JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
2, **39**, 2001

TOLERANCE AVERAGING AND BOUNDARY-LAYER EQUATIONS FOR THE HEAT TRANSFER PROBLEMS IN MICRO-PERIODIC SOLIDS

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The macroscopic mathematical models of the heat transfer in micro-periodic solids, obtained by the tolerance averaging approach, are represented by the partial differential equation for the averaged temperature field and the system of ordinary differential equations involving time derivatives of certain extra unknown fields which are called internal thermal variables, cf Woźniak (2000). It follows that in the framework of the aforementioned tolerance models the boundary conditions for temperature can be imposed exclusively on the averaged temperature field. The aim of this contribution is to show how the tolerance averaging technique can be extended in order to satisfy the boundary conditions on higher level of accuracy.

Key words: composites, modelling, tolerance

1. Introduction

The simplest mathematical models for the overall (macroscopic) behavior of micro-periodic solids can be obtained by using the results of the well known asymptotic homogenization theory, cf Bensoussan et al. (1978), Jikov et al. (1994). These models are represented by PDEs with constant coefficients, which are referred to as homogenized equations. However, the form of the homogenized equations is independent on the microstructure size and hence they are incapable of describing the effect of microstructure size on the phenomena observed on the macroscopic level. To remove this drawback a number of alternative approaches to the modelling of periodic solids was proposed; the overview of these approaches can be found in Woźniak(1999). In this contribution we shall deal with what is called the tolerance averaging of partial differential equations with highly-oscillating micro-periodic coefficients, Woźniak (2000). The tolerance averaging technique of heat transfer equations for nonstationary problems leads to a system of differential equations with constant coefficients (some of them depend on the microstructure size) for the averaged temperature field θ° and for certain extra unknown fields V^A, $A = 1, ..., N$. These fields describe the disturbances of temperature caused by the periodic microheterogeneity of the solid. The characteristic feature of the tolerance averaging technique is that the equations for V^A are ordinary differential equations involving only time derivatives of V^A. That is why V^A are called the internal thermal variables. For stationary problems V^A are governed by a system of linear algebraic equations, and the tolerance model can be reduced to the homogenized one. It follows that in the framework of the tolerance averaging the boundary conditions for the temperature can be imposed only on the averaged temperature field θ°.

The problem we are going to solve in this contribution can be stated as follows: how to extend the tolerance averaging approach to the modelling of heat transfer problems in the micro-periodic solids in order to satisfy the boundary conditions on higher level of accuracy? This problem is strictly related to the fact that the averaged equations, describing processes in micro-periodic solids on the macroscopic level, are deprived of the physical sense in a certain near boundary layer, Woźniak (1999). Hence, the question arises how to modify these equations for describing the phenomena which take place in the boundary layer. This "boundary layer" problem is well known if the averaging of equations is carried out by using the known asymptotic homogenization technique; we can mention here the results given by Sanchez-Palencia and Zaoui (1985) as well as more general asymptotic approach to this problem by Panasenko (1994). However, the aforementioned approaches to the "boundary layer" problem cannot be directly applied to the tolerance averaging of equations, where the asymptotic method of modelling is rejected.

In this paper, we propose a certain extension of the tolerance averaging of the nonstationary heat transfer equation for a micro-periodic solid, which takes into account the "boundary layer" problem. To make the contribution self-consistent we begin with some mathematical notions and we formulate the basic propositions in Section 2; the proofs of these propositions are given in the

Appendix. The general foundations of the tolerance averaging are summarized in Section 3; more detailed discussion of these results can be found in Woźniak (1999). The extension of the tolerance averaging approach is proposed in Section 4 and an example of its application is analyzed in Section 5.

2. Mathematical preliminaries

By Ω we shall denote the region in the three-dimensional reference space parametrized with the Cartesian coordinates x_1, x_2, x_3 and occupied by the solid under consideration. Setting $\Delta = (-l_1/2, l_1/2) \times (-l_2/2, l_2/2) \times (-l_3/2, l_3/2)$ it is assumed that the solid is Δ-periodic, i.e., l_α-periodic in the direction of the x_α-axis, $\alpha = 1,2,3$. Moreover, the diameter l of Δ is assumed to be very small compared with the smallest characteristic length dimension of Ω. That is why l will be referred to as the microstructure length. Denoting $\Delta(\boldsymbol{x}) := \boldsymbol{x} + \Delta$, $\Omega_\Delta := \{\boldsymbol{x} \in \Omega,\ \Delta(\boldsymbol{x}) \subset \Omega\}$, for an arbitrary integrable function f defined in Ω we shall use the known averaging formula

$$\langle f\rangle(\boldsymbol{x}) = \frac{1}{vol\Delta} \int\limits_{\Delta(\boldsymbol{x})} f(\boldsymbol{y})\ dy_1 dy_2 dy_3 \qquad\qquad \boldsymbol{x} \in \Omega_\Delta$$

Let $\mathcal{F}(\overline{\Omega})$ be a set of sufficiently smooth and bounded functions defined in Ω, which are the unknowns in the problem under consideration. Moreover, let $\varepsilon : \mathcal{F}(\overline{\Omega}) \ni \varphi \to \varepsilon_\varphi \in R^+$ be a mapping which assigns to every $\varphi \in \mathcal{F}(\overline{\Omega})$ a positive real ε_φ which will be regarded as the admissible accuracy related to the computation of the values of φ or to the measurements of a physical field (such as a temperature or a temperature gradient) represented by φ. We shall write $\varphi(\boldsymbol{x}) \cong \varphi(\boldsymbol{y})$ iff $|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{y})| \leqslant \varepsilon_\varphi$ and say that the values of φ at $\boldsymbol{x}$ and $\boldsymbol{y}$ are in a tolerance. It means that the difference between them can be neglected from the computational viewpoint. The pair $T = (\mathcal{F}(\overline{\Omega}), \varepsilon(\cdot))$ will be referred to as the tolerance system and every $\varepsilon_\varphi = \varepsilon(\varphi)$ as the tolerance parameter assigned to $\varphi \in \mathcal{F}(\overline{\Omega})$.

A sufficiently regular function $F \in \mathcal{F}(\overline{\Omega})$ will be called *slowly varying*, $F(\cdot) \in SV_\Delta(T)$, if for every $\boldsymbol{x}, \boldsymbol{y}$ from the domain of F the condition $\boldsymbol{x} - \boldsymbol{y} \in \Delta$ implies $F(\boldsymbol{y}) \cong F(\boldsymbol{x})$ and if similar conditions hold also for all derivatives of F (including time derivatives provided that F depends also on time). Hence, we deal with function $F(\cdot)$ which is slowly varying in the space not in time.

A continuous function $\psi \in \mathcal{F}(\overline{\Omega})$ will be termed *periodic-like*, $\psi \in PL_\Delta(T)$, if for every $\boldsymbol{x} \in \Omega_\Delta$ there exist a Δ-periodic function $\varphi_{\boldsymbol{x}}(\cdot)$ such that the condition $\psi(\boldsymbol{y}) \cong \psi_{\boldsymbol{x}}(\boldsymbol{y})$ holds in $B(\boldsymbol{x},l) \cap \Omega$, where $B(\boldsymbol{x},l)$ is a ball of the radius l and the center at $\boldsymbol{x}$. The function $\psi_{\boldsymbol{x}}$ is said to be the periodic approximation of ψ in $\Delta(\boldsymbol{x})$. If $\psi \in PL_\Delta(T)$ and $\langle \rho\varphi \rangle(\boldsymbol{x}) = 0$ for every $\boldsymbol{x} \in \Omega_\Delta$ and for some integrable positive-value function ρ defined on Ω then we shall write $\psi \in PL^\rho_\Delta(T)$ and refer ψ to as the oscillating periodic like function. The following assertion can be proved.

