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# ON THE DYNAMIC BEHAVIOUR OF A UNIPERIODIC FOLDED PLATES

BOHDAN MICHALAK

*Department of Structural Mechanics, Łódź University of Technology*
*e-mail: bmichala@ck-sg.p.lodz.pl*

The aim of the contribution is to formulate an averaged model describing the dynamic behaviour of periodically folded shell-like structure, called a folded plate. The contribution proposes new approach to the modelling one-dimensional periodic structure. The characteristic feature of the proposed model is that it makes possible to take into account the effect of periodicity cell size on the dynamic behaviour of the periodic structure. It will be shown that the dynamic behaviour of the plate with one-dimensional periodic structure cannot be treated as a specific case of dynamics of the plate with two-dimensional periodic structure. The main drawback of most of the existing averaged models for periodic structure is that only averaged boundary conditions can be described. On the basis of the proposed model, a more general displacement boundary conditions in the mezo-scale (in the region of the periodicity cell) can be defined.

*Key words:* shell, modelling, periodic structure, dynamics

## 1. Introduction

The subject of this paper is to formulate and investigate an averaged model describing the dynamic behaviour of a plate structure periodically folded along one direction. This structure under consideration, referred to as a folded plate, is composed of many identically repeated elements, which are periodically distributed along one direction (Fig. 1). Every element, called the periodicity cell, is made of a linear-elastic homogeneous material. The exact analysis of periodic folded plates within the theory of thin elastic shells is too complicated to constitute the basis for investigations of most engineering problems. Thus, problems of such plates are investigated in the frame-work of various approximate methods. So-called effective rigidity plate theories were presented e.g. in Caillerie (1984) and averaged homogenized models of plates with periodic structure e.g. in Lewiński (1992), while the models for plates with a technical anisotropy were presented e.g. Troitsky (1976).

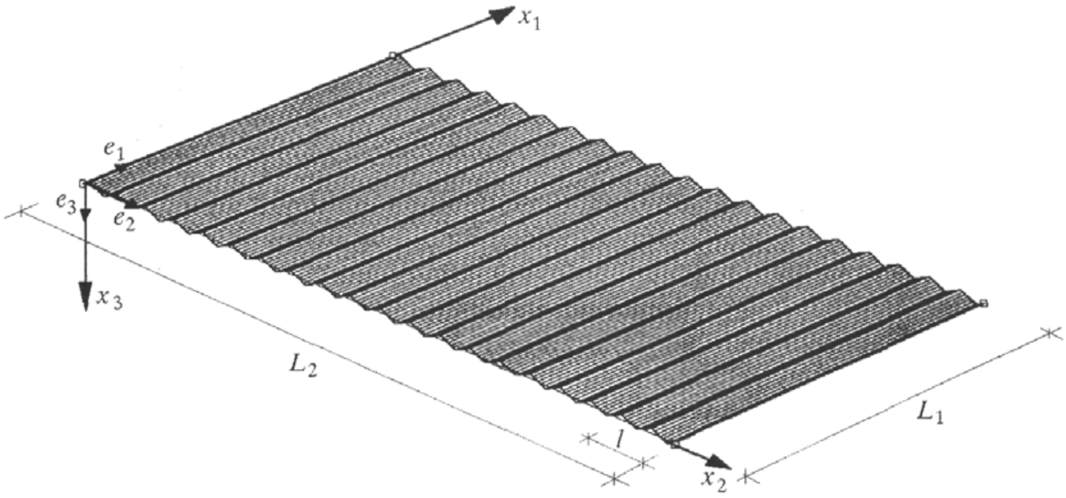


Fig. 1. A scheme of the uniperiodic folded plate

Using the asymptotic homogenization method, the effect of the periodicity cell size on the overall folded plate response is neglected. In order to investigate non-stationary problems we have applied a modelling approach presented in Woźniak and Wierzbicki (2000). This model takes into account the length-scale effect on dynamic response of a periodic structure.

The aim of this paper is two-fold. First, to formulate a non-asymptotic plate-type (averaged) model of folded plates with periodic structure along one direction (along the $x_2$-axis). Second, to investigate a free-vibration problem in the framework of non-asymptotic model of a folded plate with boundary conditions which can be defined in the mezo-scale on the boundaries $x_1 = \text{const}$. The main drawback of the effective rigidity plate theories and models for plates with a technical anisotropy, is that the displacement boundary conditions have to be imposed not on the complete displacement field $\boldsymbol{u}$ but only on the averaged part $\boldsymbol{U}$ of this field. In the proposed model, boundary conditions on the boundaries $x_1 = \text{const}$ can be imposed on the complete displacement field $\boldsymbol{u}$ (in the mezo-scale).

## 2. Modelling procedure

In this paper we will investigate a thin shell-like structure with a periodic structure along one direction (Fig. 1). The plate of this kind will be referred to as *uniperiodic folded plate*. Let the midsurface of the undeformed folded plate be given by $x^i = R^i(\theta^1, \theta^2)$, $(\theta^1, \theta^2) \in \Pi$, where $\Pi$ is a regular plane region. It is assumed that: $x^1 = \theta^1$, $x^2 = \theta^2$ and $x^3 = z(\theta^2)$, where $z(\cdot)$ is a function satisfying the condition $z(\theta^2) = z(\theta^2 + l)$. Let $l$ stand for the period of plate structure in the direction of the $x^2$-axis and hence $(0, l)$ is the periodicity interval in the plate midplane. Moreover, we assume that $l$ is sufficiently small compared to $L_\Pi$, the smallest characteristic length dimension of $\Pi$, $l \ll L_\Pi$. At the same time the thickness $\delta$ of the shell is supposed to be constant and small compared to $l$. Hence, parameter $l$ will be called *the mezostructure length parameter*.

Let us denote periodicity intervals by $\Delta L(x_2)$. For an arbitrary integrable function $f$ we define the averaging operator on $\Delta L(x_2)$ given by

$$\langle f \rangle(x_1, x_2) = \frac{1}{l} \int\limits_{\Delta L(x_2)} f(x_1, z_2)\, dz_2 \tag{2.1}$$

The direct description of folded plates is based on the well-known (Green and Zerna, 1954), linear theory for thin elastic plates. Using notations $\boldsymbol{g}_\alpha \equiv C^i{}_\alpha \boldsymbol{e}_i$, $\boldsymbol{n} = \boldsymbol{g}_1 \times \boldsymbol{g}_2 / |\boldsymbol{g}_1 \times \boldsymbol{g}_2|$, where $C^i{}_\alpha$ are given by Eq. (2.2), we obtain the metric tensor of the undeformed midsurface $a_{\alpha\beta} = C^i{}_\alpha C_{i\beta}$ and a Ricci tensor $\epsilon_{\alpha\beta}$ as $\Delta L$-periodic functions (Fig. 2) and define $a = \det a_{\alpha\beta}$

$$\begin{array}{lll} C^1{}_1 = 0 & C^2{}_1 = 0 & C^3{}_1 = 0 \\ C^1{}_2 = 0 & C^2{}_2 = 1 & C^3{}_2 = \pm \tan\alpha \\ n^1 = 0 & n^2 = \mp \sin\alpha & n^3 = \cos\alpha \end{array} \tag{2.2}$$

