

ON 3D PUNCH PROBLEMS FOR A PERIODIC TWO-LAYERED ELASTIC HALF-SPACE

ANDRZEJ KACZYŃSKI

Faculty of Mathematics and Information Science, Warsaw University of Technology
e-mail: akacz@alpha.mini.pw.edu.pl

STANISŁAW J. MATYSIAK

Faculty of Geology, University of Warsaw
e-mail: matysiak@geo.uw.edu.pl

Within the framework of the linear elasticity with microlocal parameters three-dimensional contact problems for a half-space region consisting of alternating layers of two homogeneous, isotropic and linear-elastic materials are examined. Effective results can be obtained on the basis of similarity in governing equations for the homogenized model of the laminated medium and transversely isotropic elastic solid.

Key words: periodic two-layered half-space, rigid punch, integral equation

1. Introduction

Considerable progress has been made with the modelling and analysis of contact problems. Extensive accounts can be found in the books by Shtaerman (1949), Galin (1953, 1980), Rvachev and Protsenko (1977), Gladwell (1980), Johnson (1985), Mossakovskii et al. (1985), Goryacheva and Dobykhin (1988) and in the recent proceedings by Raous et al. (1995). Willis (1966) and Hwu and Fan (1998) made significant contributions to the development of research for the contact of anisotropic bodies. Complete and new solutions to several three-dimensional contact problems were presented in two monographs by Fabrikant (1989, 1991).

This paper is devoted to the analysis of three-dimensional contact problems for a periodic two-layered elastic half-space. It is a sequel of our earlier investigations in the two-dimensional case (see Kaczyński and Matysiak, 1988, 1993).

The study is based on the use of the homogenized model of the microperiodic two-layered composite, proposed by Woźniak (1987), Matysiak and Woźniak (1988). In Section 2 we review briefly the governing equations of this model in the three-dimensional case of the linear elasticity with microlocal parameters. Due to close similarity to the fundamental equations for a transversely isotropic elastic solid, the general solutions in terms of harmonic potentials, well suited to contact problems, are constructed. Section 3 presents a general elastostatic contact problem of indentation of the two-layered half-space by a frictionless rigid smooth punch treated within the homogenized model. According to the analysis, the well-known governing integral equation of the elastic contact problem similar to that in the case of contact on a transversely isotropic half-space is obtained. This fact may be utilised to yield closed-form solutions following directly from those given, for example, by Fabrikant (1989). The complete solution in the case of a flat centrally loaded circular punch is presented in Section 4.

2. Governing equations

We consider a three-dimensional static contact problem of a two-layered microperiodic elastic half-space with a rigid smooth punch (see Fig. 1). Let λ_l, μ_l be Lamé's constants and δ_l be the thicknesses of the subsequent layers; in the following, all the quantities (material constants, stresses, etc.) with the index l or (l) are related to the layers denoted by $l = 1$ or $l = 2$. The Cartesian coordinate system (x_1, x_2, x_3) is devised with the x_3 -axis normal to the layering and the x_1x_2 -plane of boundary. Referring to this system, denote at the point $\mathbf{x} = (x_1, x_2, x_3)$ the displacement vector by $\mathbf{u} = [u_1, u_2, u_3]$ and the stresses by $\sigma_{11}, \sigma_{12}, \sigma_{22}, \sigma_{13}, \sigma_{23}, \sigma_{33}$.

To analyse the problem of a punch penetrating this layered body we take into consideration the specific homogenization procedure called microlocal modelling, proposed by Woźniak (1987) and then developed by Matysiak and Woźniak (1988), applicable to a certain macro-homogeneous model of the treated body. We present only a brief outline of its governing equations.

The homogenized model of the layered body under study is characterised by the shape function with the period $\delta = \delta_1 + \delta_2$, defined as

$$h(x_3) = \begin{cases} x_3 - \frac{\delta_1}{2} & \text{for } x_3 \in \langle 0, \delta_1 \rangle \\ \frac{\delta_1 - \eta x_3}{1 - \eta} - \frac{\delta_1}{2} & \text{for } x_3 \in \langle \delta_1, \delta \rangle \quad \eta = \frac{\delta_1}{\delta} \end{cases} \quad (2.1)$$

