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DYNAMIC RESPONSE OF A MICRO-PERIODIC BEAM UNDER MOVING LOAD – DETERMINISTIC AND STOCHASTIC APPROACH

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In the paper, the deterministic and stochastic approach to the problem of vibrations of a beam with periodically varying geometry under moving load is presented. A new averaged model for the dynamics of the periodic-like beam with a variable cross-section, Mazur-Śniady (2001), is applied. The approach to dynamics of the periodic-like beam assumed in the paper is based on concepts of the tolerance-averaged model by Woźniak (1999). The solution obtained for a single moving force is the basis of solution of stochastic vibrations caused by random train of moving forces.

Key words: dynamics, moving load, stochastic processes, periodic-like beam

1. Introduction

The determination of vibrations caused by a moving load is one of the classical problems of structural mechanics. The problem appears in dynamics of bridges, railways, roads, landing strips, etc. Numerous papers in this field were published. One of the first problems was the determination of vibrations of a beam under moving load. Well known papers by Krylow (1905) and Timoshenko (1922) describe the vibrations of a simply supported beam caused by the force moving along the beam with a constant velocity. For this beam Kączkowski (1963), using the method of superposition of deflections of the beam axis, proved that the part of solution describing aperiodic vibrations can be written in a closed form. The closed-form solutions for beams and frames with different support conditions were given by Reipert (1969, 1970). The problems of finding solutions in the closed form were presented also by Mazur and Śniady (1973) and Śniady (1976).

The deterministic solution obtained for a single moving force can be applied to the problem of stochastic vibrations of a beam under random train of forces travelling in the same direction with equal and constant velocity.

The problem of stochastic vibrations and reliability of the beam was analyzed in papers by Tung (1967), Iwankiewicz and Śniady (1984), Bryja and Śniady (1988), Śniady (1976), Sieniawska and Śniady (1990), Śniady et al. (1998, 2001). The model given above can be applied to the reliability analysis of bridge beams under traffic flow.

In this paper, the deterministic and stochastic approaches to the problem of vibrations of beam with periodically varying geometry under moving load are presented.

The standard methods of analyzing the beam dynamics are effective only if the coefficients in the well-known differential equation of the beam are constant or slowly varying. If the coefficients of this equation are varying rapidly then the solution is rather difficult to obtain.

The approach presented in the paper is an application of the tolerance-averaged model by Woźniak (1999). In this way, Mazur-Śniady (2001) has formulated equations of the periodic-like beam in the form of a system of averaged differential equations with slowly varying or (for the periodically variable beam) constant coefficients which describe the length scale effect. In contrast, in the classical homogenization theories this effect disappears, cf. for instance Bensoussan et al. (1978), Jikov et al. (1994), Sanchez-Palencia (1980).

In this paper the vibrations of the beam with periodic structure under moving load are analyzed. The solution obtained for a single moving force was used to the problem of stochastic vibrations caused by random train of moving forces.

Since the theoretical problem of obtaining the analytical solution is very complicated, the solution reduces to the form, which admits the numerical analysis by using modern computational equipment (for example the Mathematica package).

2. Periodic-like beam

We consider vibrations under moving load of the periodic-like straight beam with varying cross-section. The axis of the beam coincides with the interval $[0, L]$ of the x-axis in $0xyz$-space and the beam has the $0xy$-plane as the symmetry plane. The equation of the beam has the well known form (under assumptions of the Euler-Bernoulli linear elastic beam theory)

$$[B(x)v''(x,t)]'' + c(x)\dot{v}(x,t) + \rho(x)\ddot{v}(x,t) = p(x,t) \tag{2.1}$$

where $(\cdot)' = \partial(\cdot)/\partial x$, $\dot{(\cdot)} = \partial(\cdot)/\partial t$ and

$v(x,t)$	–	deflection of the beam axis
$B(x)$	–	flexural beam rigidity
$c(x)$	–	damping coefficient
$\rho(x)$	–	mass density per unit length
$p(x,t)$	–	loading process.

The standard methods of analyzing the beam dynamics are effective only if the coefficients in the equation (1.1) are constant or slowly varying. If the coefficients $B(\cdot)$, $c(\cdot)$, $\rho(\cdot)$ are rapidly varying functions then the solution to the equation (1.1) is rather difficult to obtain. We restrict our considerations to beams for which the rapidly varying functional coefficients $B(\cdot)$, $c(\cdot)$, $\rho(\cdot)$ are represented by periodic-like functions. It means that there exists a slowly varying function $l = l(x)$, $x \in [0,L]$, $\max l(x) \ll L$, such that in every interval $\Delta x = \big(x - l(x)/2, x + l(x)/2\big)$, $\Delta x \in [0,L]$, the functions $B(\cdot)$, $c(\cdot)$, $\rho(\cdot)$ can be approximated respectively by certain $l = l(x)$-periodic functions $B_x(\xi)$, $c_x(\xi)$, $\rho_x(\xi)$, $\xi \in [x - l/2, x + l/2]$. Moreover, all cross-sectionall dimensions of the beam must be small compared to $\max l(x)$, $x \in [0,L]$. In a special case of $l =$ const we consider beams with an l-periodic structure.