Assertion. *If $F \in SV_\Delta(T)$, $\varphi \in PL_\Delta(T)$ and $\varphi_{\boldsymbol{x}}$ is a Δ-periodic approximation of φ in $\Delta(\boldsymbol{x})$ then for every $f \in L^\infty_{per}(\Delta)$ and $h \in C^1_{per}(\overline{\Delta})$, such that $\max\{h(\boldsymbol{y}) : \boldsymbol{y} \in \overline{\Delta}\} \leqslant l$, the following propositions hold for every $\boldsymbol{x} \in \Omega_\Delta$:*

$$
\begin{array}{llll}
(T1) & \langle fF \rangle(\boldsymbol{x}) \cong \langle f \rangle F(\boldsymbol{x}) & \text{for} & \varepsilon = \langle |f| \rangle \varepsilon_F \\
(T2) & \langle f\varphi \rangle(\boldsymbol{x}) \cong \langle f\varphi_{\boldsymbol{x}} \rangle(\boldsymbol{x}) & \text{for} & \varepsilon = \langle |f| \rangle \varepsilon_\varphi \\
(T3) & \langle f\nabla(hF) \rangle(\boldsymbol{x}) \cong \langle fF\nabla h \rangle(\boldsymbol{x}) & \text{for} & \varepsilon = \langle |f| \rangle (\varepsilon_F + l\varepsilon_{\nabla F}) \\
(T4) & \langle h\nabla(f\varphi) \rangle(\boldsymbol{x}) \cong -\langle f\varphi\nabla h \rangle(\boldsymbol{x}) & \text{for} & \varepsilon = \varepsilon_G + l\varepsilon_{\nabla G} \\
 & & & G = \langle hf\varphi \rangle l^{-1}
\end{array}
$$

where ε is a tolerance parameter which defines the pertinent tolerance $\cong$ and $G, \partial_\alpha G \in \mathcal{F}(\overline{\Omega})$.

The proofs of propositions $(T1) \div (T4)$ are given in the Appendix.

The tolerance averaging (in the sequel denoted by TA), which constitutes the foundation of the proposed modelling strategy is based, roughly speaking, on replacing the tolerances in formulas $(T1) \div (T4)$ by the equalities, i.e., by neglecting the terms $O(\varepsilon_F)$, $O(\varepsilon_{\nabla F})$, $O(\varepsilon_\varphi)$ and $O(\varepsilon_G)$. This special kind of approximation is equivalent to the following assumption.

Tolerance Averaging Assumption. *In averaging of equations involving slowly varying and periodic-like functions the left-hand sides of formulae $(T1) \div (T4)$ will be approximated respectively by their right-hand sides.*

It has to be emphasized that the functions $F(\cdot)$ and $\varphi(\cdot)$, occurring in $(T1) \div (T4)$, represent unknown fields in the problem under consideration; all that is known about these functions (apart from the regularity conditions) is that their values have to be calculated within the tolerances determined respectively by certain tolerance parameters ε_F and ε_φ. If these parameters are specified then the criteria of applicability of TA can be verified *a posteriori*, i.e., after finding the functions $F(\cdot)$ and $\varphi(\cdot)$. These criteria will be written in the simple form $F(\cdot) \in SV_\Delta(T)$ and $\varphi(\cdot) \in PL_\Delta(T)$, which involves the pertinent tolerance parameters and constitutes the necessary condition for the physical reliability of the obtained solutions. If the tolerance parameters are not specified then they can be calculated after finding $F(\cdot)$ and $\varphi(\cdot)$ from the conditions $F(\cdot) \in SV_\Delta(T)$, $\varphi(\cdot) \in PL_\Delta(T)$; in this way the accuracy of the obtained solutions can be evaluated.

In the tolerance averaging of equations we shall also use the following lemmas:

($L0$) *If* $F \in SV_\Delta(T) \cap C^1(\overline{\Omega})$ *then the estimation* $l|\partial_\alpha F| \leqslant \varepsilon_F + l\varepsilon_{\nabla F}$ *holds*

($L1$) *If* $g \in PL_\Delta(T)$ *and* $g^\circ, \widetilde{g} \in \mathcal{F}(\overline{\Omega})$ *then, for an arbitrary positive valued integrable* Δ*-periodic function* ρ*, the decomposition* $g = g^\circ + \widetilde{g}$ *exists, where* $g^\circ \in SV_\Delta(T)$*,* $\widetilde{g} \in PL^\rho_\Delta(T)$

($L2$) *If* $\varphi \in PL_\Delta(T)$*,* $f \in L^\infty_{per}(\Delta)$ *and* $\langle f\varphi\rangle(\cdot) \in \mathcal{F}(\overline{\Omega})$ *then* $\langle f\varphi\rangle(\cdot) \in SV_\Delta(T)$

($L3$) *If* $F \in SV_\Delta(T)$*,* $f \in C_{per}(\overline{\Delta})$ *and* $(fF)(\cdot) \in \mathcal{F}(\overline{\Omega})$ *then* $(fF)(\cdot) \in PL_\Delta(T)$

($L4$) *If* $F \in SV_\Delta(T)$*,* $G \in SV_\Delta(T)$ *and* $kF + mG \in \mathcal{F}(\overline{\Omega})$ *for some reals* k, m*, then* $kF + mG \in SV_\Delta(T)$*.*

The proofs of lemmas ($L0$) $\div$ ($L4$) are given in the Appendix

3. Tolerance averaging

In the linear approximation the heat conduction properties of a solid material are uniquely described by the second order heat conduction tensor $\mathbf{A}$, which is symmetric and positive definite and by the specific heat scalar c, which is positive. For every periodic solid the functions $\mathbf{A} = \mathbf{A}(\cdot)$, $c = c(\cdot)$ are defined in Ω and are Δ-periodic (hence they can be also defined in E^3) where Δ is assumed to be the known cell cf Section 2. Let $\theta = \theta(\cdot, t)$ be the temperature field in Ω at the time t and $f = f(\cdot, t)$ be the known intensity of heat sources. Under the aforementioned denotations the temperature field has to satisfy the well known heat transfer equation in Ω

$$\nabla \cdot (\mathbf{A} \cdot \nabla\theta) - c\dot{\theta} = f \tag{3.1}$$

The macroscopic theory of the heat transfer phenomena in micro-periodic solids will be based on the heuristic assumption that in the problems under consideration the temperature field conforms to the periodic structure of the solid. On the assumption that a certain tolerance system $T = (\mathcal{F}(\overline{\Omega}), \varepsilon(\cdot))$ is known and that $\theta(\cdot,t) \in \mathcal{F}(\overline{\Omega})$ for every time t, the above heuristic statement can be written in the following mathematical form.

Conformability Assumption (CA). *In the modelling of the heat transfer problems in microperiodic solids every temperature field $\theta(\cdot,t)$ has to satisfy the condition*

$$\theta(\cdot,t) \in PL_{\Delta}(T)$$

This condition may be violated only near the boundary of a solid.

From (CA) and lemma $(L1)$ it follows that there exist the decomposition $\theta(\cdot,t) = \theta^{\circ}(\cdot,t) + \vartheta(\cdot,t)$, with $\theta^{\circ}(\cdot,t) \in SV_{\Delta}(T)$ and $\vartheta(\cdot,t) \in PL^{\varrho}_{\Delta}(T)$, where either $\varrho = c$ or $\varrho = 1$. It can be shown, cf Woźniak (1999), that under (CA) and $(T1)$ the tolerance averaging of (3.1) yields

$$\nabla \cdot [\langle \mathbf{A} \rangle \cdot \nabla \theta^{\circ}(\boldsymbol{x},t) + \langle \mathbf{A} \cdot \nabla \vartheta \rangle(\boldsymbol{x},t)] - \langle c \rangle \dot{\theta}^{\circ}(\boldsymbol{x},t) - \langle c\dot{\vartheta} \rangle(\boldsymbol{x},t) = \langle f \rangle(\boldsymbol{x}) \tag{3.2}$$

where $\boldsymbol{x} \in \Omega_{\Delta}$. At the same time, using $(T1)$, $(T2)$ and $(T4)$, we can prove that the following periodic variational equation for the Δ-periodic function $\vartheta_{\boldsymbol{x}}(\boldsymbol{y},t)$, $\boldsymbol{y} \in \Delta(\boldsymbol{x})$ holds

