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# ON MODELLING OF MEDIUM THICKNESS PLATES WITH A UNIPERIODIC STRUCTURE

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The aim of this contribution is to propose a new averaged 2D-model of non-homogeneous Reissner-Mindlin elastic plates with one-directional periodic structure. So far, the averaged 2D-models of periodic plates were formulated on the basis of the known asymptotic homogenization theory in the framework of which the effect of repetive cell size on the overall plate behaviour is neglected. To remove this drawback the tolerance averaging of the plate equations was applied, cf. Woźniak and Wierzbicki (2000). It is shown that the aforementioned cell size effect plays an important role not only in dynamic problems (like for plates with two-directional periodic structure) but also in the quasi-stationary and stability problems. The obtained results are compared with those derived for the plates having a periodic structure in two directions as well as for the plates described in the framework of homogenization.

*Key words:* medium thickness plates, periodic structures, modelling, length-scale effect

## 1. Introduction

By the plate with a uniperiodic structure we mean a plate having a non-homogeneous material structure and/or variable thickness which are periodic (with the period $l$) only in one direction parallel to the plate midplane. In the perpendicular direction both non-homogeneous plate material properties as well as its thickness in a general case can be arbitrary, cf. Fig. 1. It is also assumed that the period $l$ is very small as compared to the smallest characteristic length dimension of the plate midplane and is large as compared to the maximum plate thickness. In the subsequent analysis it will be also assumed that the plate under consideration can be described, with a sufficient accuracy, in the framework of the Reissner-Mindlin elastic plate theory (the medium thickness plate theory). Hence, in the known equations of this theory we shall deal with functional coefficients, which are periodic and can be noncontinuous and highly oscillating with respect to one Cartesian coordinate (say, the $x_1$-coordinate), along which the plate structure is periodic. The above coefficients can also depend on the $x_2$-coordinate, as shown in Fig. 1, or to be independent of $x_2$, which takes place in many problems met in the engineering practice, cf. Fig. 2.

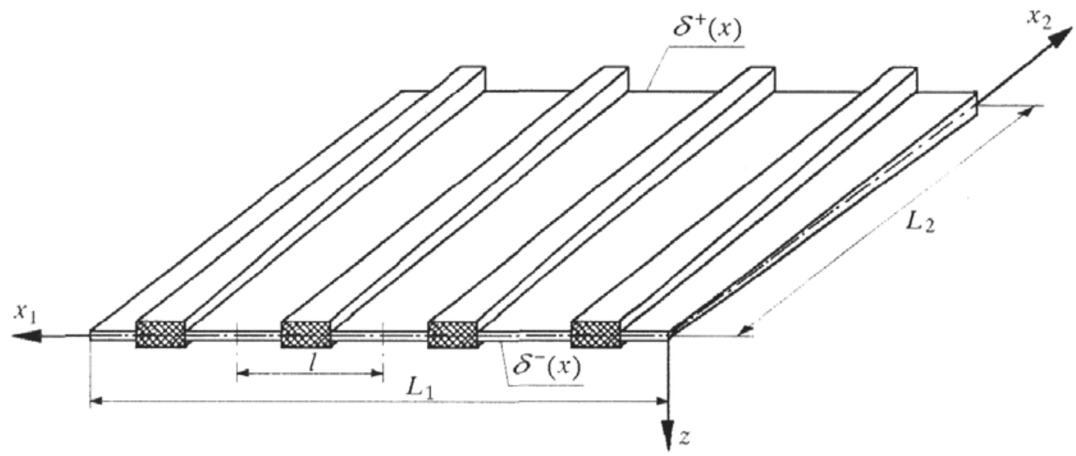


Fig. 1.

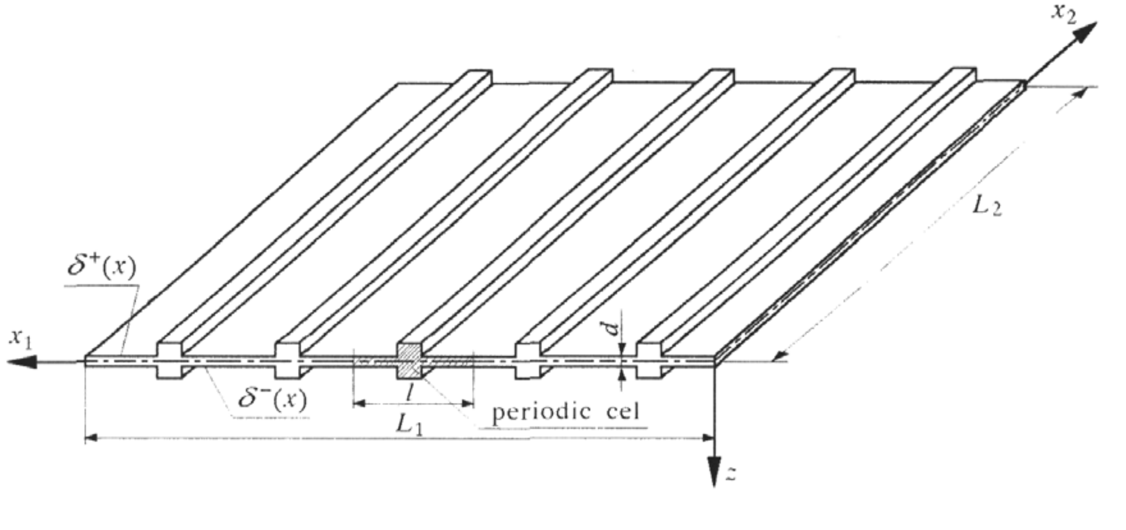


Fig. 2.