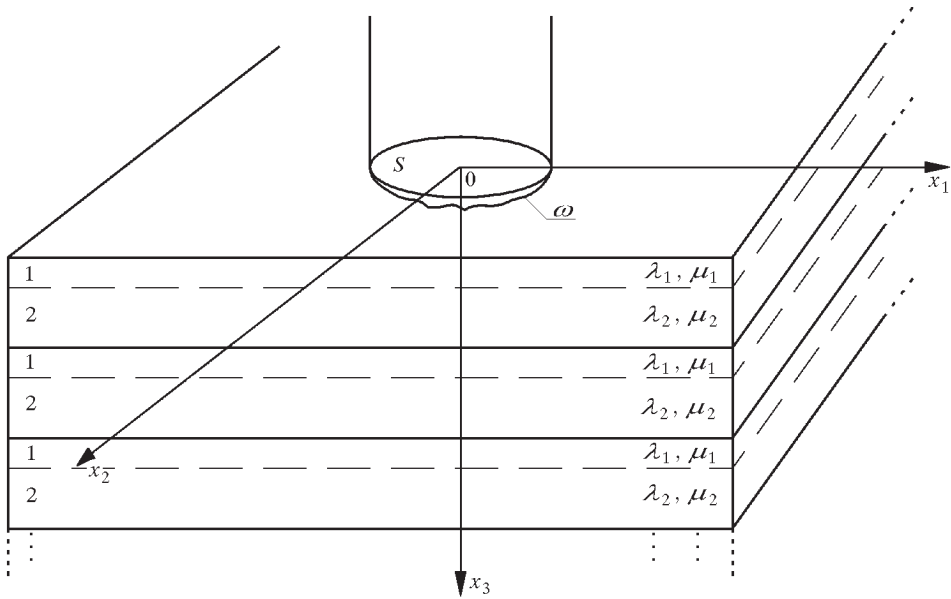


Fig. 1. Periodic two-layered half-space indented by a frictionless rigid punch

Note that the values of this function are small whereas the values of its derivative

$$h' = \begin{cases} 1 & \text{if } \mathbf{x} \text{ belongs to the 1st layer} \\ -\frac{\eta}{1-\eta} & \text{if } \mathbf{x} \text{ belongs to the 2nd layer} \end{cases}$$

are not small even for very thin layers.

The following representations and approximations are postulated within the elasticity with microlocal parameters¹

$$\begin{aligned} u_i &= w_i + h(x_3)d_i \approx w_i \\ u_{i,\alpha} &\approx w_{i,\alpha} & u_{i,3}^{(l)} &\approx w_{i,3} + h'd_i \\ \sigma_{\alpha\beta}^{(l)} &\approx \mu_l(w_{\alpha,\beta} + w_{\beta,\alpha}) + \delta_{\alpha\beta}\lambda_l(w_{i,i} + h'd_3) \end{aligned} \tag{2.2}$$

¹Indices i, j run over 1,2,3 while α, β, γ run over 1,2. They are related to the Cartesian coordinates. The summation convention holds for both kinds of the aforementioned indices.

$$\begin{aligned}\sigma_{\alpha 3}^{(l)} &\approx \mu_l(w_{\alpha,3} + w_{3,\alpha} + h'd_\alpha) \\ \sigma_{33}^{(l)} &\approx (\lambda_l + 2\mu_l)(w_{3,3} + h'd_3) + \lambda_l w_{\gamma,\gamma}\end{aligned}$$

Here $\delta_{\alpha\beta}$ is the Kronecker delta, w_i and d_i are unknown functions interpreted as the macro-displacements and microlocal parameters, respectively.

Following the special homogenization procedure (cf Woźniak, 1987), we arrive at the governing equations and constitutive relations of a certain macro-homogeneous medium (the homogenized model), given (after eliminating the microlocal parameters and in the absence of the body forces) in terms of the macro-displacements w_i as follows (see Kaczyński, 1993)

$$\begin{aligned}\frac{1}{2}(c_{11} + c_{12})w_{\gamma,\gamma\alpha} + \frac{1}{2}(c_{11} - c_{12})w_{\alpha,\gamma\gamma} + c_{44}w_{\alpha,33} + (c_{13} + c_{44})w_{3,3\alpha} &= 0 \\ (c_{13} + c_{44})w_{\gamma,\gamma 3} + c_{44}w_{3,\gamma\gamma} + c_{33}w_{3,33} &= 0 \\ \sigma_{\alpha 3} &= c_{44}(w_{\alpha,3} + w_{3,\alpha}) \quad \sigma_{33} = c_{13}w_{\alpha,\alpha} + c_{33}w_{3,3} \quad (2.3) \\ \sigma_{12}^{(l)} &= \mu_l(w_{1,2} + w_{2,1}) \\ \sigma_{11}^{(l)} &= d_{11}^{(l)}w_{1,1} + d_{12}^{(l)}w_{2,2} + d_{13}^{(l)}w_{3,3} \\ \sigma_{22}^{(l)} &= d_{12}^{(l)}w_{1,1} + d_{11}^{(l)}w_{2,2} + d_{13}^{(l)}w_{3,3}\end{aligned}$$

Positive coefficients appearing in the above equations are given in the Appendix. They depend on the material and geometrical characteristics of the subsequent layers. It is noteworthy that the condition of perfect bonding between the layers (the continuity of the stress vector at the interfaces) is satisfied. We also observe that setting $\lambda_1 = \lambda_2 \equiv \lambda$, $\mu_1 = \mu_2 \equiv \mu$ entails $c_{11} = c_{33} = \lambda + 2\mu$, $c_{12} = c_{13} = \lambda$, $c_{44} = \mu$ and the well-known equations of the elasticity for a homogeneous isotropic body with Lamé's constants λ , μ are recovered.