$$\begin{aligned} &\langle \nabla \vartheta^* \cdot \mathbf{A} \cdot \nabla \vartheta_{\boldsymbol{x}} \rangle(\boldsymbol{x},t) + \langle \vartheta^* \dot{\vartheta}_{\boldsymbol{x}} c \rangle(\boldsymbol{x},t) = \\ &= -\langle \vartheta^* f \rangle(\boldsymbol{x},t) - \langle \vartheta^* c \rangle \dot{\theta}^{\circ}(\boldsymbol{x},t) - \langle \nabla \vartheta^* \cdot \mathbf{A} \rangle \cdot \nabla \theta^{\circ}(\boldsymbol{x},t) \end{aligned} \tag{3.3}$$

where $\boldsymbol{x} \in \Omega_{\Delta}$ and $\vartheta^*(\cdot)$ is a Δ-periodic test function $\vartheta^* \in H^1_{per}(\Delta)$; here either $\langle \vartheta^* \rangle = 0$, $\langle \vartheta_{\boldsymbol{x}} \rangle = 0$ or $\langle c\vartheta^* \rangle = 0$, $\langle c\vartheta_{\boldsymbol{x}} \rangle = 0$. Eqs (3.2), (3.3) constitute the fundamentals of the tolerance averaging approach to the modelling of heat transfer problems in micro-periodic solids on the macroscopic level. In order to obtain the model equations we shall look for the approximate solutions to the periodic problems (3.3) in the form $\vartheta_{\boldsymbol{x}}(\boldsymbol{y},t) = h^A(\boldsymbol{y}) V^A(\boldsymbol{x},t)$ (summation convention over $A = 1,...,N$ holds), $\boldsymbol{y} \in V(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega_{\Delta}$, where $h^A(\cdot)$, $A = 1,...,N$, are the known V-periodic mode shape functions and $V^A(\cdot,t) \in SV_{\Delta}(T)$ are the extra unknowns. The aforementioned mode shape functions have to satisfy the condition $\langle h^A \rangle = 0$ or $\langle ch^A \rangle = 0$, and can be derived as solutions to a certain eigenvalue problem related to (3.3) or are resulting from a periodic discretization of the cell Δ, cf Woźniak (1999). In this way, setting $\vartheta^* = h^A$ and applying $(T1)$, $(T3)$, $(L2)$, $(L4)$, after many transformations, we obtain the following system of equations for θ°, V^A, $A = 1, ..., N$

$$\begin{aligned}
&\nabla \cdot [\langle \mathbf{A} \rangle \cdot \nabla \theta^\circ(\boldsymbol{x}, t) + \langle \mathbf{A} \cdot \nabla h^A \rangle V^A(\boldsymbol{x}, t)] - \langle c \rangle \dot{\theta}^\circ(\boldsymbol{x}, t) = \langle f \rangle(\boldsymbol{x}, t) \\
&\langle c h^A h^B \rangle \dot{V}^A(\boldsymbol{x}, t) + \langle \nabla h^A \cdot \mathbf{A} \cdot \nabla h^B \rangle V^B(\boldsymbol{x}, t) + \\
&+\langle \nabla h^A \cdot \mathbf{A} \rangle \cdot \nabla \theta^\circ(\boldsymbol{x}, t) = -\langle h^A f \rangle(\boldsymbol{x}, t)
\end{aligned} \tag{3.4}$$

At the same time, using ($L3$) we can prove that the temperature field can be approximated by means of the formula

$$\theta(\boldsymbol{x}, t) \simeq \theta^\circ(\boldsymbol{x}, t) + h^A(\boldsymbol{x}) V^A(\boldsymbol{x}, t) \qquad \boldsymbol{x} \in \Omega_\Delta \tag{3.5}$$

where the approximation $\simeq$ depends on the number N of terms in the formula $\vartheta_{\boldsymbol{x}}(\boldsymbol{y}, t) = h^A(\boldsymbol{y}) V^A(\boldsymbol{x}, t)$, $\boldsymbol{y} \in \Delta(\boldsymbol{x})$. The solutions to problems described by Eqs (3.4) are physically reliable only if $\theta^\circ(\cdot, t) \in SV_\Delta(T)$, $V^A(\cdot, t) \in SV_\Delta(T)$, $A = 1, ..., N$, for every time t.

Eqs (3.4), (3.5) together with the conditions $\theta^\circ(\cdot, t) \in SV_\Delta(T)$ and $V^A(\cdot, t) \in SV_\Delta(T)$, represent the *tolerance model of nonstationary heat transfer problems in a periodic microheterogeneous solid.* The main features of this model are: 1° governing Eqs (3.4) have constant coefficients (which can be calculated after obtaining the mode shape functions h^A, $A = 1, ..., N$), 2° the coefficients $\langle c h^A h^B \rangle$ depend on the microstructure length l, 3° the unknowns V^A are governed by the system of ordinary differential equations involving only time derivatives of V^A. The aforementioned equations have been derived only for every $\boldsymbol{x} \in \Omega_\Delta$, but from the formal point of view they can be assumed to hold for every $\boldsymbol{x} \in \Omega$ being deprived of the physical interpretation in $\Omega \setminus \Omega_\Delta$. The basic unknowns in (3.4) are: the averaged temperature $\theta^\circ(\cdot)$ and the functions $V^A(\cdot)$. Moreover, there are no boundary conditions for $V^A(\cdot)$; that is why the unknowns $V^A(\cdot)$ are called *the internal thermal variables.*

4. Boundary layer equations

In order to solve the "boundary layer" problem formulated in the Introduction we shall extend the results of the tolerance averaging, outlined in Section 3, by incorporating an extra term of the boundary layer type into (3.5). To this end denote by $\boldsymbol{n} = \boldsymbol{n}(\boldsymbol{x})$ the inner unit normal to $\partial\Omega$ at the points $\boldsymbol{x}$ belonging to the smooth parts of $\partial\Omega$. Let us introduce in every subregion of Ω, situated near a certain smooth part Σ of $\partial\Omega$, the system of orthogonal coordinates ξ^α, $\alpha = 1,2,3$, such that $\xi^1 = \xi^1(\boldsymbol{y})$ is the distance between the point $\boldsymbol{y}$ belonging to this subregion and Σ. Let us also assume that the smallest curvature radius of Σ is sufficiently large compared with the microstructure length parameter l. The orthogonal coordinates ξ^α, $\alpha = 1,2,3$, where $\xi^1 \geqslant 0$, assigned to an arbitrary but fixed smooth part Σ of $\partial\Omega$ will be called *the near boundary normal coordinates*. They parametrize uniquely points belonging to a certain region of the boundary layer. If the boundary $\partial\Omega$ is smooth then the whole boundary layer can be parametrized by one near boundary normal coordinate system; otherwise we have to introduce more than one system. In this case it may happen that some from the points belonging to the boundary layer will not be parametrized at all or parametrized independently by two or more normal coordinate systems generated by the adjacent smooth parts Σ_1, Σ_2 of $\partial\Omega$. The "thickness" of the boundary layer is not specified (cf Sanchez-Palencia and Zaoui, 1985) but has to include the region $\Omega \setminus \Omega_\Delta$.

Let $\{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{n}\}$ be the vector basis assigned to an arbitrary point in a region of the boundary layer and related to an arbitrary but fixed near boundary normal coordinate system. We shall assume that the vector functions $\boldsymbol{g}_1(\cdot)$, $\boldsymbol{g}_2(\cdot)$, $\boldsymbol{n}(\cdot)$ defined in the region of the boundary layer as well as their first derivatives are slowly varying. This requirement imposes certain conditions on the shape of the boundary $\partial\Omega$; and as we have stated above, the minimum curvature radius of every smooth part Σ of $\partial\Omega$ has to be sufficiently large compared with the microstructure length l. Throughout the considerations related to the boundary layer problems we shall use the following denotations

$$\partial = (\boldsymbol{n}\cdot\nabla,\ 0,\ 0) \qquad\qquad \overline{\nabla} = (0,\ \boldsymbol{g}_1\cdot\nabla,\ \boldsymbol{g}_2\cdot\nabla)$$

which are assigned to an arbitrary normal coordinate system. It means that if $F = F(\xi^1, \xi^2, \xi^3)$ then

$$\partial F = \Big(\frac{\partial F}{\partial \xi^1}, 0, 0\Big) \qquad\qquad \overline{\nabla} F = \Big(0, \frac{\partial F}{\partial \xi^2}, \frac{\partial F}{\partial \xi^3}\Big)$$

We shall also denote $\partial_n F = \partial F/\partial\xi^1$ and $\partial_n^2 F = \partial^2 F/(\partial\xi^1)^2$.

