, two examples are presented, based on a finite-element shakedown analysis of appropriately chosen RVEs of the material and a homogenisation technique of periodic media. In both cases, the analysis is restricted to two dimensions. Furthermore, the material damage as well as debonding between the matrix and fibre in the second example, are not taken into account. First, flat aluminium alloy sheets with periodically arranged slits of varying length and patterns of periodicity are investigated. The slits are considered as a material without mechanical resistance. The mechanical characteristics of the homogeneous and isotropic aluminium are as follows: Young's modulus $E = 67200$ MPa; Poisson's ratio $\nu = 0.318$ and the conventional yield stress at 0.1% of axial elongation $\sigma_Y = 137$ MPa. The following dimensionless lengths of the openings were considered: $r = l/L = 0.5$ and 0.7 with $t = 1$ mm. The results are compared with the experimental ones obtained by Litewka et al. (1984). It should be noted that the solution to the elastic reference problem for the shakedown analysis corresponds to a slit, where the sharp corners have been replaced by rounded corners with radius $t/10$. The fact that the calculated solution (Fig. 3b) is non-conservative compared to the experimental results may be related to this approximation.

In Fig. 2, two considered patterns of openings with the pitch $L = 10$ mm are specified. It follows, therefore, that two different types of symmetries regarding the rectangular openings were considered. In the first case the rectangular

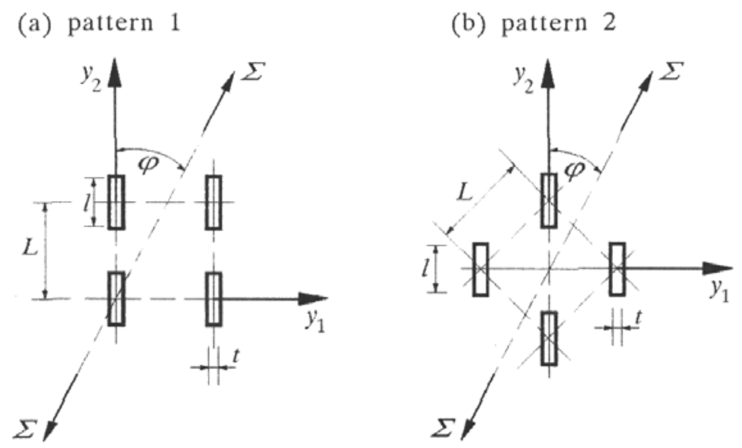

Fig. 2. Arrangement of rectangular opening

openings orientation follows the pitch, whereas in the second this orientation makes an angle $\varphi = 45°$ with the square grid specifying the pitch. In Fig. 3, a comparison of the obtained results with experimental values of the uniaxial macroscopic tensile strength as a function of the load orientation are shown for the two considered patterns.

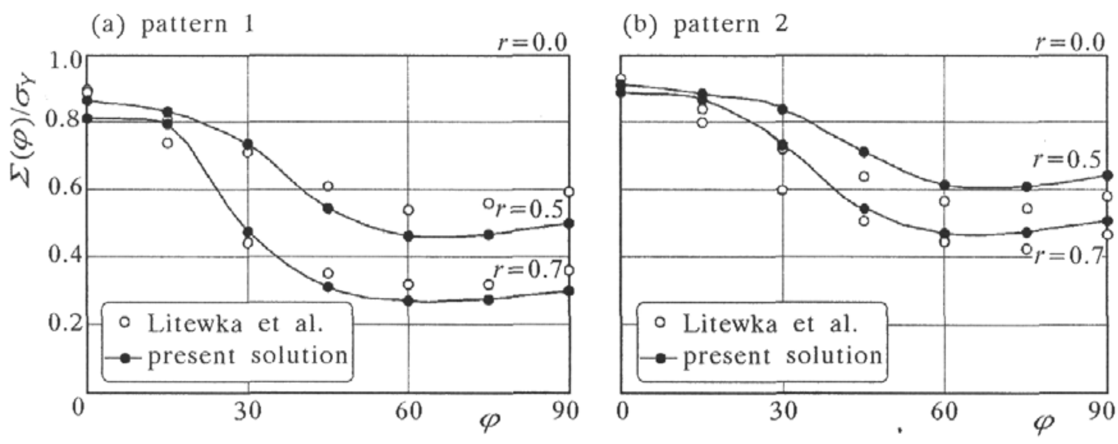

Fig. 3. Uniaxial macroscopic tensile strength versus inclinations φ

For the limit and shakedown analysis of fibre-reinforced composite materials, we consider a typical problem of an aluminium/alumina composite with perfect bonding between the fibres and matrix ($\lambda = 0$). For the given regular quadratic, rotated and hexagonal patterns of periodicity of the reinforced elastic fibres in the ductile matrix as illustrated in Fig. 4, given material properties of the fibres and matrix (Table 1), the admissible load in the space