The general solutions to governing equations (2.3) in terms of three harmonic potentials have become possible due to close similarity to the displacement and stress-displacement relations for a transversely isotropic solid (see, for example, Kassir and Sih, 1975). According to the results obtained by Kaczyński (1993), the form of the potential representations is dependent on the material constants of the layers and is given below in two cases².

²The constants t_i , m_α are defined in Appendix.

Case 1: $\mu_1 \neq \mu_2$

The displacement field can be expressed through the three potentials $\widehat{\varphi}_i(x_1, x_2, z_i)$, in which $z_i = t_i x_3$, such that

$$\nabla^2 \widehat{\varphi}_i \equiv \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial z_i^2} \right) \widehat{\varphi}_i \quad \forall i \in \{1, 2, 3\}$$

as follows

$$\begin{aligned} w_1 &= (\widehat{\varphi}_1 + \widehat{\varphi}_2)_{,1} - \widehat{\varphi}_{3,2} & w_2 &= (\widehat{\varphi}_1 + \widehat{\varphi}_2)_{,2} + \widehat{\varphi}_{3,1} \\ w_3 &= m_1 t_1 \frac{\partial \widehat{\varphi}_1}{\partial z_1} + m_2 t_2 \frac{\partial \widehat{\varphi}_2}{\partial z_2} \end{aligned} \tag{2.4}$$

From stress-displacement relations (2.3), the stresses σ_{3i} can be expressed as

$$\begin{aligned} \sigma_{31} &= c_{44} \left[(1 + m_1) t_1 \frac{\partial \widehat{\varphi}_1}{\partial z_1} + (1 + m_2) t_2 \frac{\partial \widehat{\varphi}_2}{\partial z_2} \right]_{,1} - t_3 \frac{\partial^2 \widehat{\varphi}_3}{\partial z_3 \partial x_2} \\ \sigma_{32} &= c_{44} \left[(1 + m_1) t_1 \frac{\partial \widehat{\varphi}_1}{\partial z_1} + (1 + m_2) t_2 \frac{\partial \widehat{\varphi}_2}{\partial z_2} \right]_{,2} + t_3 \frac{\partial^2 \widehat{\varphi}_3}{\partial z_3 \partial x_1} \\ \sigma_{33} &= c_{44} \left[(1 + m_1) \frac{\partial^2 \widehat{\varphi}_1}{\partial z_1^2} + (1 + m_2) \frac{\partial^2 \widehat{\varphi}_2}{\partial z_2^2} \right] \end{aligned} \tag{2.5}$$

Formulas for the remaining stresses $\sigma_{\alpha\beta}^{(l)}$ are not of immediate interest and have been omitted.

Case 2: $\mu_1 = \mu_2 \equiv \mu, \lambda_1 \neq \lambda_2$

Here, the displacement equations take the classical form

$$(B + \mu) w_{j,ji} + \mu w_{i,jj} = 0 \tag{2.6}$$

provided

$$B = \frac{\lambda_1 \lambda_2 + 2\mu[\eta \lambda_1 + (1 - \eta) \lambda_2]}{(1 - \eta) \lambda_1 + \eta \lambda_2 + 2\mu} \tag{2.7}$$

and the representation in terms of the three harmonic functions $\varphi_i(x_1, x_2, x_3)$ (satisfying $\nabla^2 \varphi_i \equiv \varphi_{i,jj} = 0 \forall i \in \{1, 2, 3\}$) is given as follows

$$\begin{aligned}
 w_1 &= (\varphi_1 + x_3\varphi_2)_{,1} - \varphi_{3,2} & w_2 &= (\varphi_1 + x_3\varphi_2)_{,2} + \varphi_{3,1} \\
 w_3 &= \varphi_{1,3} + x_3\varphi_{2,3} - \frac{B + 3\mu}{B + \mu}\varphi_2 \\
 \sigma_{31} &= 2\mu \left[\varphi_{1,3} - \frac{\mu}{B + \mu}\varphi_2 + x_3\varphi_{2,3} \right]_{,1} - \mu\varphi_{3,23} & (2.8) \\
 \sigma_{32} &= 2\mu \left[\varphi_{1,3} - \frac{\mu}{B + \mu}\varphi_2 + x_3\varphi_{2,3} \right]_{,2} + \mu\varphi_{3,13} \\
 \sigma_{33} &= 2\mu \left[\varphi_{1,33} - \frac{B + 2\mu}{B + \mu}\varphi_{2,3} + x_3\varphi_{2,33} \right]
 \end{aligned}$$

Putting in the above case $\lambda_1 = \lambda_2 \equiv \lambda$ implies $B = \lambda$, passing to the known representation for the homogeneous isotropic body with the Lamé constants λ, μ .

3. Frictionless contact problem

Consider the general problem of indentation of the two-layered periodic half-space by a frictionless smooth rigid punch. Let S is the known contact area (see Fig. 1). Within the framework of the homogenized model presented in Section 2 we can formulate the following mixed conditions on the entire plane $x_3 = 0$, denoted by Z

$$\begin{aligned}
 w_3(x_1, x_2, 0) &= \omega(x_1, x_2) & \forall (x_1, x_2) \in S \\
 \sigma_{33}(x_1, x_2, 0) &= 0 & \forall (x_1, x_2) \in Z - S \\
 \sigma_{31}(x_1, x_2, 0) &= \sigma_{32}(x_1, x_2, 0) = 0 & \forall (x_1, x_2) \in Z
 \end{aligned} \tag{3.1}$$

where $\omega(x_1, x_2)$ is a known function that describes the profile of the punch.