In the subsequent part of this section the considerations will be restricted to an arbitrary but fixed region of the boundary layer which is parametrized by the known normal coordinate system. Let the temperature field in this region be approximated by

$$\theta(\boldsymbol{x},t) \simeq \theta^\circ(\boldsymbol{x},t) + h^A(\boldsymbol{x})V^A(\boldsymbol{x},t) + \psi(\boldsymbol{x},t) \tag{4.1}$$

where the right-hand side of formula (3.5) was supplemented by a certain boundary layer term

$$\psi(\boldsymbol{x},t) = h^A(\boldsymbol{x})Y^A(\boldsymbol{x},t) \tag{4.2}$$

with $Y^A(\boldsymbol{x},t)$ as the extra unknowns which are assumed to decay in the direction of $\boldsymbol{n}(\boldsymbol{x})$, $\boldsymbol{x} \in \partial\Omega$, i.e. they decay while passing from the boundary $\partial\Omega$ to the bulk region of Ω. Hence the functions $Y^A(\cdot,t)$ and their derivatives $\partial Y^A(\cdot,t)$ are not slowly varying functions. At the same time the gradients $\overline{\nabla}Y^A(\cdot,t)$ are assumed to be slowly varying. We shall also assume that Y^A depend on the microstructure length parameter l such that $\partial_n^2 Y^A \in O(l^{-2})$, $\partial_n Y^A \in O(l^{-1})$, $Y^A \in O(1)$ and hence the functions Y^A, $l\partial_n Y^A$, $l^2\partial_n^2 Y^A$ are of $O(1)$ order. Using (4.1) and (4.2) we have to replace the sum h^AV^A in (3.5) by $h^AV^A + h^AY^A$. Denoting $\widehat{h}^A = l^{-1}h^A$ and setting

$$\begin{aligned} I &= \nabla\cdot(\langle \mathbf{A}\cdot\nabla h^AY^A\rangle + l\langle \widehat{h}^A\mathbf{A}\cdot\nabla Y^A\rangle) - l\langle c\widehat{h}^A\dot{Y}^A\rangle \\ I^A &= l\langle \widehat{h}^A\nabla\cdot(\mathbf{A}\cdot\nabla h^BY^B + l\widehat{h}^B\mathbf{A}\cdot\nabla Y^B)\rangle - l^2\langle c\widehat{h}^A\widehat{h}^B\dot{Y}^B\rangle \end{aligned} \tag{4.3}$$

after applying the procedure described by Woźniak (1999) and leading from Eqs (3.2), (3.3) to Eqs (3.4), we obtain

$$\begin{aligned} &\nabla\cdot(\langle\mathbf{A}\rangle\cdot\nabla\theta^\circ + \langle\mathbf{A}\cdot\nabla h^A\rangle V^A) - \langle c\rangle\dot{\theta}^\circ + I = \langle f\rangle \\ &\langle\nabla h^A\cdot\mathbf{A}\cdot\nabla h^B\rangle V^B + \langle\nabla h^A\cdot\mathbf{A}\rangle\cdot\nabla\theta^\circ + \langle ch^Ah^B\rangle\dot{V}^B + \langle fh^A\rangle + I^A = 0 \end{aligned} \tag{4.4}$$

Eqs (4.4) have the form similar to that of Eqs (3.4) but incorporate the extra boundary-layer type terms I, I^A. These terms can be neglected in the bulk region of Ω situated outside the boundary layer. To satisfy this requirement we shall introduce into considerations the formal asymptotic assumption that $I^A \to 0$ together with $l \to 0$. Bearing in mind that the functions Y^A, $l\partial_n Y^A$, $l^2\partial_n^2 Y^A$ are of $O(1)$ order and under the limit passage $l \to 0$ they behave like slowly varying functions, from the above asymptotic assumption we obtain

$$\begin{aligned} &l^2\langle\widehat{h}^A\widehat{h}^B\boldsymbol{n}\cdot\mathbf{A}\cdot\boldsymbol{n}\rangle\partial_n^2Y^B - l\langle(\widehat{h}^B\nabla h^A - \widehat{h}^A\nabla h^B)\cdot\mathbf{A}\cdot\boldsymbol{n}\rangle\partial_nY^B - \\ &-\langle\nabla h^A\cdot\mathbf{A}\cdot\nabla h^B\rangle Y^B = 0 \end{aligned} \tag{4.5}$$

The bounded solutions $Y^A(\cdot,t)$ to Eqs (4.5) decay across the boundary layer and will be treated, together with their derivatives, as negligibly small in the bulk region of Ω. That is why Eqs (4.5) will be referred to as the boundary layer equations. For the same reason also the terms I, I^A in Eqs (4.4) can be neglected in the bulk region. Let us also observe that if l tends to zero then the boundary layer "thickness" also tends to zero; i.e. the bulk region can be "approximated" by the region Ω. Hence the approximate description of the boundary-layer problem proposed in this contribution is based on the formal asymptotic assumption that $l \to 0$ implies $I^A \to 0$, leading to the boundary layer Eqs (4.5). The above assumption can be also supplemented by the heuristic statement that the terms I, I^A are neglected in the bulk region and the boundary layer "thickness" is negligibly small when compared with the length dimensions of Ω. Hence the proposed statement means, roughly speaking, that Eqs (4.4) can be approximated by Eqs (3.4). This simplified approach to the boundary layer phenomena is described by Eqs (3.4) and (4.5). However, more general analysis of the boundary layer phenomena can be also carried out on the basis of Eqs (4.4), (4.5).

Summarizing the obtained results we conclude that the tolerance model of the nonstationary heat transfer problems under consideration is described by Eqs (3.4) (or by more general Eqs (4.4)), by the boundary layer equation (4.5) and by formulae (4.1), (4.2) for the temperature field. This model makes it possible to satisfy the boundary conditions for the temperature field, given by $\theta(\boldsymbol{x},t) = \widetilde{\theta}(\boldsymbol{x},t)$, $\boldsymbol{x} \in \partial\Omega$, (as well as other kinds of the boundary conditions) provided that the boundary temperature $\widetilde{\theta}(\boldsymbol{x},t)$ can be approximated by

$$\theta^{\circ}(\boldsymbol{x},t) \simeq \widetilde{\theta}^{\circ}(\boldsymbol{x},t) + h^A(\boldsymbol{x})\widetilde{\theta}^A(\boldsymbol{x},t) \qquad \boldsymbol{x} \in \partial\Omega \tag{4.6}$$

where the functions $\widetilde{\theta}^{\circ}(\cdot,t)$, $\widetilde{\theta}^A(\cdot,t)$ are known. Hence the boundary conditions for the temperature are

$$\begin{aligned} &\theta^{\circ}(\boldsymbol{x},t) = \widetilde{\theta}^{\circ}(\boldsymbol{x},t) \\ &V^A(\boldsymbol{x},t) + Y^A(\boldsymbol{x},t) = \widetilde{\theta}^A(\boldsymbol{x},t) \qquad \boldsymbol{x} \in \partial\Omega \end{aligned} \tag{4.7}$$

where $Y^A(\cdot,t)$, $A = 1,...,N$, is a bounded solution to the boundary layer Eqs (4.5). It should be remembered that using Eqs (3.4) and formulae (3.5) we are able to satisfy the boundary conditions only for the averaged part θ° of the temperature field by assuming that $\theta^{\circ}(\boldsymbol{x},t) = \widetilde{\theta}^{\circ}(\boldsymbol{x},t)$, $\boldsymbol{x} \in \partial\Omega$.

It has to be emphasized that the approach to the boundary layer phenomena outlined in this section can be treated only as a certain first approximation of the boundary layer theory. The main advantage of the proposed simplified approach is that the system of the boundary layer Eqs (4.5) is not coupled with the governing Eqs (3.4) of the tolerance model of heat transfer in micro-periodic composites. A more general approach to the boundary layer problems can be based on the asymptotic analysis of this problem, Panasenko (1994).