Functions will be averaged by means of the formula

$$\langle \varphi \rangle = \frac{1}{l} \int\limits_{x-\frac{l}{2}}^{x+\frac{l}{2}} \varphi(\xi)\, d\xi \qquad x \in \Omega_0 \qquad \Omega_0 = \{x \in \Omega : \ \Delta(x) \in \Omega\} \tag{2.2}$$

where $l = l(x)$ and $\varphi(\cdot)$ is an arbitrary integrable function defined on $\Omega = (0,L)$.

If the function φ is l-periodic then $\langle \varphi \rangle =$ const. For φ depending also on the time variable, we shall also write $\langle \varphi \rangle$ instead of $\langle \varphi \rangle(x,t)$.

The proposed model is based on the physical assumption that the deflection of the l-periodic beam is an l-periodic-like function

$$v(\cdot,t) \in PL(l) \tag{2.3}$$

It means that in every interval Δx, $\Delta x \in [0,L]$, the deflection of the beam can be approximately represented by $v(\xi,t) \cong v_x(\xi,t)$, $\xi \in \Delta x$, and hence it be treated as $l = l(x)$ periodic in this interval.

Let us define the averaged deflection $w(x,t)$

$$w(x,t) = \langle\rho\rangle^{-1}(x)\langle\rho v\rangle(x,t) \qquad x \in \Omega_0 \tag{2.4}$$

Hence the total deflection of the beam can be represented by a sum

$$v(x,t) = w(x,t) + d(x,t) \qquad x \in \Omega_0 \tag{2.5}$$

The modeling decomposition (2.5) is a simple consequence of the assumption (2.3) and makes it possible to introduce two kinds of basic unknowns, namely a function $w(\cdot,t)$ which is a slowly varying function, and $d(\cdot,t)$ which is an oscillating l-periodic-like function (with the weight ρ).

The deflection disturbance function $d(\cdot,t)$ is assumed to be in the form of the series

$$d(x,t) = h^A(x)\psi_A(x,t) \qquad x \in \Omega_0 \tag{2.6}$$

(the summation convention over $A = 1,2,...$ holds), where $h^A(\cdot)$ are the a priori known oscillating l-periodic-like functions and the new unknown amplitude of the shape functions $\psi_A(x,t)$ are sufficiently regular and slowly varying functions. It was proved by Mazur-Śniady (2001) that

$$\langle\rho h^A\rangle = 0 \tag{2.7}$$

Based on concepts of the tolerance-averaged model (Woźniak, 1999), the system of $n+1$ differential equations with slowly varying or (for l-periodic beam) constant coefficients was obtained by Mazur-Śniady (2001)

$$\begin{aligned}
&\Big[\langle B\rangle w''(x,t) + \langle B(h^A)''\rangle\psi_A(x,t)\Big]'' + \langle c\rangle\dot{w}(x,t) + \langle ch^A\rangle\dot{\psi}_A(x,t) + \\
&+\langle\rho\rangle\ddot{w}(x,t) = \langle p\rangle(x,t) \\
&\langle B(h^B)''\rangle w''(x,t) + \langle B(h^B)''(h^A)''\rangle\psi_A(x,t) + \langle ch^B\rangle\dot{w}(x,t) + \\
&+\langle ch^Bh^A\rangle\dot{\psi}_A(x,t) + \langle\rho h^Bh^A\rangle\ddot{\psi}_A(x,t) = \langle ph^B\rangle(x,t)
\end{aligned} \tag{2.8}$$

Equations (2.8) hold for $x \in (0,L)$. The boundary conditions are similar to those formulated in the Euler-Bernoulli beam theory.

For the initial-value problem, suitable initial conditions for ψ_A and w should be known.

3. Vibrations of the beam under moving force

Let us consider vibrations of a simply supported beam with periodically varying cross-section under force Q moving along the beam axis with the velocity $u =$ const. In this case, in equations (2.8) the loading process is

$$p(x,t) = Q\delta(x-ut) \tag{3.1}$$

where $\delta(\cdot)$ is the Dirac function.