We now proceed to construct the potential functions well suited to the above mixed boundary conditions. This will be done by using the same potential representation as in Kaczyński (1993), for the corresponding crack problem with symmetric loading.

Case 1: $\mu_1 \neq \mu_2$

The potentials are expressed in terms of a single harmonic function $\hat{f}(x_1, x_2, x_3)$ (i.e. $\nabla^2 \hat{f} \equiv \hat{f}_{,jj} = 0$) as follows

$$\begin{aligned} \widehat{\varphi}_1(x_1, x_2, z_1) &= -\frac{1}{t_1(1+m_1)}\widehat{f}(x_1, x_2, z_1) \\ \widehat{\varphi}_2(x_1, x_2, z_2) &= \frac{1}{t_2(1+m_2)}\widehat{f}(x_1, x_2, z_2) \\ \widehat{\varphi}_3 &\equiv 0 \end{aligned} \tag{3.2}$$

Substitution Eqs (3.2) into Eqs (2.4) and making use of Eqs (2.3) yields the following representations of the displacements w_i and stresses σ_{3i} in terms of the potential \widehat{f}

$$\begin{aligned} w_\alpha(x_1, x_2, x_3) &= \sum_{k=1}^2 \frac{(-1)^k}{t_k(1+m_k)}\widehat{f}_{,\alpha}(x_1, x_2, z_k) \\ w_3(x_1, x_2, x_3) &= \frac{m_2}{1+m_2}\frac{\partial}{\partial z_2}\widehat{f}(x_1, x_2, z_2) - \frac{m_1}{1+m_1}\frac{\partial}{\partial z_1}\widehat{f}(x_1, x_2, z_1) \\ \sigma_{3\alpha}(x_1, x_2, x_3) &= c_{44}\left[\frac{\partial}{\partial z_2}\widehat{f}(x_1, x_2, z_2) - \frac{\partial}{\partial z_1}\widehat{f}(x_1, x_2, z_1)\right] \\ \sigma_{33}(x_1, x_2, x_3) &= c_{44}\left[\frac{1}{t_2}\frac{\partial^2}{\partial z_2^2}\widehat{f}(x_1, x_2, z_2) - \frac{1}{t_1}\frac{\partial^2}{\partial z_1^2}\widehat{f}(x_1, x_2, z_1)\right] \end{aligned} \tag{3.3}$$

The remaining stresses (discontinuous on the interfaces) are found to be

$$\begin{aligned} \sigma_{11}^{(l)}(x_1, x_2, x_3) &= \sum_{k=1}^2 \frac{(-1)^k}{t_k(1+m_k)} \cdot \\ &\cdot \left[d_{11}^{(l)}\widehat{f}_{,11}(x_1, x_2, z_k) + d_{22}^{(l)}\widehat{f}_{,22}(x_1, x_2, z_k) + d_{13}^{(l)}m_k\widehat{f}_{,33}(x_1, x_2, z_k) \right] \\ \sigma_{22}^{(l)}(x_1, x_2, x_3) &= \sum_{k=1}^2 \frac{(-1)^k}{t_k(1+m_k)} \cdot \\ &\cdot \left[d_{12}^{(l)}\widehat{f}_{,11}(x_1, x_2, z_k) + d_{11}^{(l)}\widehat{f}_{,22}(x_1, x_2, z_k) + d_{13}^{(l)}m_k\widehat{f}_{,33}(x_1, x_2, z_k) \right] \\ \sigma_{12}^{(l)}(x_1, x_2, x_3) &= 2\mu_l \sum_{k=1}^2 \frac{(-1)^k}{t_k(1+m_k)}\widehat{f}_{,12}(x_1, x_2, z_k) \end{aligned} \tag{3.4}$$

It easily follows from Eqs (3.3) that on the boundary $x_3 = 0$ (then $z_1 = z_2 = 0$, $\partial\widehat{f}(x_1, x_2, z_\alpha)/\partial z_\alpha = \partial\widehat{f}(x_1, x_2, x_3)/\partial x_3$) the third condition in (3.1) $\sigma_{3\alpha} = 0$ is satisfied. In addition, in view of Eqs (3.3), the components of the

displacement and stress that act along the x_3 -axis on the surface $x_3 = 0^+$ take the form

$$\begin{aligned} w_3(x_1, x_2, 0) &= \left(\frac{m_2}{1+m_2} - \frac{m_1}{1+m_1} \right) \left[\widehat{f}_{,3}(x_1, x_2, x_3) \right]_{x_3=0} \\ \sigma_{33}(x_1, x_2, 0) &= c_{44} \left(\frac{1}{t_2} - \frac{1}{t_1} \right) \left[\widehat{f}_{,33}(x_1, x_2, x_3) \right]_{x_3=0} \end{aligned} \quad (3.5)$$