The direct application of the Reissner-Mindlin plate equations to the analysis of special problems of uniperiodic plates is rather difficult due to the periodic, possibly noncontinuous and highly oscillating form of the coefficients depending on the $x_1$-coordinate. Thus, the problem arises how to formulate an approximate 2D-model of the uniperiodic plate described by equations with a certain averaged coefficients which are independent of $x_1$. This problem can be solved by using the homogenization theory of partial differential equations with periodic coefficients, Bensoussan et al. (1980), Sanchez-Palencia (1980), Caillerie (1984), Kohn and Vogelius (1984); homogenized models of the Reissner-Mindlin plates were studied by Lewiński and Telega (2000) and Lewiński (1991, 1992). However, in order to apply the homogenization as a tool of modelling of periodic structures in mechanics, we have to introduce *the heuristic assumption that the overall behaviour of the periodic structure under consideration does not depend on the periodicity cell size.* On this assumption we can apply the limit passage with the cell length dimensions to zero and to assume that the homogenized equations derived from this passage describe the problem we deal with. On the other hand, in many physical problems we are interested how the periods of inhomogeneity influence the behaviour of a microperiodic structure on the macroscopic level. To answer this question we shall replace the homogenization by a more general modelling approach called the tolerance averaging of partial differential equations with periodic coefficients, cf. the book by Woźniak and Wierzbicki (2000). So far, this method was applied in mechanics of periodic composites, cf. Woźniak (1993a,b, 1995, 1999a,b), Wągrowska and Woźniak (1996), Wierzbicki et al. (1996, 2001), in the modelling of Kirchhoff's plates, Jędrysiak and Woźniak (1995) and Jędrysiak (2000a,b), and the wavy-type plates, Michalak et al. (1996) and Michalak (1998, 2000), as well as in the continuum modelling of lattice-type structures, Cielecka et al. (2000).

The main aim of this contribution is to adapt the method of tolerance averaging to the modelling of the medium thickness uniperiodic plates. It will be shown that the resulting averaged equations after the limit passage $l \to 0$ lead to the homogenized equations. Hence, from the point of wiew of mechanics, the tolerance averaging constitutes a certain generalization of the homogenization. In the general case, the form of obtained averaged equations for uniperiodic plates is different than that derived by the tolerance averaging of the Reissner-Mindlin equations for plates with a periodic structure in two directions, given by Baron and Woźniak (1995). It will be also shown that the effect of the periodicity length on the overall behaviour of the plate plays an important role and cannot be neglegted in many problems. In the forthcoming paper the obtained plate equations will be applied to the analysis of dynamic stability and they can also find applications to many other problems described within the framework of the Reissner-Mindlin theory of uniperiodic plates.

Throughout the paper the subscripts $\alpha, \beta, ...$ $(i, j, ...)$ run over $1, 2$ (over $1, 2, 3$). The superscripts $a, b, ...$ and $A, B$ run over $1, 2, ..., n$ and $1, 2, ..., N$, respectively. Summation convention holds for all aforementioned indices.

## 2. Formulation of the modelling problem

Let $0x_1x_2x_3$ be the orthogonal Cartesian coordinate system in the physical space; subsequently we shall also use the denotations $\boldsymbol{x} = (x_1, x_2)$, $z = x_3$. The region $\Omega$ occupied by the undeformed medium thickness plate under consideration will be given by $\Omega = \{(\boldsymbol{x}, z) : \delta^-(\boldsymbol{x}) < z < \delta^+(\boldsymbol{x})$ for almost every $\boldsymbol{x} \in \Pi\}$ where $\Pi = (0, L_1) \times (0, L_2)$ and $\delta^-(\boldsymbol{x}) < 0$, $\delta^+(\boldsymbol{x}) > 0$ are functions which have the period $l$ with respect to the coordinate $x_1$ such that $l \ll \min(L_1, L_2)$. We also denote $\delta(\boldsymbol{x}) = \delta^+(\boldsymbol{x}) - \delta^-(\boldsymbol{x})$ and assume $l \gg \max \delta(\boldsymbol{x})$, $\boldsymbol{x} \in \Pi$. The plate material is assumed to be elastic and the components $A_{ijkl}$ of the elastic modulae tensor as well as the mass density $\rho$ depend on $\boldsymbol{x}, z$ and are periodic functions (with the period $l$) with respect to the $x_1$ coordinate. It is also assumed that $z =$ const are elastic symmetry planes. Subsequently, we define

$$C_{\alpha\beta\gamma\delta} := A_{\alpha\beta\gamma\delta} - A_{\alpha\beta 33}A_{33\gamma\delta}(A_{3333})^{-1} \qquad B_{\alpha\beta} := A_{\alpha 3\beta 3}$$

and

$$\mu := \int_{\delta^-}^{\delta^+} \rho \, dz \qquad J := \int_{\delta^-}^{\delta^+} z^2 \rho \, dz$$

The plate is loaded in the direction of the $x_3$-axis by the surface loading $p^+(\boldsymbol{x})$, $p^-(\boldsymbol{x})$ applied to the boundaries $x_3 = \delta^+(\boldsymbol{x})$, $x_3 = \delta^-(\boldsymbol{x})$, respectively, and by the constant body force $b$. The problem under consideration within the framework of the elasticity theory is assumed to be governed by the strain-displacement relations

$$\varepsilon_{ij} = u_{(i,j)} + \frac{1}{2}u_{3,i}u_{3,j} \tag{2.1}$$

the stress-strain relations

$$\sigma_{\alpha\beta} = \overset{\circ}{\sigma}_{\alpha\beta} + C_{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta} \qquad \sigma_{\alpha 3} = 2B_{\alpha\beta}\varepsilon_{\beta 3} \tag{2.2}$$

where $\sigma^{\circ}_{\alpha\beta}$ is the know prestressing, and by the variational equation of motion

$$\int_{\Pi}\int_{\delta^-}^{\delta^+} \sigma_{ij}\overline{\varepsilon}_{ij}\, dzd\boldsymbol{x} + \int_{\Pi}\int_{\delta^-}^{\delta^+} \rho\ddot{u}_i\overline{u}_i\, dzd\boldsymbol{x} = \tag{2.3}$$
$$= \int_{\Pi}\int_{\delta^-}^{\delta^+} \rho b\overline{u}_3\, dzd\boldsymbol{x} + \int_{\Pi}\Big[p^+\overline{u}_3(\boldsymbol{x},\delta^+) + p^-\overline{u}_3(\boldsymbol{x},\delta^-)\Big]\, d\boldsymbol{x}$$