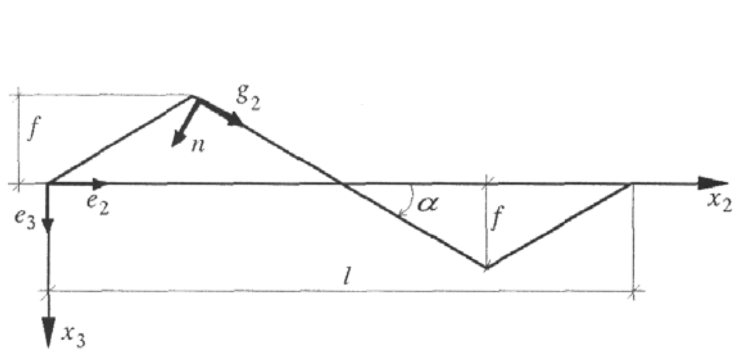


Fig. 2. The periodicity interval of the uniperiodic folded plate

By $\boldsymbol{u} = u^i(\boldsymbol{x},t)\boldsymbol{e}_i = v^\alpha \boldsymbol{g}_\alpha + w\boldsymbol{n}$ we denote the displacement vector field of the folded plate midsurface, by $\boldsymbol{p} = p^i(\boldsymbol{x},t)\boldsymbol{e}_i$ the external forces, and by $\rho$ the mass density averaged over the plate thickness related to the midsurface. In the framework of the linear approximate theory of thin elastic plate, we obtain:

— strain-displacement relations

$$\varepsilon_{\alpha\beta} = C^i{}_{(\alpha} u_{i,\beta)} \qquad \kappa_{\alpha\beta} = n^i u_{i,\alpha\beta} \tag{2.3}$$

— constitutive equations

$$n^{\alpha\beta} D H^{\alpha\beta\gamma\delta} \varepsilon_{\gamma\delta} \qquad m^{\alpha\beta} B H^{\alpha\beta\gamma\delta} \kappa_{\gamma\delta} \tag{2.4}$$

where

$$H^{\alpha\beta\gamma\mu} = \frac{1}{2}\Big[g^{\alpha\mu}g^{\beta\gamma} + g^{\alpha\gamma}g^{\beta\mu} + \nu(\epsilon^{\alpha\gamma}\epsilon^{\beta\mu} + \epsilon^{\alpha\mu}\epsilon^{\beta\gamma})\Big]$$

$$D \equiv \frac{E\delta}{1-\nu^2} \qquad B \equiv \frac{E\delta^3}{1-\nu^2}$$

— equations of motion (in the weak form)

$$\begin{aligned}\int\limits_{\Pi} (n^{\alpha\beta}\delta\varepsilon_{\alpha\beta} + m^{\alpha\beta}\delta\kappa_{\alpha\beta})\sqrt{a}\; dx^1 dx^2 + \frac{d}{dt}\int\limits_{\Pi} \rho\dot{u}^i \delta u_i \sqrt{a}\; dx^1 dx^2 = \\ = \int\limits_{\Pi} p^i \delta u_i \sqrt{a}\; dx^1 dx^2\end{aligned} \tag{2.5}$$

The modelling approach to the mezostructural theory of the folded plates is based on the *tolerance averaging* concepts given by Woźniak and Wierzbicki (2000).

### Basic kinematics hypothesis

We restrict our considerations to the motion; the displacement fields $\boldsymbol{u}(\boldsymbol{x},t)$ are periodic-like functions, $\boldsymbol{u}(\boldsymbol{x},t) \in PL_{\Delta L}(T)$, $T$ is a certain tolerance system (Woźniak and Wierzbicki, 2000). The macrodisplacements $U_i(\boldsymbol{x},t) = \langle u_i\rangle(\boldsymbol{x},t)$ describing the averaged motion of the folded plates and its all derivatives, are a *slow-varying* functions of $x_2$ only, $U_i(\boldsymbol{x},t) \in SV_{\Delta L}(T)$, i.e. they satisfy conditions of the form $\langle fU_i\rangle(\boldsymbol{x}) \cong \langle f\rangle(\boldsymbol{x})U_i(\boldsymbol{x})$ for every integrable function $f$. The local displacement oscillations defined by $\boldsymbol{v} \equiv \boldsymbol{u} - \boldsymbol{U}$, $\boldsymbol{v}(\boldsymbol{x},t) \in PL_{\Delta L}(T)$ are highly oscillating functions in $x_2$, i.e. they satisfy the conditions:

(i) $\langle (vF)_{,2} \rangle(\boldsymbol{x},t) \cong \langle F v_{,2} \rangle(\boldsymbol{x},t)$, $\boldsymbol{x} \in \Pi$, $F$ – *slow-varying* functions of $x_2$

(ii) $v(\boldsymbol{x}) \in O(l^2)$, $l v_{,\alpha}(\boldsymbol{x}) \in O(l^2)$, $l^2 v_{|\alpha\beta}(\boldsymbol{x}) \in O(l^2)$, where $O(l^2) \to 0$ with $l \to 0$.

The displacements $u_i$ will be approximated by

$$\begin{aligned} u_\alpha(\boldsymbol{x},t) &\cong U_\alpha(\boldsymbol{x},t) + h(x_2)V_\alpha(\boldsymbol{x},t) \qquad \boldsymbol{x} = (x^1,x^2) \in \Pi \qquad t \geqslant 0 \\ u_3(\boldsymbol{x},t) &\cong U_3(\boldsymbol{x},t) + g(x_2)V_3(\boldsymbol{x},t) \end{aligned} \tag{2.6}$$

Functions $U_i(x_1,x_2,t)$, $V_i(x_1,x_2,t)$ are slowly varying functions only in the periodicity direction $x_2$ (basic unknowns).

The functions $h(\cdot)V_\alpha(\cdot,t)$ and $g(\cdot)V_3(\cdot,t)$ describe the local displacement oscillations.

Functions $h(\cdot)$ and $g(\cdot)$ will be referred to as the mezo-shape functions and are obtained as an approximate solution to the eigenvalue problem on a plate segment in the periodicity interval $\Delta L(x_2)$, together with periodic boundary conditions.

The form of the mezo-shape functions is obtained as an eigenvibrations form of a plate segment in the periodicity interval $\Delta L(x_2)$.

Values $h(\boldsymbol{x})$ and $g(\boldsymbol{x})$ satisfy the conditions $h(\boldsymbol{x}) \in O(l^2)$, $h_{,\alpha}(\boldsymbol{x}) \in O(l)$, $h_{|\alpha\beta}(\boldsymbol{x}) \in O(1)$, $g(\boldsymbol{x}) \in O(l^2)$, $g_{,\alpha}(\boldsymbol{x}) \in O(l)$, $g_{|\alpha\beta}(\boldsymbol{x}) \in O(1)$.