For the beam with periodic variable cross-section, after taking $A = B = 1$, $\psi_A(x,t) = \psi_B(x,t) = \psi(x,t)$, $h_A(x) = h_B(x) = h(x)$, we obtain Eqs (2.8) in the form of the system of two equations with constant coefficients

$$\begin{aligned}
&\langle B\rangle w^{IV}(x,t) + \langle Bh''\rangle\psi''(x,t) + \langle c\rangle\dot{w}(x,t) + \langle ch\rangle\dot{\psi}(x,t) + \\
&+\langle\rho\rangle\ddot{w}(x,t) = \langle p\rangle(x,t) \\[2ex]
&\langle Bh''\rangle w''(x,t) + \langle B(h'')^2\rangle\psi(x,t) + \langle ch\rangle\dot{w}(x,t) + \\
&+\langle ch^2\rangle\dot{\psi}(x,t) + \langle\rho h^2\rangle\ddot{\psi}(x,t) = \langle ph\rangle(x,t)
\end{aligned} \tag{3.2}$$

Eqs (3.2) hold for $x \in (0,L)$. For the simply supported beam we assume functions $w(x,t)$ and $\psi(x,t)$ to be in the form of expansion in a sine series

$$w(x,t) = \sum_{k=1}^{\infty} y_k(t)\sin\frac{k\pi x}{L} \qquad\qquad \psi(x,t) = \sum_{k=1}^{\infty} q_k(t)\sin\frac{k\pi x}{L} \tag{3.3}$$

In the orthogonalization process we take into account the equation (3.3) as well as the following relations

$$\begin{aligned}
&\int\limits_0^L \langle p(x,t)\rangle\sin\frac{k\pi x}{L}\;dx = Q\sin\frac{k\pi ut}{L} \\[2ex]
&\int\limits_0^L \langle p(x,t)h(x)\rangle\sin\frac{k\pi x}{L}\;dx = 0
\end{aligned} \tag{3.4}$$

obtaining the set of Eqs (3.2) in the following form

$$\begin{aligned}
&\ddot{y}_k(t) + \frac{\langle c\rangle}{\langle\rho\rangle}\dot{y}_k(t) + \frac{\langle B\rangle}{\langle\rho\rangle}\Big(\frac{k\pi}{L}\Big)^4 y_k(t) + \frac{\langle ch\rangle}{\langle\rho\rangle}\dot{q}_k(t) + \\
&-\frac{\langle Bh''\rangle}{\langle\rho\rangle}\Big(\frac{k\pi}{L}\Big)^2 q_k(t) = \frac{2Q}{L\langle\rho\rangle}\sin\frac{k\pi ut}{L} \\[2ex]
&\ddot{q}_k(t) + \frac{\langle ch^2\rangle}{\langle\rho h^2\rangle}\dot{q}_k(t) + \frac{\langle B(h'')^2\rangle}{\langle\rho h^2\rangle}q_k(t) + \frac{\langle ch\rangle}{\langle\rho h^2\rangle}\dot{y}_k(t) + \\
&-\frac{\langle Bh''\rangle}{\langle\rho h^2\rangle}\Big(\frac{k\pi}{L}\Big)^2 y_k(t) = 0
\end{aligned} \tag{3.5}$$

The initial conditions have the form

$$\begin{aligned} y_k(0) &= 0 \qquad & \dot{y}_k(0) &= 0 \\ q_k(0) &= 0 \qquad & \dot{q}_k(0) &= 0 \end{aligned} \tag{3.6}$$

The exact analytical solution to Eqs (3.5) is very complicated and for this reason it is better to determine the numerical results using the Mathematica package.

Let us consider the undamped vibrations of the beam with periodically varying cross-section under moving force (in this case $c \equiv 0$). We introduce dimensionless variables

$$\eta = \frac{x}{L} \qquad\qquad T = \frac{ut}{L} \tag{3.7}$$

for $0 \leqslant \eta \leqslant 1$, $0 \leqslant T \leqslant 1$.

The set of Eqs (3.2) for the loading process (3.1) after simple transformations takes the following form

$$\begin{aligned} &\frac{\partial^4 w(\eta,T)}{\partial \eta^4} + \frac{\langle Bh''\rangle L^2}{\langle B\rangle}\frac{\partial^2 \psi(\eta,T)}{\partial \eta^2} + \frac{\langle \rho\rangle u^2 L^2}{\langle B\rangle}\frac{\partial^2 w(\eta,T)}{\partial T^2} = \frac{QL^3}{\langle B\rangle}\delta(\eta - T) \\ &\langle Bh''\rangle \frac{\partial^2 w(\eta,T)}{\partial \eta^2} + \langle B(h'')^2\rangle L^2 \psi(\eta,T) + \langle \rho h^2\rangle u^2 \frac{\partial^2 \psi}{\partial T^2} = 0 \end{aligned} \tag{3.8}$$