5. Example problem

In order to illustrate the model Eqs (3.4), (4.5) and formulae (4.1), (4.2) let us take into account the nonstationary problem of heat transfer in the infinite microheterogeneous half-space $x_1 \geqslant 0$ subjected on its boundary $x_1 = 0$ to the time dependent temperature oscillations $A\cos\omega t$, $t \in R$, with the known amplitude A and frequency ω. For the sake of simplicity let us confine ourselves to the simplest case of the tolerance model in which $N = 1$, taking into account only one mode shape function $h = h^1(\boldsymbol{x})$, $\boldsymbol{x} = (x_1, x_2, x_3)$, related to the smallest eigenvalue λ of the periodic eigenvalue problem related to Eqs (3.3), cf Woźniak (1999). Denoting by $\boldsymbol{n}$ the versor of the x_1-axis and setting $\overline{h} = hl^{-1}$ we shall introduce the coefficients

$$
\begin{aligned}
a_0 &= \langle \boldsymbol{n} \cdot \mathbf{A} \cdot \boldsymbol{n} \rangle = \langle A_{11} \rangle \\
a_1 &= \langle \boldsymbol{n} \cdot \mathbf{A} \cdot \nabla h \rangle = \langle A_{1\alpha} h_{,\alpha} \rangle \\
a_2 &= \langle \nabla h \cdot \mathbf{A} \cdot \nabla h \rangle = \langle A_{\alpha\beta} h_{,\alpha} h_{,\beta} \rangle \\
c_0 &= \langle c \rangle \qquad\qquad c_2 = \langle c(\overline{h})^2 \rangle \\
\alpha &= \langle \boldsymbol{n} \cdot \mathbf{A} \cdot \boldsymbol{n} (\overline{h})^2 \rangle = \langle A_{11} (\overline{h})^2 \rangle \qquad\qquad k = \frac{a_2}{\alpha}
\end{aligned}
$$

where the subscripts α, β run over $1, 2, 3$. Let us also define $V = V^1$, $Y = Y^1$. Under the aforementioned denotations Eqs (3.4) reduce to the form

$$
\begin{aligned}
a_0 \theta^\circ{}_{,11} + a_1 V_{,1} - c_0 \dot{\theta}^\circ &= 0 \\
l^2 c_2 \dot{V} + a_1 \theta^\circ{}_{,1} + a_2 V &= 0
\end{aligned}
\tag{5.1}
$$

the boundary layer equation (4.5) is given by

$$
l^2 \alpha Y_{,11} - a_2 Y = 0 \tag{5.2}
$$

and formulae (4.1), (4.2) yield

$$
\theta(\boldsymbol{x}, t) \simeq \theta^\circ(x_1, t) + h(\boldsymbol{x})[V(x_1, t) + Y(x_1, t)] \tag{5.3}
$$

where $\boldsymbol{x} = (x_1, x_2, x_3)$, $x_1 \geqslant 0$, $(x_2, x_3) \in R^2$, $t \in R$. Thus, the problem under consideration consists its finding the functions θ°, V, Y on the right-hand side of Eq (5.3) satisfying Eqs (5.1), (5.2) for $x_1 > 0$ and the following boundary conditions for $x_1 = 0$

$$
\theta^\circ(0, t) = A\cos\omega t \qquad\qquad V(0, t) + Y(0, t) = 0 \tag{5.4}
$$

To solve this problem we look for a solution to Eqs (5.1) in the complex form

$$\theta^\circ(x_1,t) = A\mathrm{e}^{\gamma x_1 + \mathrm{i}\omega t} \qquad\qquad V(x_1,t) = B\mathrm{e}^{\gamma x_1 + \mathrm{i}\omega t} \tag{5.5}$$

with the real constants A, B, γ, ω. Substituting this solution into (5.1) we arrive at the system of homogeneous linear algebraic equations for A and B. The determinant of this system has to be equal to zero which makes it possible to determine the constant γ. Under the extra denotation

$$a^\circ = a_0 - \frac{(a_1)^2}{a_2}$$

we obtain

$$\gamma^2 = \frac{\omega c_0(-l^2 c_2\omega + \mathrm{i}a_2)}{a_2 a^\circ + \mathrm{i}l^2 a_0 c_2 \omega} \tag{5.6}$$

It can be proved that the constant a° is positive and can be interpreted as the first approximation (because in (4.1), (4.2) we have assumed $N = 1$) of the effective heat transfer modulus in the direction of the x_1-axis. The square roots of (5.6) will be written down in the general form

$$\gamma(\omega) = \pm[\gamma_{\mathrm{Re}}(\omega) + \mathrm{i}\gamma_{\mathrm{Im}}(\omega)] \tag{5.7}$$

where γ_{Re} and γ_{Im} are assumed to be negative and determined by rather complicated formulae which are not given here. However, bearing in mind that l^2 in formula (5.6) can be interpreted as a small parameter, we can obtain the following asymptotic expansion for γ_{Re} and γ_{Im}

$$\begin{aligned} \gamma_{\mathrm{Re}}(\omega) &= -\sqrt{\frac{c_0\omega}{2a^\circ}}\left[1 + l^2\frac{c_2}{2a_0}\Big(\frac{a_1}{a_2}\Big)^2\omega\right] + O(l^4) \\ \gamma_{\mathrm{Im}}(\omega) &= -\sqrt{\frac{c_0\omega}{2a^\circ}}\left[1 - l^2\frac{c_2}{2a_0}\Big(\frac{a_1}{a_2}\Big)^2\omega\right] + O(l^4) \end{aligned} \tag{5.8}$$

At the same time we obtain the interrelation $B = \varphi(\omega)A$ between the real constants A and B, where

$$\varphi(\omega) = -\frac{a_1 a_2 \gamma_{\mathrm{Re}}(\omega)}{(a_2)^2 + l^4(c_2)^2\omega^2} \tag{5.9}$$

The asymptotic expansion for $\varphi(\omega)$ is given by

$$\varphi(\omega) = \frac{a_1}{a_2}\sqrt{\frac{c_0\omega}{2a^\circ}}\left[1 + l^2\frac{c_2}{2a_0}\Big(\frac{a_1}{a_2}\Big)^2\omega\right] + O(l^4) \tag{5.10}$$

Now, we shall pass to the boundary layer equation (5.2). We shall look for solutions to Eq (5.2) in the form $Y(x_1,t) = \overline{Y}(x_1)\mathrm{e}^{\mathrm{i}\omega t}$, where $\overline{Y}$ satisfies the equation

$$\overline{Y},_{11} - \Big(\frac{k}{l}\Big)^2 \overline{Y} = 0$$

Hence, the real part of Y will be given by

$$Y(x_1,t) = \overline{B}\cos(\omega t)\exp\Big(-\frac{k}{l}x_1\Big) \tag{5.11}$$

where $\overline{B}$ is an arbitrary real constant. It can be seen that the real part of $\theta^\circ(x_1,t)$ derived from formula (5.5) satisfies the first from the boundary conditions (5.4). Taking into account formulae (5.5), (5.11) it can be seen that the real part of $V(x_1,t)$ together with $Y(x_1,t)$ satisfy the second from the boundary conditions (5.4) provided that $\overline{B} = -B$. It follows that $\overline{B} = -\varphi(\omega)A$. Summarizing all the derived results, by means of (5.3) we obtain the solution to the problem under consideration in the form

$$\begin{aligned}\theta(\boldsymbol{x},t) \simeq A\Big\{&\mathrm{e}^{\gamma_{\mathrm{Re}}(\omega)x_1}\cos(\gamma_{\mathrm{Im}}(\omega)x_1 + \omega t) + \\ &+h(\boldsymbol{x})\varphi(\omega)\Big[\mathrm{e}^{\gamma_{\mathrm{Re}}(\omega)x_1}\cos(\gamma_{\mathrm{Im}}(\omega)x_1 + \omega t) - \mathrm{e}^{-\frac{k}{l}x_1}\cos(\omega t)\Big]\Big\}\end{aligned} \tag{5.12}$$

where $\boldsymbol{x} = (x_1,x_2,x_3)$, $x_1 \geqslant 0$, $(x_2,x_3) \in R^2$ and $t \in R$. In the first approximation the values of $\gamma_{\mathrm{Re}}(\omega)$, $\gamma_{\mathrm{Im}}(\omega)$ and $\varphi(\omega)$ can be calculated from (5.8) and (5.10), respectively, after neglecting the terms $O(l^4)$. This approximation can be applied only if the second term in the square brackets of formulae (5.8) and (5.10) is much smaller than 1.