## 3. Averaged description

The modelling procedure proposed in Woźniak and Wierzbicki (2000) and the aforementioned kinematics hypothesis lead to the equations of motion presented below in the coordinate form

$$\begin{aligned} & M^{i|\alpha\beta}{}_{,\alpha\beta} - N^{i|\alpha}{}_{,\alpha} + \langle\widetilde{\rho}\rangle \ddot{U}^i = \widetilde{p}^i \\ & N^\gamma - K^{\gamma|\alpha}{}_{\|\alpha} + M^\gamma + P^{\gamma|\alpha\beta}{}_{\|\alpha\beta} - 2L^{\gamma|\alpha}{}_{\|\alpha} + \langle\widetilde{\rho}hh\rangle \ddot{V}^\gamma = \langle\widetilde{p}^\gamma h\rangle \\ & N^3 - K^{3|\alpha}{}_{\|\alpha} + M^3 + P^{3|\alpha\beta}{}_{\|\alpha\beta} - 2L^{3|\alpha}{}_{\|\alpha} + \langle\widetilde{\rho}gg\rangle \ddot{V}^3 = \langle\widetilde{p}^3 g\rangle \end{aligned} \tag{3.1}$$

where we have denoted the arbitrary function $f$ defined on $\Pi$

$$f_{\|\alpha} = \begin{cases} f_{,1} & \text{for} \quad \alpha = 1 \\ 0 & \text{for} \quad \alpha = 2 \end{cases} \qquad\qquad f_{|\alpha} = \begin{cases} 0 & \text{for} \quad \alpha = 1 \\ f_{,2} & \text{for} \quad \alpha = 2 \end{cases} \tag{3.2}$$

The constitutive equations have the form

$$
\begin{aligned}
N^{i|\alpha} &= D^{i\alpha|j\beta}U_{j,\beta} + G^{i\alpha|\mu\delta}V_{\mu\|\delta} + H^{i\alpha|\mu}V_\mu + G^{i\alpha\|3\delta}V_{3\|\delta} + H^{i\alpha|3}V_3 \\
N^{\alpha} &= H^{\alpha|j\beta}U_{j,\beta} + G^{\alpha|\mu\delta}V_{\mu\|\delta} + H^{\alpha|\mu}V_\mu + G^{\alpha\|3\delta}V_{3\|\delta} + H^{\alpha|3}V_3 \\
N^{3} &= H^{3|j\beta}U_{j,\beta} + G^{3|\mu\delta}V_{\mu\|\delta} + H^{3|\mu}V_\mu + G^{3\|3\delta}V_{3\|\delta} + H^{3|3}V_3 \\
K^{\alpha|\beta} &= G^{\alpha\beta|j\delta}U_{j,\delta} + H^{\alpha\beta|\mu\delta}V_{\mu\|\delta} + G^{\alpha\beta|\mu}V_\mu + H^{\alpha\beta\|3\delta}V_{3\|\delta} + G^{\alpha\beta|3}V_3 \\
K^{3|\beta} &= G^{3\beta|j\delta}U_{j,\delta} + H^{3\beta|\mu\delta}V_{\mu\|\delta} + G^{3\beta|\mu}V_\mu + H^{3\beta\|3\delta}V_{3\|\delta} + G^{3\beta|3}V_3 \\
M^{i|\alpha\beta} &= A^{i\alpha\beta|j\gamma\delta}U_{j,\gamma\delta} + B^{i\alpha\beta|\mu\gamma\delta}V_{\mu\|\gamma\delta} + 2B^{i\alpha\beta|\mu\gamma}V_{\mu\|\gamma} + B^{i\alpha\beta|\mu}V_\mu + \\
&\quad + B^{i\alpha\beta|3\gamma\delta}V_{3\|\gamma\delta} + 2B^{i\alpha\beta|3\gamma\delta}V_{3\|\gamma} + B^{i\alpha\beta|3}V_3 \\
M^{\alpha} &= B^{\alpha|j\gamma\delta}U_{j,\gamma\delta} + C^{\alpha|\mu\gamma\delta}V_{\mu\|\gamma\delta} + 2C^{\alpha|\mu\gamma}V_{\mu\|\gamma} + B^{\alpha|\mu}V_\mu + \\
&\quad + C^{\alpha|3\gamma\delta}V_{3\|\gamma\delta} + 2C^{\alpha|3\gamma}V_{3\|\gamma} + B^{\alpha|3}V_3 \\
M^{3} &= B^{3|j\gamma\delta}U_{j,\gamma\delta} + C^{3|\mu\gamma\delta}V_{\mu\|\gamma\delta} + 2C^{3|\mu\gamma}V_{\mu\|\gamma} + B^{3|\mu}V_\mu + \\
&\quad + C^{3|3\gamma\delta}V_{3\|\gamma\delta} + 2C^{3|3\gamma}V_{3\|\gamma} + B^{3|3}V_3 \\
P^{\tau|\alpha\beta} &= B^{\tau\alpha\beta|j\gamma\delta}U_{j,\gamma\delta} + C^{\tau\alpha\beta|\mu\gamma\delta}V_{\mu\|\gamma\delta} + 2C^{\tau\alpha\beta|\mu\gamma}V_{\mu\|\gamma} + C^{\tau\alpha\beta|\mu}V_\mu + \\
&\quad + C^{\tau\alpha\beta|3\gamma\delta}V_{3\|\gamma\delta} + 2C^{\tau\alpha\beta|3\gamma}V_{3\|\gamma} + C^{\tau\alpha\beta|3}V_3 \\
P^{3|\alpha\beta} &= B^{3\alpha\beta|j\gamma\delta}U_{j,\gamma\delta} + C^{3\alpha\beta|\mu\gamma\delta}V_{\mu\|\gamma\delta} + 2C^{3\alpha\beta|\mu\gamma}V_{\mu\|\gamma} + C^{3\alpha\beta|\mu}V_\mu + \\
&\quad + C^{3\alpha\beta|3\gamma\delta}V_{3\|\gamma\delta} + 2C^{3\alpha\beta|3\gamma}V_{3\|\gamma} + C^{3\alpha\beta|3}V_3 \\
L^{\tau|\alpha} &= B^{\tau\alpha|j\gamma\delta}U_{j,\gamma\delta} + C^{\tau\alpha|\mu\gamma\delta}V_{\mu\|\gamma\delta} + 2C^{\tau\alpha|\mu\gamma}V_{\mu\|\gamma} + C^{\tau\alpha|\mu}V_\mu + \\
&\quad + C^{\tau\alpha|3\gamma\delta}V_{3\|\gamma\delta} + 2C^{\tau\alpha|3\gamma}V_{3\|\gamma} + C^{\tau\alpha|3}V_3 \\
L^{3|\alpha} &= B^{3\alpha|j\gamma\delta}U_{j,\gamma\delta} + C^{3\alpha|\mu\gamma\delta}V_{\mu\|\gamma\delta} + 2C^{3\alpha|\mu\gamma}V_{\mu\|\gamma} + C^{3\alpha|\mu}V_\mu + \\
&\quad + C^{3\alpha|3\gamma\delta}V_{3\|\gamma\delta} + 2C^{3\alpha|3\gamma}V_{3\|\gamma} + C^{3\alpha|3}V_3
\end{aligned}
\tag{3.3}
$$

The form of the coefficients in relations (3.3) is given in he Appendix.