The solution to Eqs (3.8) being functions $w(\eta,T)$ and $\psi(\eta,T)$, can be presented as the following sums (Kączkowski, 1963)

$$\begin{aligned} w(\eta,T) &= w_I(\eta,T) + w_{II}(\eta,T) \\ \psi(\eta,T) &= \psi_I(\eta,T) + \psi_{II}(\eta,T) \end{aligned} \tag{3.9}$$

where $w_I(\eta,T)$, $\psi_I(\eta,T)$ describe aperiodic vibrations and $w_{II}(\eta,T)$, $\psi_{II}(\eta,T)$ describe free vibrations of the beam. The functions $w_{II}(\eta,T)$, $\psi_{II}(\eta,T)$ enable us fulfill the apropriate initial conditions.

For the simply supported beam we assume functions $w(\eta,T)$ and $\psi(\eta,T)$ to be in the form of expansion in a sine series (similarly to the expressions (3.3))

$$w(\eta,T) = \sum_{k=1}^{\infty} y_k(T)\sin k\pi\eta \qquad\qquad \psi(\eta,T) = \sum_{k=1}^{\infty} q_k(T)\sin k\pi\eta \tag{3.10}$$

Taking into account the above relations and using orthogonalization, we obtain the set of Eqs (3.8) in the following form

$$\begin{aligned}&\frac{\langle\rho\rangle u^2 L^2}{\langle B\rangle}\frac{d^2 y_k(T)}{dT^2} + (k\pi)^4 y_k(T) - \frac{\langle Bh''\rangle L^2}{\langle B\rangle}(k\pi)^2 q_k(T) = \frac{2QL^3}{\langle B\rangle}\sin k\pi T\\ &\langle\rho h^2\rangle u^2 \frac{d^2 q_k(T)}{dT^2} + \langle B(h'')^2\rangle L^2 q_k(T) - \langle Bh''\rangle(k\pi)^2 y_k(T) = 0\end{aligned} \tag{3.11}$$

Determining the particular integral of the set of Eqs (3.11), we obtain the solution for the aperiodic vibrations of the beam

$$\begin{aligned}w_I(\eta,T) &= \frac{2QL^3}{\langle B\rangle}\sum_{k=1}^{\infty}\frac{\sin k\pi T\sin k\pi\eta}{(k\pi)^2 C}\\ \psi_I(\eta,T) &= \frac{2QL^3\langle Bh''\rangle}{\langle B\rangle}\sum_{k=1}^{\infty}\frac{(k\pi)^2}{\langle B(h'')^2\rangle - \langle\rho h^2\rangle u^2(k\pi)^2}\frac{\sin k\pi T\sin k\pi\eta}{(k\pi)^2 C}\end{aligned} \tag{3.12}$$

where

$$C = (k\pi)^2 - \frac{\langle\rho\rangle u^2 L^2}{\langle B\rangle} + \frac{(k\pi)^2\langle Bh''\rangle^2}{\langle B\rangle[\langle\rho h^2\rangle u^2(k\pi)^2 - 2\langle B(h'')^2\rangle]}$$

It is easy to see that the functions $w_I(\eta,T)$ and $\psi_I(\eta,T)$ do not satisfy the initial conditions, and that is why the solution to Eqs (3.11) should contain additional functions $w_{II}(\eta,T)$ and $\psi_{II}(\eta,T)$, fulfilling the initial conditions. We expand these functions into a sine series

$$w_{II}(\eta,T) = \sum_{k=1}^{\infty} y_{II\,k}(T)\sin k\pi\eta \qquad\qquad \psi_{II}(\eta,T) = \sum_{k=1}^{\infty} q_{II\,k}(T)\sin k\pi\eta \tag{3.13}$$

The functions $y_{II\,k}(T)$ and $q_{II\,k}(T)$ are obtained from the homogeneous set of Eqs (3.2) together with the initial conditions

$$\begin{aligned}&y_{II\,k}(0) = 0 \qquad\qquad q_{II\,k}(0) = 0\\ &\frac{dy_{II\,k}}{dT}\Big|_{T=0} = \frac{2QL^3}{\langle B\rangle k\pi C}\\ &\frac{dq_{II\,k}}{dT}\Big|_{T=0} = \frac{2QL^3\langle Bh''\rangle}{\langle B\rangle[\langle B(h'')^2\rangle - \langle\rho h^2\rangle u^2(k\pi)^2]C}\end{aligned} \tag{3.14}$$

It is worth to notice that functions $w_I(\eta,T)$ and $\psi_I(\eta,T)$ satisfy the following relations

$$\frac{\partial^2 w_I(\eta,T)}{\partial\eta^2} = \frac{\partial^2 w_I(\eta,T)}{\partial T^2} \qquad\qquad \frac{\partial^2\psi_I(\eta,T)}{\partial\eta^2} = \frac{\partial^2\psi_I(\eta,T)}{\partial T^2} \tag{3.15}$$