Now, let us compare the obtained result with that which can be derived using the homogenized model of the problem under consideration. The simplest form of this model (in which $N = 1$, i.e. if we deal with only one mode shape function h) can be directly derived from the governing Eqs (5.1) and (5.2) by neglecting in the second equation from (5.1) the term $l^2c_2\dot{V}$ involving the microstructure length parameter l. In this case, instead of (5.1) we obtain

$$a^\circ\theta^\circ,_{11} - c_0\dot{\theta}^\circ = 0 \qquad\qquad V = -\frac{a_1}{a_2}\theta^\circ,_1$$

Using the general homogenization procedure we have to calculate a°. In both cases we shall look for the solutions of the form given only by the first from Eqs (5.5). Taking into account boundary layer equation (5.2), after simple calculations we arrive at formula (5.12), in which

$$\begin{aligned} &\gamma_{\rm Re}(\omega) = \gamma_{\rm Im}(\omega) = -\sqrt{\frac{c_0\omega}{2a^\circ}} \\ &\varphi(\omega) = \frac{a_1}{a_2}\sqrt{\frac{c_0\omega}{2a^\circ}} \end{aligned} \tag{5.13}$$

It can be seen that this result is also implied by the formal limit passage $l \to 0$ in formulae (5.8) and (5.10). Hence the important conclusion that the homogenized model of the problem we deal with can be applied only if the second term in the square brackets in formulae (5.8) and (5.10) is negligibly small compared with 1. This situation takes place if

$$\Big(\frac{a_1}{a_2}\Big)^2\omega \ll \frac{2a_0}{l^2c_2}$$

Roughly speaking, the homogenized model of the problem under consideration can be applied only if the frequencies ω of the temperature oscillations are not too big; otherwise we have to use the tolerance averaging model detailed in this paper.

For a homogeneous solid $a_1 = 0$, $a^\circ = a_0 = A_{11}$, $c_0 = c$; hence $V = 0$, $Y = 0$, $\varphi = 0$ and (5.12) reduces to the known formula for the temperature θ.

Appendix

In this Appendix the proofs of propositions $(L0) \div (L4)$ and $(T1) \div (T4)$, which constitute the mathematical background of the tolerance averaging approach are given.

Lemma $(L0)$. *If* $F \in SV_\Delta(T) \cap C^1(\overline{\Omega})$ *then the estimation* $l|\partial_\alpha F| \leqslant \varepsilon_F + l\varepsilon_{\nabla F}$ *holds.*

Proof. Recall that the smooth function $F(\cdot)$ is called slowly varying, $F \in SV_\Delta(T)$, if

$$\forall \boldsymbol{x},\boldsymbol{y} \in \Omega \;\; [(\|\boldsymbol{x}-\boldsymbol{y}\| \leqslant l \Longrightarrow |DF(\boldsymbol{x}) - DF(\boldsymbol{y})| \leqslant \varepsilon_{DF}]$$

for every $DF \in \mathcal{F}(\Omega)$ where DF stands for F and for an arbitrary partial derivative $\partial_\alpha F$ which belongs to $\mathcal{F}(\overline{\Omega})$. Hence

$$\forall \boldsymbol{x},\boldsymbol{y} \in \Omega \;\; [(\|\boldsymbol{x}-\boldsymbol{y}\| \leqslant l \Longrightarrow |F(\boldsymbol{x}) - F(\boldsymbol{y})| \leqslant \varepsilon_F]$$

and

$$\forall \boldsymbol{x}, \boldsymbol{y} \in \Omega \ \ [(\|\boldsymbol{x}-\boldsymbol{y}\| < l \Longrightarrow |\partial_\alpha F(\boldsymbol{x}) - \partial_\alpha F(\boldsymbol{y})| \leqslant \varepsilon_{\nabla F}]$$

If $F \in C^1(\overline{\Omega})$ then for every vector $\boldsymbol{h}$ such that $|\boldsymbol{h}| \leqslant l$ and for $\boldsymbol{y} + \nu\boldsymbol{h} \in \Omega$, $\nu \in (0,1]$, and for some $\eta \in [0,1]$ we obtain

$$\nabla F(\boldsymbol{y} + \eta\boldsymbol{h}) \cdot \boldsymbol{h} = F(\boldsymbol{y}+\boldsymbol{h}) - F(\boldsymbol{y}) \qquad \boldsymbol{y} \in \Omega$$

Moreover, if $F \in SV_\Delta(T)$ then

$$\begin{aligned} &|\nabla F(\boldsymbol{y}) \cdot \boldsymbol{h}| = |\nabla F(\boldsymbol{y}+\eta\boldsymbol{h}) \cdot \boldsymbol{h} + [\nabla F(\boldsymbol{y}) - \nabla F(\boldsymbol{y}+\eta\boldsymbol{h})] \cdot \boldsymbol{h}| \leqslant \\ &\leqslant |\nabla F(\boldsymbol{y}+\eta\boldsymbol{h}) \cdot \boldsymbol{h}| + |[\nabla F(\boldsymbol{y}) - \nabla F(\boldsymbol{y}+\eta\boldsymbol{h})] \cdot \boldsymbol{h}| \leqslant \\ &\leqslant |F(\boldsymbol{y}+\boldsymbol{h}) - F(\boldsymbol{y})| + |[\nabla F(\boldsymbol{y}) - \nabla F(\boldsymbol{y}+\eta\boldsymbol{h})] \cdot \boldsymbol{h}| \leqslant \varepsilon_F + \varepsilon_{\nabla F}|\boldsymbol{h}| \end{aligned}$$

Let $\boldsymbol{h} = \boldsymbol{e}l$, where $\boldsymbol{e}$ is an arbitrary unit vector, $\|\boldsymbol{e}\| = 1$. Then the above estimate yields

$$|\nabla F(\boldsymbol{y}) \cdot \boldsymbol{e}| l \leqslant \varepsilon_F + l\varepsilon_{\nabla F}$$

For $\boldsymbol{e} = \boldsymbol{e}_\alpha$, where $\boldsymbol{e}_\alpha$ is the versor of the x_α-axis, $\alpha = 1,2,3$, we obtain $\nabla F(\boldsymbol{y}) \cdot \boldsymbol{e}_\alpha = \partial_\alpha F(\boldsymbol{y})$ and finally

$$l|\partial_\alpha F| \leqslant \varepsilon_F + l\varepsilon_{\nabla F}$$

which was to be proved.

Lemma ($L1$). *If $g \in PL_\Delta(T)$ and $g^\circ, \widetilde{g} \in \mathcal{F}(\overline{\Omega})$ then, for an arbitrary positive valued integrable Δ-periodic function ρ, the decomposition $g = g^\circ + \widetilde{g}$ exists, where $g^\circ \in SV_\Delta(T)$, $\widetilde{g} \in PL^\rho_\Delta(T)$.*

Proof. Setting

$$g^\circ(\cdot) = \langle\varrho\rangle^{-1}\langle\varrho g\rangle(\cdot)$$

by means of $(\varrho g)(\cdot) \in PL_\Delta(T)$ and after using ($L1$) we obtain

$$\langle\varrho g\rangle(\cdot) \in SV_\Delta(T)$$

Hence $g^\circ(\cdot)$ is a slowly varying function and the condition

$$\widetilde{g}(\boldsymbol{y}) = g(\boldsymbol{y}) - g^\circ(\boldsymbol{y}) \cong g(\boldsymbol{y}) - g^\circ(\boldsymbol{x}) \qquad \boldsymbol{y} \in B(\boldsymbol{x}, l) \cap \Omega$$

holds true for an arbitrary but fixed $\boldsymbol{x} \in \Omega$. It follows that $\widetilde{g}(\cdot)$ is also a periodic-like function. Moreover, by means of proposition ($T1$) (see below) we obtain

$$\langle\varrho\widetilde{g}\rangle = \langle\varrho g\rangle - \langle\varrho g^\circ\rangle \cong \langle\varrho g\rangle - \langle\varrho\rangle g^\circ$$

by the definition of g° the term $\langle \varrho g\rangle - \langle \varrho\rangle g^\circ$ is equal to zero and hence $\langle \varrho\widetilde{g}\rangle = 0$. Thus we jump to the conclusion that $\widetilde{g}$ is an oscillating periodic like function (with the weight ϱ) $\widetilde{g} \in PL^{\rho}_{\Delta}(T)$, which ends the proof of $(L1)$.