The above equations are the basis for investigations of the overall behaviour of uniperiodic folded plates. Substituting the right-hand sides of Eq. (3.3) into Eq. (3.1), we obtain 6 equations for 3 macrodisplacements $U_i$ and 3 disturbance variables $V_i$. The equations for disturbance variables $V_i$ obtained for plates with two-dimensional periodic structure are ordinary differential equations, while for plates with uniperiodic structure they are partial differential equations involving the derivatives of disturbance variables with respect to the time and $x_1$-coordinate. In the case of a rectangular folded plate, the midplane of which is $\Pi = (0, L_1) \times (0, L_2)$, two boundary conditions on the edges $x_1 = 0, L_1$ and $x_2 = 0, L_2$ for the macrodisplacements $U_i$ should be defined. On the edges $x_1 = 0, L_1$ we should define two boundary conditions for the disturbance variables $V_i$, which enable us to define us the boundary conditions in the mezo-scale.

## 4. Applications

We shall investigate simple problem of a cylindrical bending of a rectangular folded plate with a periodic structure along the $x_2$-axis. Let the plate band be simply supported on the opposite edges $x_1 = 0$, $x_1 = L_1$, where $L_1$ is its span. In this case the basic unknowns $U_i(\cdot)$ and $V_i(\cdot)$ depend only on arguments $x_1 = \theta^1$ and $t$. Within the framework of the structural theory ST, by substituting the right-hand sides of Eq. (3.3) into Eq. (3.1), we obtain the system of equations for $U_i = U_i(x_1, t)$ and $V_i = V_i(x_1, t)$

$$
\begin{aligned}
&M^{s|11},_{11} - N^{s|1},_1 + \langle\widetilde{\rho}\rangle\ddot{U}^s = 0 && s = 1,2,3\\
&N^s - K^{s|1},_1 + M^s + P^{s|11},_{11} - 2L^{s|1},_1 + \langle\widetilde{\rho}hh\rangle\ddot{V}^s = 0 && s = 1,2\\
&N^3 - K^{3|1},_1 + M^3 + P^{3|11},_{11} - 2L^{3|1},_1 + \langle\widetilde{\rho}gg\rangle\ddot{V}^3 = 0
\end{aligned}
\tag{4.1}
$$

The periodicity interval $\Delta L = (0, L)$ is shown in Fig. 2. Hence, the mode-shapes functions $h(\cdot)$ and $g(\cdot)$ for this cell will be assumed in the approximate forms

$$
h = g = \begin{cases}
16l^2\Big[\Big(\dfrac{1}{4}\Big)^2 - \Big(\dfrac{x}{l}\Big)^2\Big] & \text{for} \quad 0 \leqslant x \leqslant \dfrac{l}{4}\\[2ex]
16l^2\Big(\dfrac{1}{4} - \dfrac{x}{l}\Big)\Big(\dfrac{3}{4} - \dfrac{x}{l}\Big) & \text{for} \quad \dfrac{l}{4} \leqslant x \leqslant \dfrac{3l}{4}\\[2ex]
16l^2\Big(\dfrac{x}{l} - \dfrac{3}{4}\Big)\Big(\dfrac{5}{4} - \dfrac{x}{l}\Big) & \text{for} \quad \dfrac{3l}{4} \leqslant x \leqslant l
\end{cases}
\tag{4.2}
$$

After substituting the right-hand sides of constitutive equations (3.3) with notations given in the Appendix into Eq. (4.1), the system equations of motion will have a form

$$-D\langle H^{1111}C^1{}_1C^1{}_1\sqrt{a}\,\rangle U_{1,11}+\langle\widetilde{\rho}\rangle\ddot{U}^1=0$$

$$\begin{aligned}&B\langle H^{1111}n^2n^2\sqrt{a}\,\rangle U_{2,1111}+B\langle H^{1111}gn^2n^3\sqrt{a}\,\rangle V_{3,1111}+\\&+B\langle H^{1122}g_{,22}\,n^2n^3\sqrt{a}\,\rangle V_{3,11}-D\langle H^{2121}C^2{}_2C^2{}_2\sqrt{a}\,\rangle U_{2,11}-\\&-D\langle H^{2121}gC^3{}_2C^2{}_2\sqrt{a}\,\rangle V_{3,11}+\langle\widetilde{\rho}\rangle\ddot{U}^2=0\end{aligned}$$

$$\begin{aligned}&B\langle H^{1111}n^3n^3\sqrt{a}\,\rangle U_{3,1111}+B\langle H^{1111}gn^3n^2\sqrt{a}\,\rangle V_{2,1111}+\\&+B\langle H^{1122}h_{,22}\,n^3n^2\sqrt{a}\,\rangle V_{2,11}-D\langle H^{2121}C^3{}_2C^3{}_2\sqrt{a}\,\rangle U_{3,11}-\\&-D\langle H^{2121}hC^2{}_2C^3{}_2\sqrt{a}\,\rangle V_{2,11}+\langle\widetilde{\rho}\rangle\ddot{U}^3=0\end{aligned}$$

$$\begin{aligned}&-D\langle H^{1111}C^1{}_1C^1{}_1hh\sqrt{a}\,\rangle V_{1,11}+\\&+D\langle H^{1212}h_{,2}\,h_{,2}\,C^1{}_1C^1{}_1\sqrt{a}\,\rangle V_1+\langle\widetilde{\rho}hh\rangle\ddot{V}^1=0\end{aligned}$$

$$\begin{aligned}&B\langle H^{1111}n^2n^2hh\sqrt{a}\,\rangle V_{2,1111}+\Big[2B\langle H^{1122}h_{,22}\,hn^2n^2\sqrt{a}\,\rangle-\\&-4B\langle H^{1212}h_{,2}\,h_{,2}\,n^2n^2\sqrt{a}\,\rangle-D\langle H^{2121}C^2{}_2C^2{}_2hh\sqrt{a}\,\rangle\Big]V_{2,11}+\\&+\Big[B\langle H^{2222}n^2n^2h_{,22}\,h_{,22}\,\sqrt{a}\,\rangle+D\langle H^{2222}h_{,2}\,h_{,2}\,C^2{}_2C^2{}_2\sqrt{a}\,\rangle\Big]V_2+\\&+B\langle H^{1111}n^2n^3h\sqrt{a}\,\rangle U_{3,1111}+\Big[B\langle H^{2211}n^2n^3h_{,22}\,\sqrt{a}\,\rangle-\\&-D\langle H^{2121}hC^3{}_2C^2{}_2\sqrt{a}\,\rangle\Big]U_{3,11}+\langle\widetilde{\rho}hh\rangle\ddot{V}^2=0\end{aligned}\tag{4.3}$$