It is reason (cf Śniady, 1976; Mazur and Śniady, 1973), that the functions, describing the aperiodic vibrations $w_I(\eta,T)$ and $\psi_I(\eta,T)$, satisfy not only the set of the partial differential equations (3.11) but also the following set of the ordinary differential equations

$$\frac{d^4 w_I(\eta,T)}{d\eta^4}+\frac{\langle\rho\rangle u^2L^2}{\langle B\rangle}\frac{d^2 w_I(\eta,T)}{d\eta^2}+\frac{\langle Bh''\rangle L^2}{\langle B\rangle}\frac{d^2\psi_I(\eta,T)}{d\eta^2}=\frac{QL^3}{\langle B\rangle}\delta(\eta-T) \tag{3.16}$$

$$\langle Bh''\rangle\frac{d^2 w_I(\eta,T)}{d\eta^2}+\langle B(h'')^2\rangle L^2\psi_I(\eta,T)+\langle\rho h^2\rangle u^2L^2\frac{\partial^2\psi_I(\eta,T)}{\partial\eta^2}=0$$

In Eqs (3.16) the variable T is a parameter of the position of the moving force in the basis of dimensionless variables η, T. The ordinary differential equations (3.16) enable us obtain the aperiodic vibrations determinated by functions $w_I(\eta,T)$ and $\psi_I(\eta,T)$ to find the closed form solution (Kączkowski, 1963).

Analyzing the vibrations of the beam by means of modern computational methods (for example the Mathematica package) is also easier to use the set of the ordinary differential equations (3.16) instead of the set of partial differential equations (3.8).

Finally let us find the critical velocities of the moving force. We determine the lowest critical velocity. In the case of $k=1$ we obtain two critical velocities. The first one is characteristic of the segment of periodicity of the beam and is equal to

$$u_{kr}^m=\frac{1}{\pi}\sqrt{\frac{\langle B(h'')^2\rangle}{\langle\rho h^2\rangle}} \tag{3.17}$$

the second one is characteristic of the whole beam and fulfills the equation

$$\pi^2-\frac{\langle\rho\rangle u_{kr}^2L^2}{\langle B\rangle}+\frac{\pi^2\langle Bh''\rangle^2}{\langle B\rangle[\langle\rho h^2\rangle u_{kr}^2\pi^2-\langle B(h'')^2\rangle]}=0 \tag{3.18}$$

If the we take under consideration the following relation

$$\frac{\pi^2\langle Bh''\rangle^2}{\langle\rho h^2\rangle u_{kr}^2\pi^2-\langle B(h'')^2\rangle}\ll\langle\rho\rangle L^2 \tag{3.19}$$

then we obtain the aproximate value of critical velocity

$$u_{kr}\cong\frac{\pi}{L}\sqrt{\frac{\langle B\rangle}{\langle\rho\rangle}} \tag{3.20}$$

4. Stochastic vibrations of the beam

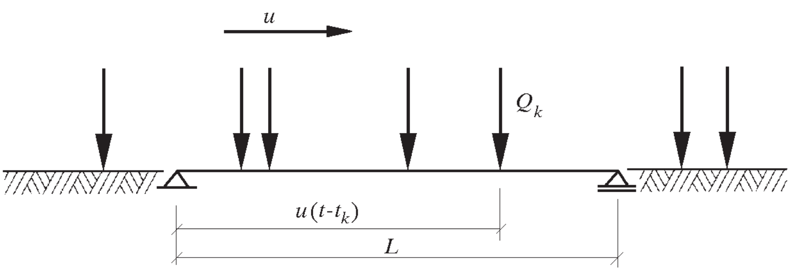

Fig. 1. The beam loaded by a random train of forces travelling in the same direction, all with equal, constant velocities u

Let us consider stochastic vibrations of a beam caused by a random train of forces travelling in the same direction, all with equal, constant velocities u (see Fig. 1). The forces Q_i arrive at the beam at random times t_i, and this constitutes a Poisson process $N(t)$, and $dN(t)$ denotes the number of forces arrivaling within time intervals $(0,t)$ and $(t, t+dt)$, respectively, and $P\{\cdot\}$ denotes the probability of the event and $E[\cdot]$ denotes the expected value of the quantity in brackets. The properties of the Poisson process are as follows

$$\begin{aligned} P\{dN(t) = 1\} &= \lambda dt + o(dt) \\ P\{dN(t) = 0\} &= 1 - \lambda dt + o(dt) \\ P\{dN(t) > 1\} &= o(dt) \end{aligned} \tag{4.1}$$

and consequently

$$E[dN^k(t)] = \lambda dt \qquad\qquad k = 1, 2, ... \tag{4.2}$$

where the parameter λ is the expected arrival rate of moving forces.