where $\varepsilon_{ij} = \overline{u}_{(i,j)} + u_{3,(i}\overline{u}_{3,j)}$, which holds for every virtual displacement field $\overline{u}_i$. We shall also take into account the Reissner-Mindlin kinematic assumption

$$u_\alpha(\boldsymbol{x},z,t) = zv_\alpha(\boldsymbol{x},t) \qquad\qquad u_3(\boldsymbol{x},z,t) = w(\boldsymbol{x},t) \tag{2.4}$$

where $w(\boldsymbol{x},t)$ is the deflection of the plate in the $x_3$-axis direction and $v_\alpha(\boldsymbol{x},t)$ are indepedent rotations. Combining (2.1)-(2.4) and denoting

$$G_{\alpha\beta\gamma\delta} := \int_{\delta^-}^{\delta^+} z^2 C_{\alpha\beta\gamma\delta}\, dz \qquad\qquad D_{\alpha\beta} := \int_{\delta^-}^{\delta^+} B_{\alpha\beta}\, dz$$
$$N^{\circ}_{\alpha\beta} := \int_{\delta^-}^{\delta^+} \sigma^{\circ}_{\alpha\beta}\, dz \qquad\qquad p = p^+ - p^- + b\mu$$

after linearisation and under assumption that $N_{\alpha\beta,\beta} = 0$, we obtain the following system of equations for $w$ and $\vartheta_\alpha$

$$\begin{aligned}&\big(G_{\alpha\beta\gamma\delta}\vartheta_{(\gamma,\delta)}\big)_{,\beta} - D_{\alpha\beta}\vartheta_\beta - J\ddot{\vartheta}_\alpha = 0\\ &N^{\circ}_{\alpha\beta}w_{,\alpha\beta} + \big[D_{\alpha\beta}(\vartheta_\beta + w_{,\beta})\big]_{,\alpha} - \mu\ddot{w} + p = 0\end{aligned} \tag{2.5}$$

representing the Reissner-Mindlin 2D-model of a medium thickness plate. The coefficients $G_{\alpha\beta\gamma\delta}$, $D_{\alpha\beta}$, $J$, $\mu$ in (2.5) are periodic functions of the argument $x_1$ with the period $l$; in the general case they can also depend on $x_2$. In the subsequent analysis, functions like these will be called *uniperiodic*. Hereafter, any function which depends only on $x_1$ and has the period $l$ will be referred to as periodic.

The modelling problem we are going to solve is to derive from (2.5) a system of equations with coefficients which are independent of $x_1$; moreover, some from these coeficients should depend on the period $l$. Bearing in mind the comments given in Section 1 the above modelling problem cannot be formulated within the framework of the homogenization and will be solved by using the tolerance averaging procedure. To make this paper self consistent in the subsequent section, following Woźniak and Wierzbicki (2000), we outline the basic concepts and assertions of this procedure.

## 3. Mathematical preliminaries

Let $\mathcal{F}(\Pi)$ be a system of functions defined in $\Pi$ (which can also depend on time $t$), which are treated as unknowns in the problem under consideration; it is assumed that to every $F \in \mathcal{F}(\Pi)$ there is assigned a certain unit measure. Moreover, let $\varepsilon(\cdot)$ be a mapping which assigns to every $F \in \mathcal{F}(\Pi)$ the positive number $\varepsilon_F = \varepsilon(F)$ which characterises a certain admissible accuracy related to calculations of values of the function $F(\cdot)$ or to the measurements of physical quantities (displacements, strains, stresses, etc) represented by $F(\cdot)$.

For any pair $\boldsymbol{x}$, $\boldsymbol{y}$ from the domain of $F(\cdot)$ we shall write $F(\boldsymbol{x}) \overset{\varepsilon}{\approx} F(\boldsymbol{y})$ if and only if $|F(\boldsymbol{x}) - F(\boldsymbol{y})| \leqslant \varepsilon_F$, it means that the values of $F(\cdot)$ at $\boldsymbol{x}$ and $\boldsymbol{y}$ can be treated as indiscernible in the problem under consideration. The constant $\varepsilon_F$ is called the tolerance parameter (related to the values of function $F$) and the symbol $\overset{\varepsilon}{\approx}$ represents a certain tolerance relation which is the reflexive and symmetric binary relation defined on the set $\boldsymbol{R}^2$ of real numbers endowed with a certain unit measure.

Let us denote $\Lambda = \{\boldsymbol{x} \in \boldsymbol{R}^2 : \ x_1 \in (-l/2, l/2), \ x_2 = 0\}$ and assume that there is known the triple $\mathcal{T} = \mathcal{F}(\Pi), \varepsilon(\cdot), \Lambda$. Let $DF$ stand for a certain function $F(\cdot) \in \mathcal{F}(\Pi)$ as well as for all its derivatives (including time derivatives) which occur in the problem under consideration and hence, also belong to $\mathcal{F}(\Pi)$. The function $F(\cdot)$ will be called *slowly varying* (with respect to $\mathcal{T}$), $F(\cdot) \in SV(\mathcal{T})$, if for every $x_1'$, $x_1''$ such that $|x_1' - x_1''| \leqslant 2l$ and every $x_2$ the condition $|DF(x_1', x_2) - DF(x_1'', x_2)| \leqslant \varepsilon_F$ holds in the whole domain of the definition of every $DF$.