$$\begin{aligned}&B\langle H^{1111}n^3n^3gg\sqrt{a}\,\rangle V_{3,1111}+\Big[2B\langle H^{1122}g_{,22}\,gn^3n^3\sqrt{a}\,\rangle-\\&-4B\langle H^{1212}g_{,2}\,g_{,2}\,n^3n^3\sqrt{a}\,\rangle-D\langle H^{2121}C^3{}_2C^3{}_2gg\sqrt{a}\,\rangle\Big]V_{3,11}+\\&+\Big[B\langle H^{2222}n^3n^3g_{,22}\,g_{,22}\,\sqrt{a}\,\rangle+D\langle H^{2222}g_{,2}\,g_{,2}\,C^3{}_2C^3{}_2\sqrt{a}\,\rangle\Big]V_3+\\&+B\langle H^{1111}n^2n^3g\sqrt{a}\,\rangle U_{2,1111}+\Big[B\langle H^{2211}n^2n^3g_{,22}\,\sqrt{a}\,\rangle-\\&-D\langle H^{2121}gC^3{}_2C^2{}_2\sqrt{a}\,\rangle\Big]U_{2,11}+\langle\widetilde{\rho}gg\rangle\ddot{V}^3=0\end{aligned}$$

where for the assumed cell we have

$$
\begin{array}{lll}
C^1{}_1 = 0 & C^2{}_1 = 0 & C^3{}_1 = 0 \\
C^1{}_2 = 0 & C^2{}_2 = 1 & C^3{}_2 = \pm\dfrac{4f}{l} \\
n^1 = 0 & n^2 = \mp\dfrac{4f}{l}\dfrac{1}{\sqrt{1+16\left(\frac{f}{l}\right)^2}} & n^3 = \dfrac{1}{\sqrt{1+16\left(\frac{f}{l}\right)^2}}
\end{array}
\tag{4.4}
$$

**Boundary conditions**

Solution to Eq. (4.3) will be assumed in the form satisfying the boundary conditions for a simply supported plate. For transverse vibration, boundary conditions on a mezo-scale have the form (cf. Fig. 3):

$$
\begin{gathered}
w = v_2 = m^{11} = 0 \qquad \text{for} \quad x_1 = 0, L \\
\left.\begin{aligned}
w &= n^2 u_2 + n^3 u_3 = 0 \\
v_2 &= C^2{}_2 u_2 + C^3{}_2 u_3 = 0
\end{aligned}\right\} \Rightarrow u_2 = u_3 = 0
\end{gathered}
\tag{4.5}
$$

where $u_2$, $u_3$ the folded plate displacements, and $m^{11}$ being the bending moment along these edge, are given by equations

$$
\begin{gathered}
\left.\begin{aligned}
u_2(\boldsymbol{x},t) &= U_2(\boldsymbol{x},t) + h(x_2)V_2(\boldsymbol{x},t) = 0 \\
u_3(\boldsymbol{x},t) &= U_3(\boldsymbol{x},t) + g(x_2)V_3(\boldsymbol{x},t) = 0
\end{aligned}\right\} \quad \text{for} \quad \left\{\begin{array}{l} \boldsymbol{x} = ((x_1 = 0, L), x_2) \\ t \geqslant 0 \end{array}\right. \\
\begin{aligned}
m^{11} = {} & BH^{1111}n^2U_{2,11} + BH^{1111}n^3U_{3,11} + BH^{1111}n^2hV_{2,11} + \\
& + BH^{1111}n^3gV_{3,11} + BH^{1122}n^2h_{,22}V_2 + BH^{1122}n^3g_{,22}V_3 = 0
\end{aligned}
\end{gathered}
\tag{4.6}
$$

The boundary conditions present above are valid for any functions $h(x_2)$, $g(x_2)$, $n^2(x_2)$, $n^3(x_2)$.

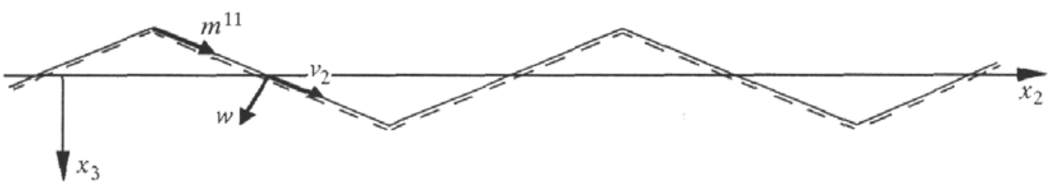


Fig. 3. Boundary conditions on the edges $x_1 = \text{const}$

Hence, we arrive at the following boundary conditions for the macrodisplacements $U_2$, $U_3$ and the disturbance variable $V_2$, $V_3$ for $x_1 = 0, L$

$$U_2 = U_3 = V_2 = V_3 = 0 \qquad U_{2,11} = U_{3,11} = 0 \qquad V_{2,11} = V_{3,11} = 0 \tag{4.7}$$

Functions satisfying the above conditions can be assumed as

$$\begin{aligned} U_2 &= A_2 \sin(kx_1)\cos(\omega_2 t) \qquad & U_3 &= A_3 \sin(kx_1)\cos(\omega_3 t) \\ V_2 &= C_2 \sin(kx_1)\cos(\omega_3 t) \qquad & V_3 &= C_3 \sin(kx_1)\cos(\omega_2 t) \end{aligned} \tag{4.8}$$

where $k := \pi/L$ is the wave number and $A_2$, $A_3$, $C_2$, $C_3$ are the corresponding amplitudes.