The loading process assumed can be presented as follows

$$p(x,t) = \sum_{i=1}^{N(t)} Q_i \delta[x - u(t - t_i)] \tag{4.3}$$

The amplitudes Q_i are assumed to be random variables that are mutually independent and independent of the times t_i, hence we shall assume the expected values $E[Q_i^r] = E[Q^r] =$ const $(r = 1, 2, ...)$ to be known.

Let the dynamic influence function $H(x, t - t_i)$ denotes the response of the beam at time t to the moving forces $Q_i = 1$, their arrival times being t_i.

The dynamic influence function depends on the velocity u and has two different forms.

If $t_i \leqslant t \leqslant t_i + L/u$ (i.e. the force is on the beam), $H(x,t-t_i) = H_1(x,t-t_i)$, and if $t > t_i + L/u$ (i.e. the force has left the beam – free vibrations), $H(x,t-t_i) = H_2(x,t-t_i-L/u)$.

The influence function $H(x,t-t_i) = H_1(x,t-t_i)$ is equal to the function $v(x,t)$ found in Section 3 for vibrations of the beam when $Q = 1$, and instead of the t we should introduce the time $t-t_i$. The second part of the influence function $H(x,t-t_i) = H_2(x,t-t_i-L/u)$ satisfies the homogeneous system of equations (3.2) (for $p(x,t) \equiv 0$) and the initial conditions for $t = t_i + L/u$ respectively

$$H_2(x,0) = H_1\Big(x,\frac{L}{u}\Big) \qquad\qquad \dot{H}_2(x,0) = \dot{H}_1\Big(x,\frac{L}{u}\Big) \tag{4.4}$$

The stochastic deflection $v(x,t)$ of the beam is a filtered Poisson process and can be presented in the form of the Stieltjes integral

$$\begin{aligned} v(x,t) &= \sum_{i=1}^{N(t)} QH(x,t-t_i) = \int\limits_0^t H(x,t-\tau)\,dN(\tau) = \\ &= \int\limits_{t-\frac{L}{u}}^{t} QH_1(x,t-\tau)\,dN(\tau) = \int\limits_0^{t-\frac{L}{u}} QH_2(x,t-\tau)\,dN(\tau) \end{aligned} \tag{4.5}$$

Taking into account relations (2.5) and (2.6) we obtains the dynamic influence function in the form (for $Q = 1$)

$$H(x,t-t_i) = w(x,t-t_i) + h(x)\psi(x,t-t_i) \tag{4.6}$$

and in view of (4.5) we have

$$\begin{aligned} v(x,t) &= \int\limits_0^t Q(\tau)H(x,t-\tau)\,dN(\tau) = \\ &= \int\limits_0^t Q(\tau)[w(x,t-\tau) + h(x)\psi(x,t-\tau)]\,dN(\tau) \end{aligned} \tag{4.7}$$

The expected value and variance of the deflection $v(x,t)$ can be obtained by taking into account equations (4.1) and (4.2). This yields

$$
\begin{aligned}
E[v(x,t)] &= E[Q]\lambda \int\limits_0^t H(x,t-\tau)\,dN(\tau) = \\
&= E[Q]\lambda \int\limits_0^t [w(x,t-\tau)+h(x)\psi(x,t-\tau)]\,d(\tau)
\end{aligned}
\tag{4.8}
$$

and the variance

$$
\sigma_v^2(x) = E[Q^2]\lambda \int\limits_0^t [w(x,t-\tau)+h(x)\psi(x,t-\tau)]^2\,d(\tau) \tag{4.9}
$$

The general, the cumulants of order k have the form

$$
\kappa_v^{(k)}(x) = E[Q^k]\lambda \int\limits_0^t [w(x,t-\tau)+h(x)\psi(x,t-\tau)]^k\,d(\tau) \tag{4.10}
$$

By analogy to equation (4.9) the variance of the velocity of the beam has the form

$$
\sigma_{\dot{v}}^2(x) = E[Q^2]\lambda \int\limits_0^t \Big[\frac{dw(x,t-\tau)}{dt} + h(x)\frac{d\psi(x,t-\tau)}{dt}\Big]^2\,d(\tau) \tag{4.11}
$$

The above formulae for the beam with periodic structure were obtained in a similar way as for the beam with a constant cross-section, Iwankiewicz and Śniady (1984), Sieniawska and Śniady (1990).