The above relations reduce the contact problem given by Eqs (3.1) to the classical mixed problem (cf Sneddon, 1966) for finding the harmonic function \widehat{f} in the half-space $x_3 \geq 0$, which vanishes at infinity and satisfies the boundary conditions

$$\begin{aligned} \left[\widehat{f}_{,3}(x_1, x_2, x_3) \right]_{x_3=0} &= \frac{(1+m_1)(1+m_2)}{m_2-m_1} \omega(x_1, x_2) & \forall (x_1, x_2) \in S \\ \left[\widehat{f}_{,33}(x_1, x_2, x_3) \right]_{x_3=0} &= 0 & \forall (x_1, x_2) \in Z-S \end{aligned} \quad (3.6)$$

Case 2: $\mu_1 = \mu_2 \equiv \mu$, $\lambda_1 \neq \lambda_2$

The solution to Eqs (2.6) in terms of one harmonic function f with the assumption that the boundary $x_3 = 0$ is free from tangential stresses is achieved by taking in Eqs (2.8)

$$\varphi_1 = \frac{\mu}{B+\mu} f \quad \varphi_2 = f_{,3} \quad \varphi_3 = 0 \quad (3.7)$$

Then it follows from Eqs (2.8) that the displacement and stress components are

$$\begin{aligned} w_\alpha &= \frac{\mu}{B+\mu} f_{,\alpha} + x_3 f_{,3\alpha} & w_3 &= -\frac{B+2\mu}{B+\mu} f_{,3} + x_3 f_{,33} \\ \sigma_{3\alpha} &= 2\mu x_3 f_{,\alpha 33} & \sigma_{33} &= 2\mu(-f_{,33} + x_3 f_{,333}) \\ \sigma_{11}^{(l)} &= 2\mu(D_{11}^{(l)} f_{,11} + D_{12}^{(l)} f_{,22} + x_3 f_{,113}) & & \\ \sigma_{22}^{(l)} &= 2\mu(D_{12}^{(l)} f_{,11} + D_{11}^{(l)} f_{,22} + x_3 f_{,223}) & & \\ \sigma_{12}^{(l)} &= 2\mu_l \left(\frac{\mu}{B+\mu} f_{,12} + x_3 f_{,123} \right) & & \end{aligned} \quad (3.8)$$

where

$$D_{11}^{(l)} = 1 + \frac{2\mu(\lambda_1 - B)}{(B + \mu)(\lambda_1 + 2\mu)} \quad D_{12}^{(l)} = \frac{\lambda_1}{B + \mu} \left(1 + \frac{\lambda_1 - B}{\lambda_1 + 2\mu} \right) \quad (3.9)$$

The following expressions are found on the plane $x_3 = 0$

$$w_3 = -\frac{B + 2\mu}{B + \mu} f_{,3} \quad \sigma_{33} = -2\mu f_{,33} \quad (3.10)$$

Application of conditions (3.1) yields a similar problem to that appearing in Eqs (3.6) in finding the harmonic function f

$$\begin{aligned} \left[\widehat{f}_{,3}(x_1, x_2, x_3) \right]_{x_3=0} &= -\frac{B + 2\mu}{B + \mu} \omega(x_1, x_2) & \forall (x_1, x_2) \in S \\ \left[\widehat{f}_{,33}(x_1, x_2, x_3) \right]_{x_3=0} &= 0 & \forall (x_1, x_2) \in Z - S \end{aligned} \quad (3.11)$$

The mixed boundary-value problems for the harmonic functions \widehat{f} in Case 1 and f in Case 2 can be reduced to integral equations by using the representations of their first x_3 -derivatives through the potentials of the simple layer, namely

$$\begin{aligned} \widehat{f}_{,3}(x_1, x_2, x_3) &= \iint_S \frac{\widehat{\sigma}_0(x, y) \, dx dy}{\sqrt{(x_1 - x)^2 + (x_2 - y)^2 + x_3^2}} \\ f_{,3}(x_1, x_2, x_3) &= \iint_S \frac{\sigma_0(x, y) \, dx dy}{\sqrt{(x_1 - x)^2 + (x_2 - y)^2 + x_3^2}} \end{aligned} \quad (3.12)$$

where the unknown functions $\widehat{\sigma}_0$ and σ_0 will be determined from the well-known properties of these potentials

$$\begin{aligned} \widehat{f}_{,33} \Big|_{x_3=0} &= \begin{cases} -2\pi \widehat{\sigma}_0(x_1, x_2) & \forall (x_1, x_2) \in S \\ 0 & \forall (x_1, x_2) \in Z - S \end{cases} \\ f_{,33} \Big|_{x_3=0} &= \begin{cases} -2\pi \sigma_0(x_1, x_2) & \forall (x_1, x_2) \in S \\ 0 & \forall (x_1, x_2) \in Z - S \end{cases} \end{aligned} \quad (3.13)$$