Let use define $\Lambda(\boldsymbol{x}) = \boldsymbol{x} + \Lambda$ and $\Pi_\Lambda = \{\boldsymbol{x} \in \Pi : \ \Lambda(\boldsymbol{x}) \subset \Pi\}$. The function $\varphi(\cdot) \in \mathcal{F}(\Pi)$ will be called *periodic like* (with respect to $\mathcal{T}$), $\varphi(\cdot) \in PL(\mathcal{T})$, if for every $\boldsymbol{x} = (x_1, x_2) \in \Pi_\Lambda$ there exist a periodic function $\varphi_{\boldsymbol{x}}$ of $y_1$ such for every $(y_1, x_2) \in \Lambda(\boldsymbol{x})$ the condition $|\varphi_{\boldsymbol{x}}(y_1, x_2) - \varphi(y_1, x_2)| \leqslant \varepsilon_\varphi$ holds. It means that in every subset $\Lambda(\boldsymbol{x})$, $\boldsymbol{x} \in \Pi_\Lambda$ of $\Pi$, the function $\varphi$ can be approximated by a certain periodic function $\varphi_{\boldsymbol{x}}(\cdot)$ which will be called a uniperiodic approximation of $\varphi(\cdot)$ in $\Lambda(\boldsymbol{x})$.

We shall define the averaging operator $\langle\cdot\rangle$ setting

$$\langle f\rangle = \langle f\rangle(\boldsymbol{x}) = \frac{1}{l}\int\limits_{x_1-l/2}^{x_1+l/2} f(y_1,x_2)\,dy_1 \qquad \boldsymbol{x}=(x_1,x_2)\in \Pi_\Lambda$$

where $f(\cdot)$ is an arbitrary integrable function defined in $\Pi$, which can also depend on time $t$. If $f(\cdot)$ is uniperiodic then $\langle f\rangle(\cdot)$ is independent of $x_1$.

Let $\rho$ be an integrable positive valued uniperiodic function. The function $\psi(\cdot)\in\mathcal{F}(\Pi)$ will be called *oscillating* (with the weight $\rho$), $\psi(\cdot)\in PL^\rho(\mathcal{T})$ if $\psi(\cdot)\in PL(\mathcal{T})$ and $\langle\rho\psi\rangle(\boldsymbol{x})=0$ for every $\boldsymbol{x}\in\Pi_\Lambda$.

It can be proved that if $\varphi(\cdot)\in PL(\mathcal{T})$ then there exist the unique decomposition

$$\varphi(\cdot)=\varphi^\circ(\cdot)+\varphi^*(\cdot) \tag{3.1}$$

where $\varphi^\circ(\cdot)\in SV(\mathcal{T})$, and $\varphi^*(\cdot)\in PL^\rho(\mathcal{T})$ where $\varphi^\circ(\cdot)$, $\varphi^*(\cdot)$ will be referred to as the averaged and oscillatory parts of $\varphi(\cdot)$, respectively; here $\varphi^\circ(\boldsymbol{x})=\langle\rho\varphi\rangle(\boldsymbol{x})[\langle\rho\rangle(x_2)]^{-1}$.

If $f$ is an arbitrary integrable uniperiodic function, $F$ is a slowly varying function, $\varphi$ is a periodic-like function and $h(\cdot)$ is a differentiable uniperiodic function such that $\max\{|h(\boldsymbol{y})|:\ y_1\in\Lambda\}\leqslant l$, then it can be proved that the following relations hold

$$\begin{aligned}
&\langle fF\rangle(\boldsymbol{x}) \overset{\varepsilon}{\approx} \langle f\rangle F(\boldsymbol{x}) && \text{for}\quad \varepsilon=\langle|f|\rangle\varepsilon_F\\
&\langle f\varphi\rangle(\boldsymbol{x}) \overset{\varepsilon}{\approx} \langle f\varphi_x\rangle(\boldsymbol{x}) && \text{for}\quad \varepsilon=\langle|f|\rangle\varepsilon_\varphi\\
&\langle f(hF)_{,1}\rangle(\boldsymbol{x}) \overset{\varepsilon}{\approx} \langle fFh_{,1}\rangle(\boldsymbol{x}) && \text{for}\quad \varepsilon=\langle|f|\rangle(\varepsilon_F+l\varepsilon_{F,1})\\
&\langle f(h\varphi)_{,1}\rangle(\boldsymbol{x}) \overset{\varepsilon}{\approx} -\langle f\varphi h_{,1}\rangle(\boldsymbol{x}) && \text{for}\quad \varepsilon=\varepsilon_F+l\varepsilon_{F,1} \qquad F=\langle hf\varphi\rangle l^{-1}
\end{aligned} \tag{3.2}$$

For the detailed discussion related to the concept of tolerance and its applications the reader is referred to Woźniak and Wierzbicki (2000).

## 4. Modelling approach

The tolerance averaging of equations (2.5), leading to equations with coefficients which are independent of $x_1$, is based on two assumptions. The first is the heuristic *conformability assumption* which states that *the unknown kinematics fields* $w(\cdot)$, $\vartheta_\alpha(\cdot)$ in (2.5) *conform to the uniperiodic structure of the* *plate*. It means that these fields (together with their derivatives) are periodic-like functions: $w(\cdot) \in PL(\mathcal{T})$, $\vartheta_\alpha(\cdot) \in PL(\mathcal{T})$. The second assumption states that in the course of modelling the left-hand sides of relations (3.2) can be approximated by their right-hand sides; this assumption was referred by Woźniak and Wierzbicki (2000) to as *the tolerance averaging assumption.*

From the conformability assumption and (3.1) we obtain

$$\begin{aligned} \vartheta_\alpha(\cdot, x_2, t) &= \vartheta^\circ_\alpha(\cdot, x_2, t) + \vartheta^*_\alpha(\cdot, x_2, t) \\ w(\cdot, x_2, t) &= w^\circ(\cdot, x_2, t) + w^*(\cdot, x_2, t) \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} \vartheta^\circ_\alpha(\cdot, x_2, t) &= [\langle J\rangle(x_2)]^{-1}\langle J\vartheta_\alpha\rangle(\cdot, x_2, t) \\ w^\circ(\cdot, x_2, t) &= [\langle \mu\rangle(x_2)]^{-1}\langle \mu w\rangle(\cdot, x_2, t) \end{aligned} \tag{4.2}$$