For the steady-state case $(t \to \infty)$ the solutions (4.8), (4.9), (4.11) have the following forms

$$
\begin{aligned}
E[v(x,\infty)] &= E[Q]\lambda \int\limits_0^\infty [w(x,\xi)+h(x)\psi(x,\psi)]\,d(\xi) \\
\sigma_v^2(x,\infty) &= E[Q^2]\lambda \int\limits_0^\infty [w(x,\xi)+h(x)\psi(x,\xi)]^2\,d(\xi) \\
\sigma_{\dot{v}}^2(x,\infty) &= E[Q^2]\lambda \int\limits_0^\infty \Big[\frac{dw(x,\xi)}{d\xi} + h(x)\frac{d\psi(x,\xi)}{d\xi}\Big]^2\,d\xi
\end{aligned}
\tag{4.12}
$$

The deflection $v(x,t)$ is a non-normal process as the filtered Poisson process. For increasing parameter λ, the process $v(x,t)$ as the sum of many independent processes can be approximated by the normal process. For this reason, for the steady-state case the crossing rate $n_+(x)$ of the threshold a can be given by the Rice formula

$$n_+(a,x) = \frac{1}{2\pi}\frac{\sigma_{\dot{v}}(x,\infty)}{\sigma_v(x,\infty)}\exp\Big(-\frac{a^2}{\sigma_v^2(x,\infty)}\Big) \tag{4.13}$$

The reliability of the beam, as the condition of not crossing the threshold a by the deflection of the beam, can be given by the formula

$$p_s(x,t) = \exp[n_+(a,t)t] \tag{4.14}$$

5. Numerical example

As an example, let us consider undamped vibrations of the beam with l-periodic structure (in this case $c(x) \equiv 0$). For simplicity we restrict the numerical analysis only to the influence of these forces which at the moment t are on the beam $(t_i \in [t - L/u, t])$, on the probabilistic characteristics of the deflection of the beam.

The typical segment of the beam has a piece-wise constant rigidity $B(\cdot)$ and a mass density $\rho(\cdot)$. For $\xi \in (-a,a)$ we have $B(\xi) = B_1$, $\rho(\xi) = \rho_1$, for $\xi \in [-l/2,-a]$ and $[a,l/2]$ we have $B(\xi) = B_2$, $\rho(\xi) = \rho_2$, where B_1, B_2, ρ_1, ρ_2 are constants. For the beam of a periodic structure, the mode shape function $h(\cdot)$ is l-periodic, hence this function is uniquely determined by the function $h_0(\xi)$, $\xi \in [-l/2,l/2]$, where $h(x) = h(sl+\xi) = h_0(\xi)$, $s = 1,2,,...$ with $x = sl+\xi$. Mazur-Śniady (2001) found the mode shape functions being the solution of the eigenproblem with periodic boundary conditions at $x \pm l/2$ together with the corresponding jump conditions.

For the following data: $a = l/4$, $\beta_1/\beta_2 = 8$, $\rho_1/\rho_2 = 2$, the first even mode shape functions have the form:

— for $\xi \in (-l/4,l/4)$

$$h_1(\xi) = l^2\cos\Big(\frac{5.64768}{l}\xi\Big) - 0.094638l^2\cosh\Big(\frac{5.64768}{l}\xi\Big)$$

— for $\xi \in (l/4,l/2)$

$$h_2(\xi) = -1.70403l^2\cos\Big[\frac{7.98703}{l}\Big(\xi-\frac{l}{2}\Big)\Big] - 0.20041\cosh\Big[\frac{7.98703}{l}\Big(\xi-\frac{l}{2}\Big)\Big]$$

— for $\xi \in (-l/2, -l/4)$

$$h_2(\xi) = -h_2(-\xi)$$

For $Q = 10\,\mathrm{N}$, $L = 20\,\mathrm{m}$, $l = 0.4\,\mathrm{m}$, $B_2 = 8 \cdot 10^6\,\mathrm{Nm}$, $\rho_2 = 500\,\mathrm{kg/m}$, $u = 30\,\mathrm{m/s}$ and using the Mathematica package, we obtain the solution to Eqs (3.5) for the general coordinate $y_1(t)$ as presented in Fig. 2 and $q_1(t)$ as presented in Fig. 3. These coordinates describe the run of the beam vibrations for a single term in the expansion (3.3).