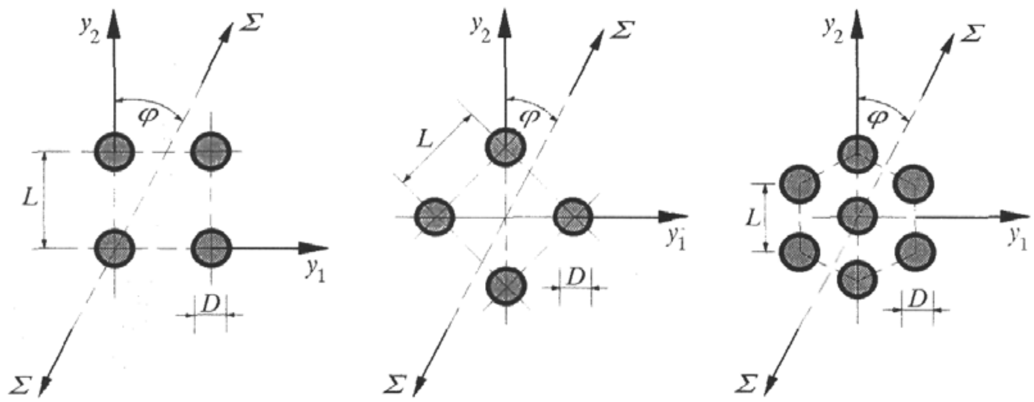

Fig. 4. Arrangement of fibre-reinforced composite

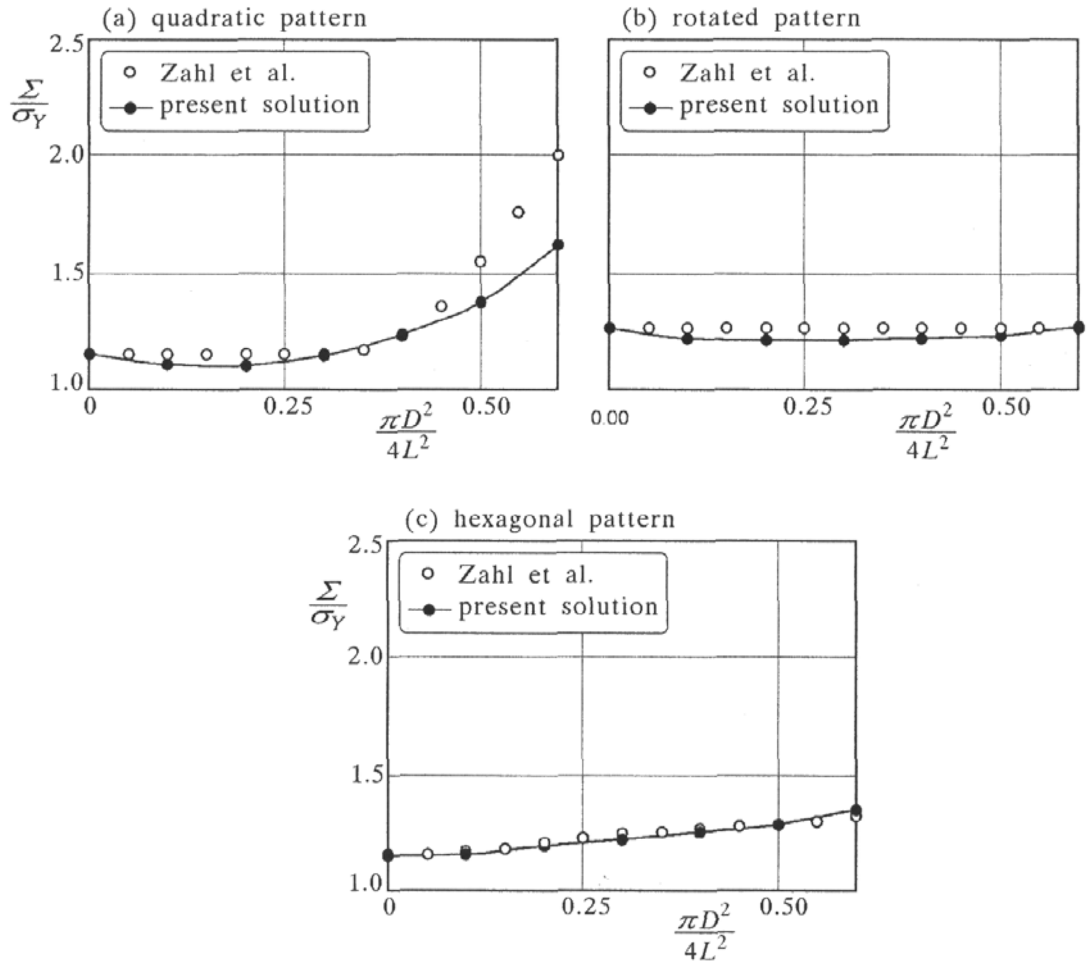

Fig. 5. Variation of macroscopic stress with fibre volume fraction

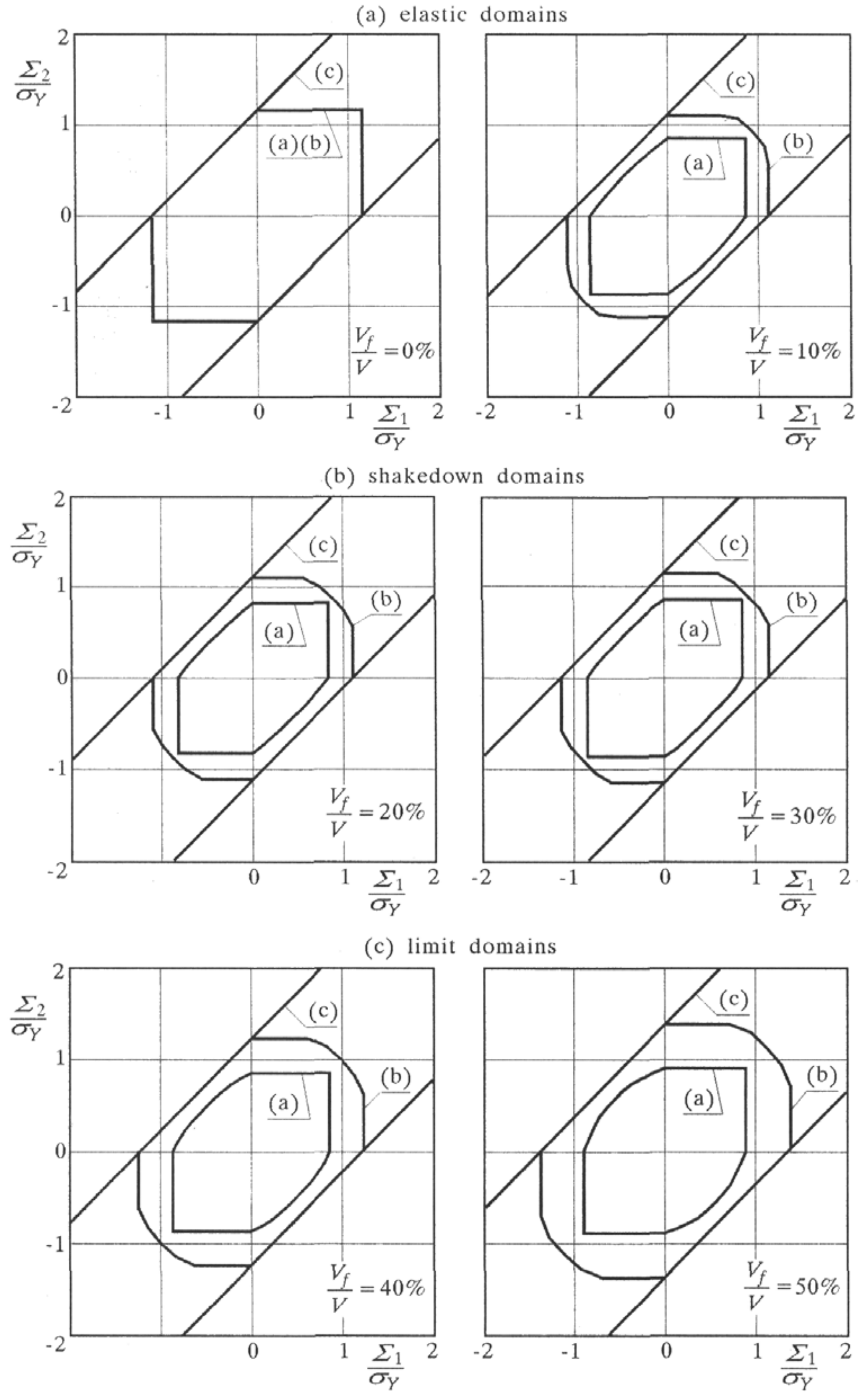

Fig. 6. Admissible domains of macroscopic stresses

of macroscopic stresses is determined. The adopted mechanical characteristics for the matrix and the fibres are:

Table 1. Mechanical characteristics of fibre-reinforced composite

	Matrix	Fibre
Material	Al	Al_2O_3
Young's modulus E [GPa]	70	370
Poisson's ratio ν	0.3	0.3
Yield stress σ_Y [MPa]	80	—

Fig. 5 presents variation of the admissible value of the macroscopic stress ($\Sigma_1 = \Sigma_2 = \Sigma$), normalised by the yield stress σ_Y of the matrix, with fibre volume fraction, where the results obtained by Zahl and Schmauder (1994) for, quadratic, rotated and hexagonal patterns are represented. The admissible rectangular macroscopic domains for the quadratic pattern are shown in Fig. 6, where the bounds of elastic, limit and shakedown domains are represented. These bounds are obtained for different values of the fibre volume fraction ($V_f/V = \pi D^2/4L^2 = 0\% - 50\%$).

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Analiza przystosowania i wytrzymałości granicznej kompozytów o strukturze periodycznej

Streszczenie

W pracy zaprezentowano metodologię oceny kompozytów o strukturze periodycznej przenoszących zmienne, powtarzalne obciążenia na podstawie teorii przystosowania w zastosowaniach do kompozytów z metalową osnową. Metodologię oparto na lokalnej analizie przystosowania dla reprezentatywnego elementu kompozytu i zastosowaniu technik uśredniających w celu określenia dopuszczalnych naprężeń na poziomie makroskali.

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