Substituting the right-hand sides of Eq. (4.8) into Eq. (4.3) we obtain for the out-of-plane vibrations non-trivial solutions only if

$$\begin{vmatrix} (\omega_3)^2\langle\widetilde{\rho}\rangle - C_{33} & C_{35} \\ C_{53} & (\omega_3)^2\langle\widetilde{\rho}hh\rangle - C_{55} \end{vmatrix} = 0 \tag{4.9}$$

where for a constant thickness $\delta$, with notation $\lambda := \delta/l$, $\gamma := kl$

$$\begin{aligned} C_{33} &= \frac{E}{\delta(1-\nu^2)}C'_{33} = \\ &= \frac{E}{\delta(1-\nu^2)}\Big[\frac{1}{12\sqrt{1+16(f/l)^2}}\lambda^4\gamma^4 + \frac{8(f/l)^2(1-\nu)}{\sqrt{1+16(f/l)^2}}\lambda^2\gamma^2\Big] \\ C_{35} &= C_{53} = \frac{E\delta}{1-\nu^2}C'_{35} = \\ &= \frac{-E\delta}{1-\nu^2}\frac{4f/l}{\sqrt{1+16(f/l)^2}}\Big[\frac{\lambda^2\gamma^4}{18} + \frac{32\nu\lambda^2\gamma^2}{12[1+16(f/l)^2]} - \frac{(1-\nu)\gamma^2}{3}\Big] \\ C_{55} &= \frac{E\delta^3}{1-\nu^2}C'_{55} = \frac{E\delta^3}{1-\nu^2}\frac{1}{\sqrt{1+16(f/l)^2}}\Big[0.711111\Big(\frac{f}{l}\Big)^2\gamma^4 + \\ &+ 56.88888\frac{(f/l)^2\gamma^2}{\sqrt{1+16(f/l)^2}} + 0.26666\frac{(1-\nu)\gamma^2}{\lambda^2} + \\ &+ 1365.333\frac{(f/l)^2}{\sqrt{1+16(f/l)^2}} + \frac{1}{12\lambda^2}\Big] \end{aligned} \tag{4.10}$$

From Eq. (4.9) we conclude that for the above form of vibrations we have two free vibration frequencies: a lower vibration frequency $\omega'_3$ and a higher one $\omega''_3$

$$(\omega_3')^2 = \frac{E}{\rho\delta(1-\nu^2)} \frac{\langle\widetilde{\rho}hh\rangle C'_{33} + \langle\widetilde{\rho}\rangle C'_{55}\lambda^4 - \sqrt{\Delta}}{2\langle\widetilde{\rho}\rangle\langle\widetilde{\rho}hh\rangle}$$
$$(\omega_3'')^2 = \frac{E}{\rho\delta(1-\nu^2)} \frac{\langle\widetilde{\rho}hh\rangle C'_{33} + \langle\widetilde{\rho}\rangle C'_{55}\lambda^4 + \sqrt{\Delta}}{2\langle\widetilde{\rho}\rangle\langle\widetilde{\rho}hh\rangle} \tag{4.11}$$

where

$$\Delta = (\langle\widetilde{\rho}hh\rangle C'_{33} + \langle\widetilde{\rho}\rangle C'_{55}\lambda^4)^2 - 4\langle\widetilde{\rho}\rangle\langle\widetilde{\rho}hh\rangle(C'_{33}C'_{55} - C'_{53}C'_{35})\lambda^4$$

Now we consider free vibrations of the uniperiodic folded plate in the framework of *the homogenized model* in which the structure of the folded plate is scaled down to $l \to 0$. Keeping in mind that $\delta/l =$ const, we shall neglect mezoinertial terms $\langle\widetilde{\rho}hh\rangle \to 0$, $\langle\widetilde{\rho}gg\rangle \to 0$ in the equations of motion. From Eq. (3.1) we obtain the governing equations in the form

$$\begin{aligned} &M^{i|\alpha\beta}{}_{;\alpha\beta} - N^{i|\alpha}{}_{;\alpha} + \langle\widetilde{\rho}\rangle\ddot{U}^i = 0 \\ &N^{\gamma} - K^{\gamma|\alpha}{}_{\|\alpha} + M^{\gamma} + P^{\gamma|\alpha\beta}{}_{\|\alpha\beta} - 2L^{\gamma|\alpha}{}_{\|\alpha} = 0 \\ &N^3 - K^{3|\alpha}{}_{\|\alpha} + M^3 + P^{3|\alpha\beta}{}_{\|\alpha\beta} - 2L^{3|\alpha}{}_{\|\alpha} = 0 \end{aligned} \tag{4.12}$$

The solution to the above equations can be assumed in the form (4.8). Now formula (4.9) leads to

$$\begin{vmatrix} (\omega_3)^2\langle\widetilde{\rho}\rangle - C_{33} & C_{35} \\ C_{53} & -C_{55} \end{vmatrix} = 0 \tag{4.13}$$

From Eq. (4.13) we can obtain the relation for lower vibration frequency

$$(\omega_3)^2 = \frac{E}{\rho\delta(1-\nu^2)}\Big[C'_{33} - \frac{(C'_{35})^2}{C'_{55}}\Big] \tag{4.14}$$

Figure 4 show the diagrams of the free vibration frequency $\omega$ versus the non-dimensional wave number $\gamma$, where the continuous line concerns the free vibration frequency for structural theory, while the dashed line describes the free vibration frequency for the homogenized model.

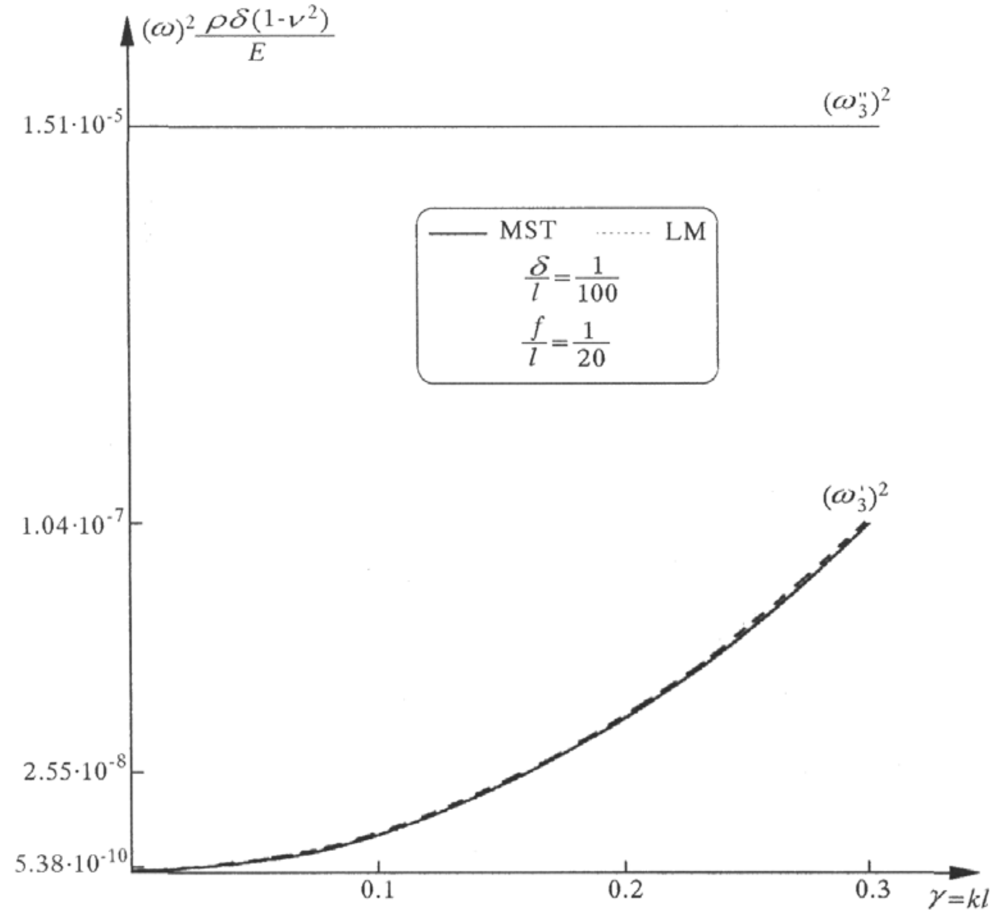


Fig. 4. Free vibration frequencies $\omega_3'$, $\omega_3''$ versus the non-dimensional wave number $\gamma = kl$