Notice that the second condition in Eqs (3.6) and (3.11) is satisfied and the first one gives, in view of Eqs (3.5) and (3.10), the equations for $\widehat{\sigma}_0$ and σ_0

$$\begin{aligned} \frac{t_1 t_2}{c_{44}(t_1 - t_2)} \sigma_{33}(x_1, x_2, 0) &= -2\pi \widehat{\sigma}_0(x_1, x_2) & \text{Case 1} \\ -\frac{1}{2\mu} \sigma_{33}(x_1, x_2, 0) &= -2\pi \sigma_0(x_1, x_2) & \text{Case 2} \end{aligned} \quad (3.14)$$

Hence, denoting the normal contact traction $\sigma_{33}(x_1, x_2, 0) \equiv p_3(x_1, x_2)$ on S , we obtain (see Appendix)

$$\begin{aligned} \hat{\sigma}_0(x_1, x_2) &= \frac{t_1 t_2}{2\pi t - c_{44}} p_3(x_1, x_2) && \text{Case 1} \\ \sigma_0(x_1, x_2) &= \frac{1}{4\pi\mu} p_3(x_1, x_2) && \text{Case 2} \end{aligned} \tag{3.15}$$

Substitution of Eqs (3.15) in Eqs (3.12) leads to the potentials expressed by $p_3(x_1, x_2)$ (unknown function) as follows

$$\begin{aligned} \hat{f}_{,3}(x_1, x_2, x_3) &= \frac{t_1 t_2}{2\pi t - c_{44}} \iint_S \frac{p_3(x, y) \, dx dy}{\sqrt{(x_1 - x)^2 + (x_2 - y)^2 + x_3^2}} && \text{Case 1} \\ f_{,3}(x_1, x_2, x_3) &= \frac{1}{4\pi\mu} \iint_S \frac{p_3(x, y) \, dx dy}{\sqrt{(x_1 - x)^2 + (x_2 - y)^2 + x_3^2}} && \text{Case 2} \end{aligned} \tag{3.16}$$

Finally, satisfaction of the first conditions in Eqs (3.6) and (3.11) yields the governing integral equation of the considered contact problem for a two-layered periodic half-space

$$-H \iint_S \frac{p_3(x, y) \, dx dy}{\sqrt{(x_1 - x)^2 + (x_2 - y)^2}} = \omega(x_1, x_2) \tag{3.17}$$

where H is the same constant as used by Fabrikant (1989) in study of contact on a transversely isotropic half-space, here taking on the values

$$H = \begin{cases} \frac{t_1 t_2}{2\pi c_{44} t -} \frac{m_1 - m_2}{(1 + m_1)(1 + m_2)} = \frac{t + \sqrt{c_{11} c_{33}}}{2\pi(c_{11} c_{33} - c_{13}^2)} && \text{Case 1} \\ \frac{1}{4\pi\mu} \frac{B + 2\mu}{B + \mu} && \text{Case 2} \end{cases} \tag{3.18}$$

Once the contact stresses $p_3(x_1, x_2)$ are known from the solution of the above integral equation, the complete displacement and stress fields can be written down using Eqs (3.3), (3.4) in Case 1 and Eqs (3.8) in Case 2 with the main potentials \hat{f} and f , determined from Eqs (3.16) by integrating with respect to x_3

$$\begin{aligned} \hat{f}(x_1, x_2, x_3) &= \\ &= \frac{t_1 t_2}{2\pi t - c_{44}} \iint_S \ln \left[\sqrt{(x_1 - x)^2 + (x_2 - y)^2 + x_3^2} + x_3 \right] p_3(x, y) \, dx dy \end{aligned}$$

$$\begin{aligned}
 f(x_1, x_2, x_3) &= \tag{3.19} \\
 &= \frac{1}{4\pi\mu} \iint_S \ln \left[\sqrt{(x_1 - x)^2 + (x_2 - y)^2 + x_3^2} + x_3 \right] p_3(x, y) \, dx dy
 \end{aligned}$$

Integral equation (3.17) has been widely known, but its solution presents considerable difficulties. However, marked progress has been made by Fabrikant (1989, 1991) in obtaining exact and complete solutions to various contact problems (in elementary functions) for a circular punch of any polynomial profile. Owing to the same governing equation (3.17), these solutions will be used for solving the corresponding problems of contact on a periodic two-layered half-space within the framework of the elasticity with microlocal parameters, presented in Section 2. For the sake of simplicity, the results will be presented for the simplest case of indentation by a flat punch in the next section.

4. Example: flat centrally loaded circular punch

Consider the case when a flat rigid circular punch of the radius a is pressed against a two-layered periodic elastic half-space $x_3 \geq 0$ by the centrally applied normal force P . This problem is characterised by mixed boundary conditions (3.1) with the contact area $S = \{(x_1, x_2) : \rho^2 \equiv x_1^2 + x_2^2 \leq a^2\}$ and the punch settlement $\omega(x_1, x_2) = \text{const} \equiv \omega_0 > 0$.