Lemma $(L2)$. *If* $\varphi \in PL_{\Delta}(T)$, $f \in L^{\infty}_{per}(\Delta)$ *and* $\langle f\varphi\rangle(\cdot) \in \mathcal{F}(\overline{\Omega})$ *then* $\langle f\varphi\rangle(\cdot) \in SV_{\Delta}(T)$.

Proof. If $\varphi \in PL_{\Delta}(T)$ then for every $\boldsymbol{y}_1$, $\boldsymbol{y}_2$ such that $\|\boldsymbol{y}_1 - \boldsymbol{y}_2\| \leqslant l$ and for $\boldsymbol{x} = (\boldsymbol{y}_1 + \boldsymbol{y}_2)/2$ we obtain

$$|\langle \varphi f\rangle(\boldsymbol{y}_1) - \langle \varphi_{\boldsymbol{x}} f\rangle(\boldsymbol{y}_1)| \leqslant \langle |f|\rangle \varepsilon_{\varphi}$$

$$|\langle \varphi f\rangle(\boldsymbol{y}_2) - \langle \varphi_{\boldsymbol{x}} f\rangle(\boldsymbol{y}_2)| \leqslant \langle |f|\rangle \varepsilon_{\varphi}$$

where $\varphi_{\boldsymbol{x}}(\cdot)$ is a periodic approximation of $\varphi(\cdot)$ on $\Delta(\boldsymbol{x})$. Because of

$$\langle \varphi_{\boldsymbol{x}} f\rangle(\boldsymbol{y}_1) = \langle \varphi_{\boldsymbol{x}} f\rangle(\boldsymbol{y}_2) = \text{const}$$

we conclude that the condition

$$|\langle \varphi f\rangle(\boldsymbol{y}_1) - \langle \varphi f\rangle(\boldsymbol{y}_2)| \leqslant 2\langle |f|\rangle \varepsilon_{\varphi}$$

holds for every $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \Omega$ such that $\|\boldsymbol{y}_1 - \boldsymbol{y}_2\| \leqslant l$. It follows that $\langle \varphi f\rangle(\cdot) \in SV_{\Delta}(T)$ with $\varepsilon_{\langle \varphi f\rangle} = 2\langle |f|\rangle \varepsilon_{\varphi}$, which ends the proof of $(L2)$.

Lemma $(L3)$. *If* $F \in SV_{\Delta}(T)$, $f \in C_{per}(\overline{\Delta})$ *and* $(fF)(\cdot) \in \mathcal{F}(\overline{\Omega})$ *then* $(fF)(\cdot) \in PL_{\Delta}(T)$.

Proof. If $F(\cdot)$ is a slowly varying function defined in Ω then for every $\boldsymbol{y} \in B(\boldsymbol{x}, l) \cap \Omega$ and every $\boldsymbol{x} \in \Omega$

$$|f(\boldsymbol{y})F(\boldsymbol{y}) - f(\boldsymbol{y})F(\boldsymbol{x})| = |f(\boldsymbol{y})||F(\boldsymbol{y}) - F(\boldsymbol{x})| \leqslant |f(\boldsymbol{y})|\varepsilon_F$$

Bearing in mind that $f \in C_{per}(\overline{\Delta})$ and setting

$$\varepsilon_{fF} = \varepsilon_F \max\{|f(\boldsymbol{y})| : \ \boldsymbol{y} \in \overline{\Delta}\}$$

we obtain

$$|f(\boldsymbol{y})F(\boldsymbol{y}) - f(\boldsymbol{y})F(\boldsymbol{x})| \leqslant \varepsilon_{fF} \qquad \boldsymbol{y} \in B(\boldsymbol{x}, l) \cap \Omega$$

It follows that $f(\boldsymbol{y})F(\boldsymbol{x})$, $\boldsymbol{y} \in B(\boldsymbol{x}, l) \cap \Omega$ can be treated as a certain periodic approximation of the function $(fF)(\cdot)$ on $\Delta(\boldsymbol{x})$, i.e., $f(\boldsymbol{y})F(\boldsymbol{x}) = (fF)_{\boldsymbol{x}}(\boldsymbol{y})$, $\boldsymbol{y} \in B(\boldsymbol{x}, l) \cap \Omega$; hence $(fF)(\cdot)$ is a periodic-like function. This ends the proof of $(L3)$.

Lemma ($L4$). *If* $F \in SV_\Delta(T)$, $G \in SV_\Delta(T)$ *and* $kF + mG \in \mathcal{F}(\overline{\Omega})$ *for some reals* k, m, *then* $kF + mG \in SV_\Delta(T)$.

Proof. It is easy to see that

$$|(kF+mG)(\boldsymbol{x}) - (kF+mG)(\boldsymbol{y})| = |k[F(\boldsymbol{x}) - F(\boldsymbol{y})] + m[G(\boldsymbol{x}) - G(\boldsymbol{y})]| \leqslant$$

$$\leqslant |k||F(\boldsymbol{x}) - F(\boldsymbol{y})| + |m||G(\boldsymbol{x}) - G(\boldsymbol{y})|$$

holds for every $\boldsymbol{x}, \boldsymbol{y} \in \Omega$. If $\|\boldsymbol{x} - \boldsymbol{y}\| \leqslant l$ then by means of $F, G \in SV_\Delta(T)$ we obtain

$$|(kF+mG)(\boldsymbol{x}) - (kF+mG)(\boldsymbol{y})| \leqslant |k|\varepsilon_F + |m|\varepsilon_G$$

If $kF(\cdot) + mG(\cdot) \in \mathcal{F}(\overline{\Omega})$ then setting

$$\varepsilon_{kF+mG} = |k|\varepsilon_F + |m|\varepsilon_G$$

we conclude that $kF(\cdot) + mG(\cdot)$ is a slowly varying function. This ends the proof of ($L4$).

Assertion. *If* $F \in SV_\Delta(T)$, $\varphi \in PL_\Delta(T)$ *and* $\varphi_{\boldsymbol{x}}$ *is a* Δ*-periodic approximation of* φ *on* $\Delta(\boldsymbol{x})$ *then for every* $f \in L^\infty_{per}(\Delta)$ *and* $h \in C^1_{per}(\overline{\Delta})$, *such that* $\max\{|h(\boldsymbol{y})| : \boldsymbol{y} \in \overline{\Delta}\} \leqslant l$, *the following propositions hold for every* $\boldsymbol{x} \in \Omega_\Delta$:

($T1$)	$\langle fF\rangle(\boldsymbol{x}) \cong \langle f\rangle F(\boldsymbol{x})$	*for*	$\varepsilon = \langle |f|\rangle \varepsilon_F$
($T2$)	$\langle f\varphi\rangle(\boldsymbol{x}) \cong \langle f\varphi_{\boldsymbol{x}}\rangle(\boldsymbol{x})$	*for*	$\varepsilon = \langle |f|\rangle \varepsilon_\varphi$
($T3$)	$\langle f\nabla(hF)\rangle(\boldsymbol{x}) \cong \langle fF\nabla h\rangle(\boldsymbol{x})$	*for*	$\varepsilon = \langle |f|\rangle (\varepsilon_F + l\varepsilon_{\nabla F})$
($T4$)	$\langle h\nabla(f\varphi)\rangle(\boldsymbol{x}) \cong -\langle f\varphi\nabla h\rangle(\boldsymbol{x})$	*for*	$\varepsilon = \varepsilon_G + l\varepsilon_{\nabla G}$ $G = \langle hf\varphi\rangle l^{-1}$

where ε *is the tolerance parameter which defines the pertinent tolerance* $\cong$ *and* $G, \partial_\alpha G \in \mathcal{F}(\overline{\Omega})$.