## 5. Conclusions

In this presentation, the dynamic behaviour of folded plate with one-dimensional periodic structure is analysed. This model, called the *uniperiodic folded plate* model, is represented by the system of differential equations for the macrodisplacements $\boldsymbol{U}$ and the disturbance variables $\boldsymbol{V}$, with coefficients which are functions of $x_1$-coordinate and are independent of $x_2$, the periodicity direction. The characteristic feature of these equations is that in the case of a rectangular folded plate, at the boundaries $x_1 = \text{const}$ the displacement boundary conditions have to be imposed on the complete displacement field $\boldsymbol{u}$. In this model, at the boundaries $x_1 = \text{const}$, the boundary conditions in the mezo-scale can be defined. In the typical approach for the refined models of periodic bodies which were presented in a series of papers (Baron and Woźniak, 1995; Wierzbicki, 1995; Jędrysiak, 1998; Michalak, 1998; Woźniak, 1999) and called internal variable models, the considered structures have periodic structure in both directions. In the framework of these models, the displacement boundary conditions have to be imposed only on the averaged part $\boldsymbol{U}$ of the displacement field. Hence, internal variable models are not sufficient for the analysis of *uniperiodic folded plate*, this plate cannot be treated as a specific case of those with a periodic structure in both directions.

Analysing the obtained results, it can be observed that this model makes it possible to investigate higher-order resonance vibration of folded plates which cannot be obtained within the framework of homogenized models, while the lower frequency coincides with those obtained from the homogenized model.

## Appendix

$$D^{i\alpha|j\beta} \equiv D\langle H^{\delta\alpha\gamma\beta} C^i{}_\delta C^j{}_\gamma \sqrt{a}\,\rangle$$

$$H^{i\alpha|\mu} = H^{\mu|i\alpha} \equiv D\langle H^{\delta\alpha\gamma\beta} C^i{}_\delta C^\mu{}_\gamma h_{|\beta} \sqrt{a}\,\rangle$$

$$H^{i\alpha|3} = H^{3|i\alpha} \equiv D\langle H^{\delta\alpha\gamma\beta} C^i{}_\delta C^3{}_\gamma g_{|\beta} \sqrt{a}\,\rangle$$

$$H^{\alpha|\mu} \equiv D\langle H^{\tau\beta\gamma\delta} C^\alpha{}_\tau C^\mu{}_\gamma h_{|\beta} h_{|\delta} \sqrt{a}\,\rangle$$

$$H^{\alpha|3} = H^{3|\alpha} \equiv D\langle H^{\tau\beta\gamma\delta} C^\alpha{}_\tau C^3{}_\gamma g_{|\delta} h_{|\beta} \sqrt{a}\,\rangle$$

$$H^{3|3} \equiv D\langle H^{\alpha\beta\gamma\delta} C^3{}_\alpha C^3{}_\gamma g_{|\beta} g_{|\delta} \sqrt{a}\,\rangle$$

$$H^{\tau\beta|\mu\delta} \equiv D\langle H^{\alpha\beta\gamma\delta} C^\tau{}_\alpha C^\mu{}_\gamma hh \sqrt{a}\,\rangle$$

$$H^{3\beta|3\delta} \equiv D\langle H^{\alpha\beta\gamma\delta} C^3{}_\alpha C^3{}_\gamma gg \sqrt{a}\,\rangle$$

$$H^{\tau\beta|3\delta} = H^{3\delta|\tau\beta} \equiv D\langle H^{\alpha\beta\gamma\delta} C^\tau{}_\alpha C^3{}_\gamma hg \sqrt{a}\,\rangle$$

$$B^{i\alpha\beta|\mu\gamma} = B^{\mu\gamma|i\alpha\beta} \equiv B\langle H^{\alpha\beta\gamma\delta} h_{|\delta} n^\mu n^i \sqrt{a}\,\rangle$$

$$B^{i\alpha\beta|\mu} = B^{\mu|i\alpha\beta} \equiv B\langle H^{\alpha\beta\gamma\delta} h_{|\gamma\delta} n^\mu n^i \sqrt{a}\,\rangle$$

$$B^{i\alpha\beta|3\gamma} = B^{3\gamma|i\alpha\beta} \equiv B\langle H^{\alpha\beta\gamma\delta} g_{|\delta} n^3 n^i \sqrt{a}\,\rangle$$

$$B^{i\alpha\beta|\mu\gamma\delta} = B^{\mu\gamma\delta|i\alpha\beta} \equiv B\langle H^{\alpha\beta\gamma\delta} hn^\mu n^i \sqrt{a}\,\rangle$$

$$B^{i\alpha\beta|3\gamma\delta} = B^{3\gamma\delta|i\alpha\beta} \equiv B\langle H^{\alpha\beta\gamma\delta} gn^3 n^i \sqrt{a}\,\rangle$$

$$B^{i\alpha\beta|3} = B^{3|i\alpha\beta} \equiv B\langle H^{\alpha\beta\gamma\delta} g_{|\gamma\delta} n^3 n^i \sqrt{a}\,\rangle$$

$$B^{\tau|\mu} \equiv B\langle H^{\alpha\beta\gamma\delta} h_{|\alpha\beta} h_{|\gamma\delta} n^\mu n^\tau \sqrt{a}\,\rangle$$

$$B^{\tau|3} = B^{3|\tau} \equiv B\langle H^{\alpha\beta\gamma\delta} h_{|\alpha\beta} g_{|\gamma\delta} n^3 n^\tau \sqrt{a}\,\rangle$$

$$B^{3|3} \equiv B\langle H^{\alpha\beta\gamma\delta} g_{|\alpha\beta} g_{|\gamma\delta} n^3 n^3 \sqrt{a}\,\rangle$$

$$A^{i\alpha\beta|j\gamma\delta} \equiv B\langle H^{\alpha\beta\gamma\delta} n^j n^i \sqrt{a}\,\rangle$$

$$C^{\tau\alpha\beta|\mu\gamma\delta} \equiv B\langle H^{\alpha\beta\gamma\delta} hh n^\mu n^\tau \sqrt{a}\,\rangle$$