Several methods of solving axisymmetric punch problems were reported in the literature (see for example a review by Barber, 1992). A wide range of new investigations in the field of contact problems related to a transversely isotropic body and directed to obtain complete solutions has been carried out by Fabrikant (1989, 1991). At present, by making use of his results, we present the exact solution of the posed problem within the framework of the homogenized model.

The solution to governing integral equation (3.17) gives the contact stresses

$$p_3(x_1, x_2) = \sigma_3(x_1, x_2, 0) = -\frac{\omega_0}{\pi^2 H \sqrt{a^2 - x_1^2 - x_2^2}} \tag{4.1}$$

The total force P is related with the punch settlement ω_0 by the relationship

$$P = - \iint_S p_3(x_1, x_2) \, dx_1 dx_2 = \frac{2\omega_0 a}{\pi H} \tag{4.2}$$

Now we substitute Eq. (4.1) into Eqs (3.16) in order to find the main potential functions giving a complete solution. The method of Fabrikant yields the results in elementary functions as follows (for Case 1 and Case 2, respectively)

$$\begin{aligned} \widehat{f}(x_1, x_2, x_3) &= -\frac{t_1 t_2 \omega_0}{\pi^2 H t_- c_{44}} \left[x_3 \arcsin \frac{a}{l_2} - \sqrt{a^2 - l_1^2} + a \ln(l_2 + \sqrt{l_2^2 - \rho^2}) \right] \\ f(x_1, x_2, x_3) &= -\frac{\omega_0}{2\pi^2 H \mu} \left[x_3 \arcsin \frac{a}{l_2} - \sqrt{a^2 - l_1^2} + a \ln(l_2 + \sqrt{l_2^2 - \rho^2}) \right] \end{aligned}$$

where in his notation

$$\begin{aligned} l_1 &\equiv l_1(a, \rho, x_3) = \frac{1}{2} \left[\sqrt{(\rho + a)^2 + x_3^2} - \sqrt{(\rho - a)^2 + x_3^2} \right] \\ l_2 &\equiv l_2(a, \rho, x_3) = \frac{1}{2} \left[\sqrt{(\rho + a)^2 + x_3^2} + \sqrt{(\rho - a)^2 + x_3^2} \right] \end{aligned} \tag{4.3}$$

Appropriate differentiation of the above potentials (see Appendix 5 in the book by Fabrikant, 1991) and then making use of Eqs (3.3) in Case 1 and Eqs (3.8) in Case 2 give the complete displacement and stress field in the following concise form:

Case 1

$$\begin{aligned} w_\alpha &= \frac{2\omega_0 a x_\alpha}{\pi \rho^2} \sum_{k=1}^2 \frac{1}{t_k(m_k - 1)} \left[1 - \frac{\sqrt{a^2 - l_{1k}^2}}{a} \right] \\ w_3 &= \frac{2\omega_0}{\pi} \sum_{k=1}^2 \frac{m_k}{m_k - 1} \arcsin \frac{a}{l_{2k}} \\ \sigma_{3\alpha} &= \frac{\omega_0 t_1 t_2 x_\alpha}{\pi^2 H t_- \rho} \sum_{k=1}^2 (-1)^2 \frac{l_{1k} \sqrt{a^2 - l_{1k}^2}}{l_{2k}(l_{2k}^2 - l_{1k}^2)} \\ \sigma_{33} &= \frac{\omega_0 t_1 t_2}{\pi^2 H t_-} \sum_{k=1}^2 (-1)^2 \frac{\sqrt{a^2 - l_{1k}^2}}{t_k(l_{2k}^2 - l_{1k}^2)} \end{aligned} \tag{4.4}$$

Here the notations l_{1k} and l_{2k} for $k = 1, 2$ are understood as $l_1(a, \rho, z_k)$ and $l_2(a, \rho, z_k)$, respectively. The evaluation of $\sigma_{\alpha\beta}^{(l)}$ is not given because of the complexity.

Case 2

$$w_\alpha = \frac{2\omega_0 x_\alpha}{\pi \rho} \frac{B + \mu}{B + 2\mu} \left[-\frac{\mu}{B + \mu} \frac{a - \sqrt{a^2 - l_1^2}}{\rho} + \frac{x_3 l_1 \sqrt{l_2^2 - a^2}}{l_2(l_2^2 - l_1^2)} \right]$$

$$\begin{aligned}
 w_3 &= \frac{2\omega_0}{\pi} \left[\arcsin \frac{a}{l_2} + \frac{B + \mu}{B + 2\mu} \frac{x_3 \sqrt{a^2 - l_1^2}}{l_2(l_2^2 - l_1^2)} \right] \\
 \sigma_{3\alpha} &= -\frac{\omega_0 \mu x_\alpha x_3 (B + \mu)}{\pi (B + 2\mu)} \frac{\sqrt{a^2 - l_1^2} (3l_2^2 + l_1^2 - 4a^2)}{(l_2^2 - l_1^2)^3} \\
 \sigma_{33} &= \frac{\omega_0 \mu (B + \mu)}{\pi (B + 2\mu)} \left\{ -\frac{\sqrt{a^2 - l_1^2}}{l_2^2 - l_1^2} + \frac{x_3^2 [l_1^4 + a^2(\rho^2 - 2a^2 - 2x_3^2)]}{\sqrt{a^2 - l_1^2} (l_2^2 - l_1^2)^3} \right\}
 \end{aligned}
 \tag{4.5}$$