are slowly varying functions and $\vartheta^*_\alpha(\cdot, x_2, t)$, $w^*(\cdot, x_2, t)$ are oscillating functions with the weights $J$, $\mu$, respectively. Subsequently, in order to simplify the calculations, in equations (2.5) we shall approximate the term $N^\circ_{\alpha\beta}w_{,\alpha\beta}$ by $N^\circ_{\alpha\beta}w^\circ_{,\alpha\beta}$ neglecting the effect of oscillations $w^*$ on the stability of the plate (this effect will be studied in a separated paper). Substituting the right-hand sides of (4.1) into (2.5), and averaging the resulting equations over $\Lambda(\boldsymbol{x})$, $\boldsymbol{x} \in \Pi_\Lambda$, after applying the tolerance averaging assumption we obtain *the averaged equation of motion*

$$\begin{aligned} & [\langle G_{\alpha\beta\gamma\delta}\rangle\vartheta^\circ_{(\gamma,\delta)} + \langle G_{\alpha\beta\gamma\delta}\vartheta^*_{(\gamma,\delta)}\rangle]_{,\beta} - \langle D_{\alpha\beta}\rangle(\vartheta^\circ_\beta + w^\circ_{,\beta}) - \\ & -\langle D_{\alpha\beta}(\vartheta^*_\beta + w^*_{,\beta})\rangle - \langle J\rangle\ddot{\vartheta}^\circ_\alpha = 0 \\ & N^\circ_{\alpha\beta}w^\circ_{,\alpha\beta} + [\langle D_{\alpha\beta}\rangle(\vartheta^\circ_\beta + w^\circ_{,\beta})]_{,\alpha} + [\langle D_{\alpha\beta}(\vartheta^*_\beta + w^*_{,\beta})\rangle]_{,\alpha} - \langle\mu\rangle\ddot{w}^\circ + \langle p\rangle = 0 \end{aligned} \tag{4.3}$$

Now, let us restrict the domain of definition of functions in equations (2.5) to an arbitrary but fixed interval $\Lambda(\boldsymbol{x})$, $\boldsymbol{x} \in \Pi_\Lambda$. Multiplying these equations by the periodic test functions $v_\alpha(\cdot)$ and $\eta(\cdot)$ of the argument $y_1$, taking into account decomposition (4.1) and using the tolerance averaging assumption, after some transformations we formulate the following *periodic cell problem*: for the fixed $\boldsymbol{x} = (x_1, x_2) \in \Pi_\Lambda$ find the uniperiodic functions $\vartheta^*_{\boldsymbol{x}\alpha}(\cdot, x_2, t)$, $w^*_{\boldsymbol{x}}(\cdot, x_2, t)$ of the argument $y_1 \in [x_1 - l/2, x_1 + l/2]$, satisfying the conditions $\langle J\vartheta^*_{\boldsymbol{x}\alpha}\rangle(\boldsymbol{x}) = 0$, $\langle \mu w^*_{\boldsymbol{x}}\rangle(\boldsymbol{x}) = 0$ and the variational equation

$$
\begin{aligned}
&\langle v_{\alpha,1} G_{\alpha 1\gamma\delta} \vartheta^*_{\boldsymbol{x}(\gamma,\delta)}\rangle - \langle v_\alpha (G_{\alpha 2\gamma\delta}\vartheta^*_{\boldsymbol{x}(\gamma,\delta)})_{,2}\rangle + \\
&+\langle v_\alpha D_{\alpha\beta}(\vartheta^*_{\boldsymbol{x}\beta} + w^*_{\boldsymbol{x},\beta})\rangle + \langle v_\alpha J \ddot{\vartheta}^*_{\boldsymbol{x}\alpha}\rangle = \\
&= -\langle v_{\alpha,1} G_{\alpha 1\gamma\delta}\rangle \vartheta^\circ_{(\gamma,\delta)} + \langle v_\alpha (G_{\alpha 2\gamma\delta}\rangle \vartheta^\circ_{(\gamma,\delta)})_{,2} - \langle v_\alpha D_{\alpha\beta}\rangle(\vartheta^\circ_\beta + w^\circ_{,\beta}) \\
&\langle \eta_{,1} D_{1\beta}(\vartheta^*_{\boldsymbol{x}\beta} + w^*_{\boldsymbol{x},\beta})\rangle - \langle \eta [D_{2\beta}(\vartheta^*_{\boldsymbol{x}\beta} + w^*_{\boldsymbol{x},\beta})]_{,2}\rangle + \langle \eta\mu \ddot{w}^*_{\boldsymbol{x}}\rangle = \\
&= N^\circ_{\alpha\beta}\langle \eta\rangle w^\circ_{,\alpha\beta} - \langle \eta_{,1} D_{1\beta}\rangle(\vartheta^\circ_\beta + w^\circ_{,\beta}) + \langle \eta [D_{2\beta}\rangle(\vartheta^\circ_\beta + w^\circ_{,\beta})]_{,2} + \langle \eta p\rangle
\end{aligned}
\tag{4.4}
$$

which holds for arbitrary periodic functions $v_\alpha$, $\eta$ of $y_1$ such that $\langle J v_\alpha\rangle(\boldsymbol{x}) = 0$, $\langle \mu\eta\rangle(\boldsymbol{x}) = 0$. It has to be emphasized that in (4.4) we deal with values of the functions $\vartheta^\circ_\alpha$, $w^\circ$ and their derivatives at the point $\boldsymbol{x} = (x_1, x_2)$, where $x_1$, $x_2$ are treated as parameters.

Averaged equation of motion (4.3) and the periodic cell problem related to variational equation (4.4) constitute the fundamentals of the applied modelling approach.