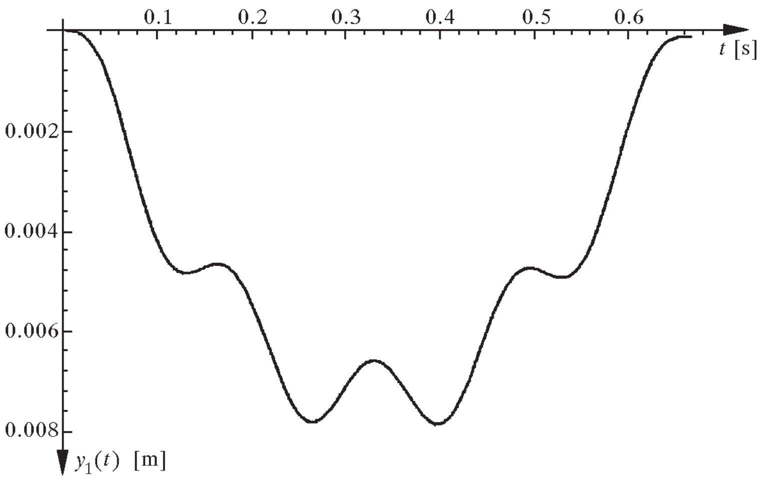

Fig. 2. The graph of the general coordinate $y_1(t)$ [m]

For above data, the expected value and the variance of the middle point of the beam $(x = L/2)$ are equal for stochastic vibrations of the beam caused by a random train of moving forces

$$E[v(L/2,\infty)] = 3.1 \cdot 10^{-8} E[Q]\lambda$$
$$\sigma_v^2(L/2,\infty) = 1.85 \cdot 10^{-5} E[Q^2]\lambda$$

For the intensity $\lambda = 0.3\,\mathrm{s}^{-1}$ assuming $E[Q] = 10^5\,\mathrm{N}$ and $E[Q^2] = 1.2\ E^2[Q] = 1.2 \cdot 10^{10}\,\mathrm{N}^2$, we obtain the values of above expressions equal to

$$E[v(L/2,\infty)] = 0.9 \cdot 10^{-3}\ \mathrm{m}$$
$$\sigma_v^2(L/2,\infty) = 0.666 \cdot 10^{-5}\ \mathrm{m}^2$$

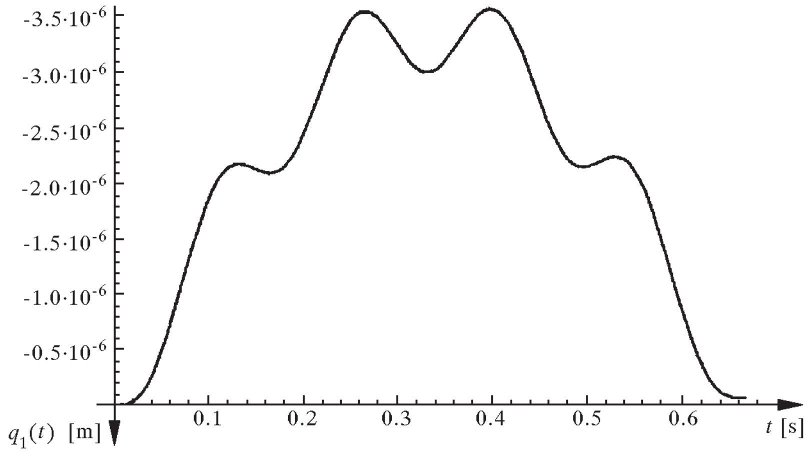

Fig. 3. The graph of the general coordinate $q_1(t)$ [m]

6. Conclusions

In the paper the deterministic and stochastic approach to the problem of vibrations of a beam with periodically varying geometry under moving load is presented.

This approach is an application of the tolerance-averaged model (Woźniak, 1999). In this way, Mazur-Śniady (2001) has formulated equations of the structured beam in the form of the system of averaged differential equations with slowly varying (for periodic-like) or constant (for the periodically variable beam) coefficients which describe the length scale effect.

For the l-periodic beam we reduce the system of partial differential equations to the system of differential equations by expansion into the eigenfunctions. The solution of this system was obtained using the Mathematica package. The solution for a single moving force was adapted to the problem of stochastic vibrations caused by a random train of moving forces. In this case we obtain the formulas for the probabilistic characteristics response of the beam. The presented solutions can be applied in the analysis of dynamics and reliability of bridges.

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Deterministyczne i stochastyczne drgania belki o okresowo zmiennej geometrii wywołane ruchomym obciążeniem

Streszczenie

W pracy rozpatruje się drgania belki o okresowo zmiennej geometrii wywołane działaniem ruchomych obciążeń. Wykorzystuje się model belki o prawie periodyczne strukturze (Mazur-Śniady, 2001), otrzymany metodą uśredniania tolerancyjnego (Woźniak, 1999).

Podano rozwiązanie zagadnienia drgań belki o okresowo zmiennej sztywności wywołanych poruszającą się ze stałą prędkością siłą skupioną. Powyższe rozwiązanie wykorzystano wyznaczając probabilistyczne charakterystyki przemieszczeń belki obciążonej losowym ciągiem ruchomych sił skupionych.

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