It is of interest to record the normal displacement and stress distribution of the boundary $x_3 = 0$. Taking into account that

$$\begin{aligned}
 l_1 \Big|_{x_3=0} &= l_{1k} \Big|_{x_3=0} = \min(a, \rho) \\
 l_2 \Big|_{x_3=0} &= l_{2k} \Big|_{x_3=0} = \max(a, \rho)
 \end{aligned}$$

one obtains

$$\begin{aligned}
 w_3(x_1, x_2, 0) &= \begin{cases} \omega_0 & \text{if } x_1^2 + x_2^2 \leq a^2 \\ \frac{2\omega_0}{\pi} \arcsin \frac{a}{\sqrt{x_1^2 + x_2^2}} & \text{if } x_1^2 + x_2^2 > a^2 \end{cases} \\
 \sigma_{33}(x_1, x_2, 0) &= \begin{cases} -\frac{\omega_0}{\pi^2 H \sqrt{a^2 - x_1^2 - x_2^2}} & \text{if } x_1^2 + x_2^2 < a^2 \\ 0 & \text{if } x_1^2 + x_2^2 > a^2 \end{cases}
 \end{aligned}
 \tag{4.6}$$

with H defined by Eqs (3.18). Assuming $\lambda_1 = \lambda_2 \equiv \lambda$ in Case 2 we obtain the well-known solution of the contact problem under study for a homogeneous isotropic elastic half-space with Lamé's constants λ and μ .

5. Conclusion

The three-dimensional contact problem for a periodic two-layered half-space has been investigated within the homogenized model with microlocal parameters. The governing integral equation of this problem turns out to have the classical form well known from the consideration of the corresponding

problem of an arbitrary frictionless rigid punch pressed against a transversely isotropic elastic half-space. Hence, complete solutions to several punch problems, which were included in Fabrikant (1989, 1991), may be extended and adopted in the case of contact on the laminar half-space under study.

A. Appendix

- Denoting by $\eta = \delta_1/\delta$, $b_l = \lambda_l + 2\mu_l$ ($l = 1, 2$), $b = (1 - \eta)b_1 + \eta b_2$, the positive coefficients in governing equations (2.3) are given by the following formulae

$$c_{11} = c_{33} + \frac{4\eta(1 - \eta)(\mu_1 - \mu_2)(\lambda_1 - \lambda_2 + \mu_1 - \mu_2)}{b}$$

$$c_{13} = \frac{(1 - \eta)\lambda_2 b_1 + \eta\lambda_1 b_2}{b} \quad c_{33} = \frac{b_1 b_2}{b}$$

$$c_{12} = \frac{\lambda_1 \lambda_2 + 2[\eta\mu_2 + (1 - \eta)\mu_1][\eta\lambda_1 + (1 - \eta)\lambda_2]}{b}$$

$$c_{44} = \frac{\mu_1 \mu_2}{\eta\mu_2 + (1 - \eta)\mu_1} \quad d_{13}^{(l)} = \frac{\lambda_l c_{33}}{b_l}$$

$$d_{11}^{(l)} = \frac{4\mu_l(\lambda_l + \mu_l) + \lambda_l c_{13}}{b_l} \quad d_{12}^{(l)} = \frac{2\mu_l \lambda_l + \lambda_l c_{13}}{b_l}$$

- The constants appearing in Eqs (2.4) and (2.5) are given as follows

$$t_1 = \frac{1}{2}(t_+ - t_-) \quad t_2 = \frac{1}{2}(t_+ + t_-)$$

$$t_3 = \sqrt{\frac{\eta\mu_1 + (1 - \eta)\mu_2}{c_{44}}} \quad m_\alpha = \frac{c_{11}t_\alpha^{-2} - c_{44}}{c_{13} + c_{44}} \quad \forall \alpha \in \{1, 2\}$$

where

$$t_\pm = \sqrt{\frac{(A_\pm \pm 2c_{44})A_\mp}{c_{33}c_{44}}} \quad A_\pm = \sqrt{c_{11}c_{33}} \pm c_{13}$$

Note that $t_1 t_2 = \sqrt{c_{11}/c_{33}}$, $m_1 m_2 = 1$.

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O trójwymiarowych zagadnieniach kontaktowych dla periodycznej dwuwarstwowej półprzestrzeni sprężystej

Streszczenie

W ramach liniowej teorii sprężystości z parametrami mikrolokalnymi zbadano kontaktowe zagadnienia przestrzenne dotyczące wciskania stempla w periodycznie dwuwarstwową półprzestrzeń sprężystą. Efektywne wyniki uzyskano dzięki podobieństwu rządzących równań modelu zhomogenizowanego półprzestrzeni z równaniami dla ciała sprężystego z poprzeczną izotropią.

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