The approximate solution to the periodic cell problem will be obtained by the discretization of the cell $[x_1 - l/2, x_1 + l/2]$ and applying the Galerkin approximation. To this end let us introduce two systems of the linear independent periodic shape functions $h^a(y_1)$, $a = 1, ..., n$ and $g^A(y_1)$, $A = 1, ..., N$, such that $\langle J h^a\rangle(\boldsymbol{x}) = 0$, $\langle \mu g^A\rangle(\boldsymbol{x}) = 0$; in the special case the above systems of functions can coincide. In the simplest case the shape functions can be assumed as continuous linear ones similar to those applied in the one-dimensional finite element method but satisfying the aforementioned conditions. For the detailed discussion of the cell problem the reader is referred to Woźniak and Wierzbicki (2000). The approximate solution to the periodic cell problem will be assumed in the form of the finite sums

$$
\begin{aligned}
\vartheta^*_{\boldsymbol{x}\alpha}(y_1, x_2, t) &\cong h^a(y_1)\Theta^a_\alpha(\boldsymbol{x}, t) \\
w^*_{\boldsymbol{x}}(y_1, x_2, t) &\cong g^A(y_1) W^A(\boldsymbol{x}, t)
\end{aligned}
\tag{4.5}
$$

where $y_1 \in [x_1 - l/2, x_1 + l/2]$, $\boldsymbol{x} = (x_1, x_2) \in \Pi_\Lambda$, and $\Theta^a_\alpha(\boldsymbol{x}, t)$, $W^A(\boldsymbol{x}, t)$ are the new unknowns and the approximation $\cong$ depends on the number of terms on the right-hand sides of (4.5). It follows that for the unknowns $\vartheta^*_\alpha(\cdot)$, $w^*(\cdot)$ we obtain the approximation formulae

$$
\begin{aligned}
\vartheta^*_\alpha(\boldsymbol{x}, t) &\cong h^a(x_1)\Theta^a_\alpha(\boldsymbol{x}, t) \\
w^*(\boldsymbol{x}, t) &\cong g^A(x_1) W^A(\boldsymbol{x}, t)
\end{aligned}
\tag{4.6}
$$

only if $\Theta_\alpha^a(\cdot, x_2, t)$ and $W^A(\cdot, x_2, t)$ are slowly varying functions

$$\Theta_\alpha^a(\cdot, x_2, t),\ W^A(\cdot, x_2, t) \in SV(\mathcal{T}) \tag{4.7}$$

Substituting the right-hand sides of (4.5) into (4.4), using (4.7) and assuming that $v_\alpha = h^a(y_1)\overline{\Theta}_\alpha^a$, $\eta = g^A(y_1)\overline{W}^A$, where $\overline{\Theta}_\alpha^a$, $\overline{W}^A$ are arbitrary constants, after some transformations we obtain equations for the unknowns $\Theta_\alpha^a(\cdot)$, $W^A(\cdot)$, which also depend on the unknowns $\vartheta_\alpha^\circ(\cdot)$ and $w^\circ(\cdot)$. Similarly, substituting the right-hand sides of (4.6) into (4.3) by means of (4.7) we obtain equations involving $\vartheta_\alpha^\circ(\cdot)$, $w^\circ(\cdot)$ and $\Theta_\alpha^a(\cdot)$, $W^A(\cdot)$ as the basic unknowns. Obviously, in both cases the tolerance averaging assumption has to be taken into account. Assuming that $h^a(x_1) \in O(l)$, $g^A(x_1) \in O(l)$ we shall also introduce the functions

$$\overline{h}^a = l^{-1}h^a \qquad\qquad \overline{g}^A = l^{-1}g^A$$

the values of which can be treated as independent of the period $l$. Setting aside rather lengthy calculations, we can prove that the resulting system of equations can be represented by the equations of motion

$$\begin{aligned} &M_{\alpha\beta,\beta} - Q_\alpha - \langle J\rangle\ddot{\vartheta}_\alpha^\circ = 0 \\ &N_{\alpha\beta}^\circ w_{,\alpha\beta}^\circ + Q_{\alpha,\alpha} - \langle\mu\rangle\ddot{w}^\circ + p = 0 \end{aligned} \tag{4.8}$$

by the following system of equations for $\Theta^a$, $W^A$

$$\begin{aligned} &l^2\langle J\overline{h}^a\overline{h}^b\rangle\ddot{\Theta}_\alpha^b + M_\alpha^a - l\widetilde{M}_{\alpha,2}^a = 0 \\ &l^2\langle\mu\overline{g}^A\overline{g}^B\rangle\ddot{W}^B + Q^A - l\widetilde{Q}_{,2}^A - lN_{\alpha\beta}^\circ\langle\overline{g}^A\rangle w_{,\alpha\beta}^\circ - l\langle\overline{g}^A p\rangle = 0 \end{aligned} \tag{4.9}$$

and by the constitutive equations

$$\begin{aligned} M_{\alpha\beta} &= \langle G_{\alpha\beta\gamma\delta}\rangle\vartheta_{(\gamma,\delta)}^\circ + \langle h_{,1}^a G_{\alpha\beta 1\delta}\rangle\Theta_\delta^a + l\langle\overline{h}^a G_{\alpha\beta 2\delta}\rangle\Theta_{\delta,2}^a \\ Q_\alpha &= \langle D_{\alpha\beta}\rangle(\vartheta_\beta^\circ + w_{,\beta}^\circ) + l\langle\overline{h}^a D_{\alpha\beta}\rangle\Theta_\beta^a + \langle g_{,l}^A D_{\alpha 1}\rangle W^A + l\langle\overline{g}^A D_{\alpha 2}\rangle W_{,2}^A \\ M_\alpha^a &= \langle h_{,1}^a h_{,1}^b G_{\alpha 11\delta}\rangle\Theta_\delta^b + \langle h_{,1} G_{\alpha 1\gamma\delta}\rangle\vartheta_{(\gamma,\delta)}^\circ + l\langle h_{,1}^a\overline{h}^b G_{\alpha 12\delta}\rangle\Theta_{\delta,2}^b + \\ &+ l^2\langle\overline{h}^a\overline{h}^b D_{\alpha\beta}\rangle\Theta_\beta^b + l\langle\overline{h}^a D_{\alpha\beta}\rangle(\vartheta_\beta^\circ + w_{,\beta}^\circ) + \\ &+ l\langle\overline{h}^a g_{,1}^A D_{\alpha 1}\rangle W^A + l^2\langle\overline{h}^a\overline{g}^A D_{\alpha 2}\rangle W_{,2}^A \qquad\qquad (4.10) \\ \widetilde{M}_\alpha^a &= \langle\overline{h}^a h_{,1}^b G_{\alpha 21\delta}\rangle\Theta_\delta^b + \langle\overline{h}^a G_{\alpha 2\gamma\delta}\rangle\vartheta_{(\gamma,\delta)}^\circ + l\langle\overline{h}^a\overline{h}^b G_{\beta 22\delta}\rangle\Theta_{\delta,2}^b \end{aligned}$$