Proof

($T1$) If $F \in SV_\Delta(T)$ and $f \in L^\infty_{per}(\Delta)$ then for every $\boldsymbol{x} \in \Omega_\Delta$ we obtain

$$|\langle fF\rangle(\boldsymbol{x}) - \langle f\rangle F(\boldsymbol{x})| =$$

$$= \frac{1}{|\Delta|}\Big|\int_\Delta [f(\boldsymbol{x}+\boldsymbol{y})F(\boldsymbol{x}+\boldsymbol{y}) - f(\boldsymbol{x}+\boldsymbol{y})F(\boldsymbol{x})]\, d\Delta(\boldsymbol{y})\Big| \leqslant$$

$$\leqslant \frac{1}{|\Delta|}\int_\Delta |f(\boldsymbol{x}+\boldsymbol{y})[F(\boldsymbol{x}+\boldsymbol{y}) - F(\boldsymbol{x})]|\, d\Delta(\boldsymbol{y})$$

By means of $F \in SV_\Delta(T)$, for every $\boldsymbol{y} \in \Delta$ and for every $\boldsymbol{x} \in \Omega_\Delta$

$$|F(\boldsymbol{x}+\boldsymbol{y}) - F(\boldsymbol{x})| \leqslant \varepsilon_F$$

Hence

$$|\langle fF\rangle(\boldsymbol{x}) - \langle f\rangle F(\boldsymbol{x})| \leqslant \langle |f|\rangle \varepsilon_F$$

for every $\boldsymbol{x} \in \Omega_\Delta$. This ends the proof of $(T1)$

$(T2)$ If $f \in L^\infty_{per}(\Delta)$, $\varphi \in PL_\Delta(T)$ and $\varphi_{\boldsymbol{x}}$ is a certain Δ-periodic approximation of φ on $\Delta(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega_\Delta$, then we obtain

$$\begin{aligned} &|\langle f\varphi\rangle(\boldsymbol{x}) - \langle f\varphi_{\boldsymbol{x}}\rangle(\boldsymbol{x})| = \\ &= \frac{1}{|\Delta|}\Big|\int_\Delta [f(\boldsymbol{x}+\boldsymbol{y})\varphi(\boldsymbol{x}+\boldsymbol{y}) - f(\boldsymbol{x}+\boldsymbol{y})\varphi_{\boldsymbol{x}}(\boldsymbol{x}+\boldsymbol{y})]\; d\Delta(\boldsymbol{y})\Big| \leqslant \\ &\leqslant \frac{1}{|\Delta|}\int_\Delta |f(\boldsymbol{x}+\boldsymbol{y})[\varphi(\boldsymbol{x}+\boldsymbol{y}) - \varphi_{\boldsymbol{x}}(\boldsymbol{x}+\boldsymbol{y})]|\; d\Delta(\boldsymbol{y}) \end{aligned}$$

By means of $\varphi \in PL_\Delta(T)$ we obtain $|\varphi(\boldsymbol{x}+\boldsymbol{y}) - \varphi_{\boldsymbol{x}}(\boldsymbol{x}+\boldsymbol{y})| \leqslant \varepsilon_\varphi$ for every $\boldsymbol{x} \in \Omega_\Delta$ and every $\boldsymbol{y} \in \Delta$. Hence

$$|\langle f\varphi\rangle(\boldsymbol{x}) - \langle f\varphi_{\boldsymbol{x}}\rangle(\boldsymbol{x})| \leqslant \langle |f|\rangle \varepsilon_\varphi$$

which ends the proof of $(T2)$.

$(T3)$ If $F \in SV_\Delta(T)$, $f \in L^\infty_{per}(\Delta)$ and $h \in C^1_{per}(\overline{\Delta})$, such that $\max\{|h(\boldsymbol{y})| : \ \boldsymbol{y} \in \overline{\Delta}\} \leqslant l$ then, bearing in mind that $l|\partial_\alpha F| \leqslant \varepsilon_F + l\varepsilon_{\nabla F}$ (see $(L0)$), for every $\boldsymbol{x} \in \Omega_\Delta$ we obtain

$$\begin{aligned} &|\langle f\partial_\alpha(hF)\rangle(\boldsymbol{x}) - \langle fF\partial_\alpha h\rangle(\boldsymbol{x})| = \\ &= \frac{1}{|\Delta|}\Big|\int_\Delta [f(\boldsymbol{x}+\boldsymbol{y})\partial_\alpha(hF)(\boldsymbol{x}+\boldsymbol{y}) - f(\boldsymbol{x}+\boldsymbol{y})(F\partial_\alpha h)(\boldsymbol{x}+\boldsymbol{y})]\; d\Delta(\boldsymbol{y})\Big| \leqslant \\ &\leqslant \frac{1}{|\Delta|}\int_\Delta |f(\boldsymbol{x}+\boldsymbol{y})[\partial_\alpha(hF)(\boldsymbol{x}+\boldsymbol{y}) - (F\partial_\alpha h)(\boldsymbol{x}+\boldsymbol{y})]|\; d\Delta(\boldsymbol{y}) = \\ &= \frac{1}{|\Delta|}\int_\Delta |(fh\partial_\alpha F(\boldsymbol{x}+\boldsymbol{y})|\; d\Delta(\boldsymbol{y}) \leqslant \langle |f|\rangle(\varepsilon_F + l\varepsilon_{\nabla F}) \end{aligned}$$

This shows that

$$|\langle f\partial_\alpha(hF)\rangle(\boldsymbol{x}) - \langle fF\partial_\alpha h\rangle(\boldsymbol{x})| \leqslant \langle |f|\rangle(\varepsilon_F + l\varepsilon_{\nabla F}) \qquad \alpha = 1,2,3$$

which ends the proof of $(T3)$.

$(T4)$ The difference between both sides of $(T4)$ is equal to

$$|\langle h\partial_\alpha(f\varphi)\rangle(\boldsymbol{x}) + \langle f\varphi\partial_\alpha h\rangle(\boldsymbol{x})| = |\langle\partial_\alpha(f\varphi h)\rangle(\boldsymbol{x})|$$

Setting $G = \langle hf\varphi\rangle l^{-1}$ and using $(L2)$ we conclude that $G \in SV_\Delta(T)$. Bearing in mind that $l|\partial_\alpha G| \leqslant \varepsilon_G + l\varepsilon_{\nabla G}$ (see $(L0)$) we obtain

$$|\langle h\partial_\alpha(f\varphi)\rangle(\boldsymbol{x}) + \langle f\varphi\partial_\alpha h\rangle(\boldsymbol{x})| = l|\partial_\alpha G(\boldsymbol{x})| \leqslant \varepsilon_G + l\varepsilon_{\nabla G} \qquad \alpha = 1,2,3$$

which ends the proof of $(T4)$.

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Uśrednianie tolerancyjne i równania warstwy brzegowej dla przewodnictwa ciepła w ośrodku mikroperiodycznym

Streszczenie

Modele makroskopowe przewodnictwa ciepła w ośrodku mikroperiodycznym, sformułowane na drodze uśredniania tolerancyjnego (modele tolerancyjne), są reprezentowane równaniem różniczkowym cząstkowym dla uśrednionego pola temperatury oraz równaniami rózniczkowymi zwyczajnymi zawierającymi pochodne czasowe pewnych

dodatkowych niewiadomych, zwanych termicznymi zmiennymi wewnętrznymi, Woźniak (2000). Niewiadome te opisują zaburzenia pola temperatury spowodowane periodyczną mikroniejednorodnością ośrodka. Tym samym w ramach modeli tolerancyjnych, warunki brzegowe mogą być określone tylko dla uśrednionego pola temperatury. Celem opracowania jest pokazanie, jak metoda tolerancyjnego uśredniania równań może zostać rozszerzona w celu zapewnienia dokładniejszego spełnienia warunków brzegowych.

Manuscript received October 23, 2000; accepted for print January 4, 2001