$$C^{\tau\alpha\beta|3\gamma\delta} = C^{3\gamma\delta|\tau\alpha\beta} \equiv B\langle H^{\alpha\beta\gamma\delta} hg n^3 n^\tau \sqrt{a}\,\rangle$$

$$C^{\tau\alpha|\mu\gamma\delta} = C^{\mu\gamma\delta|\tau\alpha} \equiv B\langle H^{\alpha\beta\gamma\delta} h_{|\beta} h n^\mu n^\tau \sqrt{a}\,\rangle$$

$$C^{3\alpha\beta|3\gamma\delta} \equiv B\langle H^{\alpha\beta\gamma\delta} gg n^3 n^3 \sqrt{a}\,\rangle$$

$$C^{3\alpha|\mu\gamma\delta} = C^{\mu\gamma\delta|3\alpha} \equiv B\langle H^{\alpha\beta\gamma\delta} g_{|\beta} h n^\mu n^3 \sqrt{a}\,\rangle$$

$$C^{\tau|3\gamma} = C^{3\gamma|\tau} \equiv B\langle H^{\alpha\beta\gamma\delta} g_{|\delta} h_{|\alpha\beta} n^\tau n^3 \sqrt{a}\,\rangle$$

$$C^{\tau|3\gamma\delta} = C^{3\gamma\delta|\tau} \equiv B\langle H^{\alpha\beta\gamma\delta} g h_{|\alpha\beta} n^\tau n^3 \sqrt{a}\,\rangle$$

$$C^{3\alpha\beta|3\gamma} = C^{3\gamma|3\alpha\beta} \equiv B\langle H^{\alpha\beta\gamma\delta} g g_{|\delta} n^3 n^3 \sqrt{a}\,\rangle$$

$$C^{3\gamma\delta|3} = C^{3|3\gamma\delta} \equiv B\langle H^{\alpha\beta\gamma\delta} g g_{|\alpha\beta} n^3 n^3 \sqrt{a}\,\rangle$$

$$C^{\tau|\mu\gamma\delta} = C^{\mu\gamma\delta|\tau} \equiv B\langle H^{\alpha\beta\gamma\delta} h_{|\alpha\beta} h n^\mu n^\tau \sqrt{a}\,\rangle$$

$$C^{\tau|\mu\gamma} = C^{\mu\gamma|\tau} \equiv B\langle H^{\alpha\beta\gamma\delta} h_{|\alpha\beta} h_{|\delta} n^\mu n^\tau \sqrt{a}\,\rangle$$

$$C^{3|\mu\gamma\delta} = C^{\mu\gamma\delta|3} \equiv B\langle H^{\alpha\beta\gamma\delta} g_{|\alpha\beta} h n^\mu n^3 \sqrt{a}\,\rangle$$

$$C^{3|\mu\gamma} = C^{\mu\gamma|3} \equiv B\langle H^{\alpha\beta\gamma\delta} g_{|\alpha\beta} h_{|\delta} n^\mu n^3 \sqrt{a}\,\rangle$$

$$C^{3|3\gamma} = C^{3\gamma|3} \equiv B\langle H^{\alpha\beta\gamma\delta} g_{|\alpha\beta} g_{|\delta} n^3 n^3 \sqrt{a}\,\rangle$$

$$C^{\tau\alpha|3\gamma\delta} = C^{3\gamma\delta|\tau\alpha} \equiv B\langle H^{\alpha\beta\gamma\delta} h_{|\beta} g n^3 n^\tau \sqrt{a}\,\rangle$$

$$C^{\tau\alpha|\mu\gamma} \equiv B\langle H^{\alpha\beta\gamma\delta} h_{|\beta} h_{|\delta} n^\mu n^\tau \sqrt{a}\,\rangle$$

$$C^{\tau\alpha|3\gamma} = C^{3\gamma|\tau\alpha} \equiv B\langle H^{\alpha\beta\gamma\delta} h_{|\beta} g_{|\delta} n^3 n^\tau \sqrt{a}\,\rangle$$

$$C^{3\alpha|3\gamma} = C^{3\gamma|3\alpha} \equiv B\langle H^{\alpha\beta\gamma\delta} g_{|\beta} g_{|\delta} n^3 n^3 \sqrt{a}\,\rangle$$

$$G^{\tau\beta|\mu} = G^{\mu|\tau\beta} \equiv D\langle H^{\alpha\beta\gamma\delta} C^{\tau}{}_{\alpha} C^{\mu}{}_{\gamma} h_{|\delta} h\sqrt{a}\,\rangle$$

$$G^{i\beta|3\delta} = G^{3\delta|i\beta} \equiv D\langle H^{\alpha\beta\gamma\delta} C^{i}{}_{\alpha} C^{3}{}_{\gamma} g\sqrt{a}\,\rangle$$

$$G^{3\beta|\mu} = G^{\mu|3\beta} \equiv D\langle H^{\alpha\beta\gamma\delta} C^{3}{}_{\alpha} C^{\mu}{}_{\gamma} h_{|\delta} g\sqrt{a}\,\rangle$$

$$G^{i\beta|\mu\delta} = G^{\mu\delta|i\beta} \equiv D\langle H^{\alpha\beta\gamma\delta} C^{i}{}_{\alpha} C^{\mu}{}_{\gamma} h\sqrt{a}\,\rangle$$

$$G^{\tau\beta|3} = G^{3|\tau\beta} \equiv D\langle H^{\alpha\beta\gamma\delta} C^{\tau}{}_{\alpha} C^{3}{}_{\gamma} g_{|\delta} h\sqrt{a}\,\rangle$$

$$G^{3\beta|3} = G^{3|3\beta} \equiv D\langle H^{\alpha\beta\gamma\delta} C^{3}{}_{\alpha} C^{3}{}_{\gamma} g_{|\delta} g\sqrt{a}\,\rangle$$

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**O dynamicznym zachowaniu się jednokierunkowo periodycznych płyt tarczownicowych**

Streszczenie

Celem pracy jest otrzymanie uśrednionego modelu opisującego dynamiczne zachowanie się płyt jednokierunkowo periodycznych zbudowanych z tarczownicowych elementów powłokowych. Rozważania oparte są na metodzie uśredniania tolerancyjnego przedstawionej w pracy Woźniaka i Wierzbickiego (2000). Otrzymane równania uwzględniają efekt skali, co pozwala opisać zjawiska dyspersji. Przedstawiony model pokazuje, że dynamika płyty jednokierunkowo periodycznej nie może być traktowana jako szczególny przypadek płyty o strukturze dwukierunkowo periodycznej. Wadą większości istniejących uśrednionych modeli struktur periodycznych jest to, że pozwalają one na zapisanie warunków brzegowych tylko dla wartości uśrednionych. Proponowany model pozwala zapisać na brzegach prostopadłych do kierunku periodyczności warunki brzegowe dla pól całkowitych, a nie tylko dla ich uśrednionych części.

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