$$
\begin{aligned}
Q^A &= \langle g^A_{,1} g^B_{,1} D_{11}\rangle W^B + \langle g^A_{,1} D_{1\beta}\rangle(\vartheta^\circ_\beta + w^\circ_{,\beta}) + l\langle g^A_{,1}\overline{h}^a D_{1\beta}\rangle \Theta^a_\beta + \\
&+ \; l\langle g^A_{,1}\overline{g}^B D_{12}\rangle W^B_{,2} \\
\widetilde{Q}^A &= \langle \overline{g}^A g^B_{,1} D_{21}\rangle W^B + \langle \overline{g}^A D_{2\beta}\rangle(\vartheta^\circ_\beta + w^\circ_{,\beta}) + l\langle \overline{g}^A h^a D_{2\beta}\rangle \Theta^a_\beta + \\
&+ \; l\langle \overline{g}^A \overline{g}^B D_{22}\rangle W^B_{,2}
\end{aligned}
$$

Equations (4.8)-(4.10) have the physical sense only if the basic unknowns are slowly varying functions

$$
\Theta^\circ_\alpha(\cdot, x_2, t),\; w^\circ(\cdot, x_2, t),\; \Theta^a_\alpha(\cdot, x_2, t),\; W^A(\cdot, x_2, t) \in SV(\mathcal{T}) \tag{4.11}
$$

Equations (4.8)-(4.11) together with formulae (4.1), (4.6) constitute the proposed averaged 2D-model of a uniperiodic plate. It can be seen that the coefficients in (4.8)-(4.10) are independent of the $x_1$-coordinate; at the same time the dependence of some from these coefficients on the period $l$ is shown in the explicit form. It means that (4.8)-(4.11) represent the averaged model of the uniperiodic plate under consideration which depends on $l$, and hence the modelling problem formulated at the end of Section 2 has been solved. If the functions $G_{\alpha\beta\gamma\delta}(\cdot)$, $D_{\alpha\beta}(\cdot)$, $\delta^-(\cdot)$, $\delta^+(\cdot)$ describing the plate under consideration are independent of the $x_2$-coordinate then the obtained model is governed by the equations with constant coefficients.

Comparing equations (4.8)-(4.10) for the uniperiodic plates with the equations derived by Baron and Woźniak (1995) for the plates with periodic structure with respect to the coordinates $x_1$ and $x_2$, it can be easily seen that the averaged equations for the uniperiodic plates are more complicated. It follows from the fact that the definitions of the slowly varying and periodic-like functions introduced for the modelling of uniperiodic structures are less restrictive than those introduced for the modelling of structures which are periodic in two directions, cf. Woźniak and Wierzbicki (2000).

At the end of this section it has to be emphasized that the definition of a slowly varying function depends on the mapping $\varepsilon(\cdot)$, and hence on the choice of the tolerance parameters assigned to the calculation of values of the unknowns $\vartheta^\circ_\alpha$, $w^\circ$, $\Theta^a_\alpha$, $W^A$ and their derivatives.

After obtaining the solution to the initial-boundary value problem for equations (4.8)-(4.10) the tolerance parameters can be determined directly from (4.11). It means that conditions (4.11) can be treated as certain *a posteriori* estimates of the solutions to the problems described by averaged equations (4.8)-(4.10). It also means that the heuristic conformability assumption introduced in Section 4 can be verified (within a certain tolerance) in every problem under consideration.

## 5. Asymptotic model

By the asymptotic 2D-model of a uniperiodic plate under consideration we shall mean the model derived from (4.8)-(4.10) by the formal limit passage $l \to 0$. In this case, from (4.9) and (4.10) we obtain the following system of linear algebraic equations for the functions $\Theta^a_\alpha(\cdot)$, $W^A$

$$\begin{aligned} &\langle h^a_{,1} h^b_{,1} G_{\alpha 1 1 \delta}\rangle \Theta^b_\delta + \langle h^a_{,1} G_{\alpha 1 \gamma\delta}\rangle \vartheta^\circ_{(\gamma,\delta)} = 0 \\ &\langle g^A_{,1} g^B_{,1} D_{11}\rangle W^B + \langle g^A_{,1} D_{1\beta}\rangle (\vartheta^\circ_\beta + w^\circ_{,\beta}) = 0 \end{aligned} \tag{5.1}$$

It can be shown that the linear mappings determined by $\langle h^a_{,1} h^b_{,1} G_{\alpha 1 1 \delta}\rangle$ and $\langle g^A_{,1} g^B_{,1} D_{11}\rangle$ are invertible and hence the solutions to (5.1) are unique and can be written in the form

$$\begin{aligned} \Theta^a_\alpha &= -K^{ab}_{\alpha\beta} \langle h^b_{,1} G_{\beta 1 \gamma\delta}\rangle \vartheta^\circ_{(\gamma,\delta)} \\ W^A &= -L^{AB} \langle g^B_{,1} D_{1\beta}\rangle (\vartheta^\circ_\beta + w^\circ_{,\beta}) \end{aligned} \tag{5.2}$$

where $K^{ab}_{\alpha\beta}$, $L^{AB}$ represent the corresponding inverse mappings. Thus, we conclude that the unknowns $\Theta^a_\alpha$, $W^A$ can be eliminated from the model equations, and after the denotations

$$\begin{aligned} \widetilde{G}_{\alpha\beta\gamma\delta} &= \langle G_{\alpha\beta\gamma\delta}\rangle - \langle h^a_{,1} G_{\alpha\beta 1\nu}\rangle K^{ab}_{\nu\rho} \langle h^b_{,1} G_{\rho 1 \gamma\delta}\rangle \\ \widetilde{D}_{\alpha\beta} &= \langle D_{\alpha\beta}\rangle - \langle g^A_{,1} D_{\alpha 1}\rangle L^{AB} \langle g^B_{,1} D_{1\beta}\rangle \end{aligned} \tag{5.3}$$

we obtain the governing equations of the asymptotic model in the form of the equations of motion

$$\begin{aligned} &M_{\alpha\beta,\beta} - Q_\alpha - \langle J\rangle \ddot{\vartheta}^\circ_\alpha = 0 \\ &N^\circ_{\alpha\beta} w^\circ_{,\alpha\beta} + Q_{\alpha,\alpha} - \langle \mu\rangle \ddot{w}^\circ + p = 0 \end{aligned} \tag{5.4}$$

and the constitutive equations

$$\begin{aligned} M_{\alpha\beta} &= \widetilde{G}_{\alpha\beta\gamma\delta} \vartheta^\circ_{(\gamma,\delta)} \\ Q_\alpha &= \widetilde{D}_{\alpha\beta} (\vartheta^\circ_\beta + w^\circ_{,\beta}) \end{aligned} \tag{5.5}$$

If the functions describing the material properties and geometry of the plate are independent of the $x_2$-coordinate (i.e. they are periodic functions of $x_1$) then material modulae (5.3) are constant. In this special case $\widetilde{G}_{\alpha\beta\gamma\delta}$, $\widetilde{D}_{\alpha\beta}$ represent a certain approximation of the well known effective modulae of the homogenization theory. The aforementioned approximation is caused by the approximate form of solutions (4.5) to the periodic cell problem; this cell problem in the framework of the asymptotic approximation $l \to 0$ reduces to the periodic cell problem of the homogenization theory provided that the plate properties depend only on the $x_1$-coordinate.

The advantage of the asymptotic model is the relatively simple form of equations (5.4), (5.5) in contrast to general equations (4.8)-(4.10) derived by the tolerance averaging approach. However, in the framework of the asymptotic model we are not able to investigate the effect of the periodicity cell size $l$ on the overall behaviour of the plate. Moreover, for the homogenized model the initial and boundary conditions can be imposed only on the averaged fields $\vartheta^{\circ}_{\alpha}$, $w^{\circ}$ in contrast to the model derived in this contribution where the initial conditions can be imposed also on the oscillations $\vartheta^{*}_{\alpha}$, $w^{*}$ by means of formulae (4.5). It can be shown that by using the proposed model the boundary conditions for the oscillations $\vartheta^{*}_{\alpha}$, $w^{*}$ can be formulated on the boundaries $x_2 = 0$, $x_2 = L_2$.

## 6. Conclusions

An averaged 2D-model of a nonhomogeneous medium thickness elastic plate with a uniperiodic structure has been obtained using the tolerance averaging procedure and is given by:

- equations (4.8)-(4.10) for the unknown functions $\vartheta^{\circ}_{\alpha}(\cdot)$, $w^{\circ}(\cdot)$, $\Theta^{a}_{\alpha}(\cdot)$, $W^{A}(\cdot)$,
- physical applicability conditions (4.11), on the basis of which the tolerance parameters $\varepsilon(\cdot)$ related $\vartheta^{\circ}_{\alpha}(\cdot)$, $w^{\circ}(\cdot)$, $\Theta^{a}_{\alpha}(\cdot)$, $W^{A}(\cdot)$, and their derivatives can be calculated,
- approximate relations (4.1), (4.6) for the displacements $\vartheta_{\alpha}(\cdot)$ and $w(\cdot)$ of the Reissner-Mindlin plate.

Equations (4.8)-(4.10) cannot be obtained as a special case of equations derived by Baron and Woźniak (1995) for the medium thickness plates with two-directional periodic structure. The equations for the uniperiodic plates are more complicated. It follows from the fact that the conditions for the modelling of the uniperiodic plates are less restrictive than those introduced for the modelling of the plates with two-directional periodic structure.

The main feature of the proposed model of the medium thickness uniperiodic plate is that it describes the effect of the repetitive cell size $l$ on the plate behaviour.

Comparing equations (4.8)-(4.10) with equations of the asymptotic model (5.4) and (5.5) it can be easily seen that the tolerance averaging model enables analysing a large class of problems. In the framework of the asymptotic model we are not able to investigate the effect of the repetive cell size on the plate behaviour. For this model the initial and boundary conditions can be imposed only on the averaged fields $\vartheta_\alpha^\circ(\cdot)$, $w^\circ(\cdot)$ in contrast to the model derived in this contribution where the initial conditions can be imposed also on the oscillations $\vartheta_\alpha^*(\cdot)$ and $w^*(\cdot)$.

Examples of applications of the proposed model to some dynamic stability problems will be given in a forthcoming paper.

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**Modelowanie średniej grubości płyt o strukturze uniperiodycznej**

Streszczenie

Celem pracy jest przedstawienie nowego, uśrednionego dwuwymiarowego modelu niejednorodnych, średniej grubości (wg hipotezy Reissnera-Mindlina) liniowosprężystych płyt o jednokierunkowej strukturze periodycznej. Dotychczas zagadnienia płyt tego typu były najczęściej rozwiązywane metodą homogenizacji asymptotycznej. Metoda ta pomija jednak wpływ powtarzalnego segmentu płyty na jej makromechaniczne własności. Dlatego też zastosowano metodę uśredniania tolerancyjnego równań płyty, która ten wpływ uwzględnia, a opisana jest np. przez Woźniaka i Wierzbickiego (2000). Wpływ wymiaru powtarzalnego segmentu płyty odgrywa istotną rolę nie tylko w dynamice, ale również w niektórych zagadnieniach quasi-stacjonarnych i zagadnieniach stateczności. Uzyskane równania porównano z równaniami wyprowadzonymi tą samą metodą modelowania dla płyt o średniej grubości o dwukierunkowej strukturze periodycznej oraz z modelem zhomogenizowanym asymptotycznie.

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