

ON ALGEBRAIC EQUATIONS OF ELASTIC TRUSSES, FRAMES AND GRILLAGES

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The algebraic equations of elastic frames are obtained by imposing natural constraints on the trial displacement fields. The obtained set of equations describe deformations of trusses, frames made of both compressible and incompressible bars, grillages of rigid joints as well as pin-jointed grillages. A new form of the set of algebraic equations of frames with a diagonal constitutive matrix is put forward. The presented approach is well-suited for educational purposes.

Key words: elastic frames, grillages, trusses, natural approach of Argyris

1. Introduction

The present paper refers to the linear theory of frames, trusses and grillages, the foundations of which were given to the present author by Professor Zbigniew Kączkowski, in his courses on Structural Mechanics at the Faculty of Civil Engineering, Warsaw University of Technology, in the academic year 1976/1977. Although a long period of time elapsed, just these lectures were an inspiration of the present article.

The equilibrium equations of a frame are implied by the variational equilibrium equation of an elastic body. It expresses equality of the work of unknown stresses on trial strains and the work of given loadings on the trial displacement field, the equality being valid for all kinematically admissible displacement fields. Imposition of the plane cross-sections assumption results in the variational equilibrium equation for a space frame; in particular one obtains the equations for plane frames, trusses and grillages, depending on the geometry and loading. For a plane frame such an equation is given by Eq (2.3) given below. Thus one variational equation describes the equilibrium of a skeletal

structure. The aim of the present paper is construction of algebraic equations following from the variational equation of equilibrium. A similar approach can be found in Borkowski (1985). However, in this book the variational equations are required independently for each bar and the stiffness matrix of the frame is constructed by aggregation. In the approach presented here the stiffness matrices for bars are not necessary. The aggregation assumes tacitly that the bars cut out from a frame undergo the rules of equilibrium known from the theory of solid body mechanics. These rules do not contradict the method applied in the present paper but are not referred to. The aim is to find all equations as a mere consequence of the variational equilibrium equation of the frame. This is the only approach that makes it possible to find algebraic equilibrium equations of more complicated elastic systems in which some degrees of freedom are neither displacements nor angles of rotation. This question is probably better analysed in the plate theories than in the theories of skeletal structures.

The algebraic systems obtained in the present paper are similar but not identical with those reported by Kączkowski (1988, 1984). The definitions of deformations of the frame coincide with those used in the natural approach of Argyris (see Argyris et al., 1964; Argyris and Mlejnek, 1986; Borkowski, 1985).

Let us note that the article by Kączkowski (1984) was then dwarfed in the second edition of the same book, see Kączkowski (1991), such that the readers were deprived of the matrix natural formulation of the displacement method. One of the aims of this paper is to retrieve this approach, stressing some of its points important for the educational practice and indicate its application in the optimum design of skeletal structures.

A passage from the variational equilibrium equation to the algebraic equilibrium equations can be performed by several methods, by choosing special forms of the trial displacement fields. The rigid trial fields method (Sec. 2, (i)) is outlined only, since its applications are limited and its theory – subtle. Nevertheless, just this method is taught to our students because of its reasonable applications for plane frames with non-orthogonal and incompressible bars. The elastic trial field method (Sec. 2, (ii)) is given here in greater detail, emphasizing its links to the finite element method. Note, that the latter, in the case of the bars being treated as finite elements and for the uniform loading, provides incorrect moment distributions and requires the well-known post processing.

The approach presented makes it possible for a uniform algebraic treatment of statics of frames with compressible and incompressible bars. Moreover, the analogy between the equations of frames of incompressible bars and equations

of pin-jointed grillages is disclosed. The last section deals with rearranging the algebraic system of space frames to the form characteristic for the trusses in which the constitutive matrix is diagonal and the stiffness matrix assumes a dyadic form.

2. Plane elastic frames made of compressible bars

Deformation of a plane frame composed of e straight and prismatic bars is determined by the axial displacement field $u(x)$ and the deflection field $w(x)$, x being a coordinate measured along the neutral axes of the bars. Thus $0 \leq x \leq L$, L being the sum of the lengths l_K of the bars $K = 1, \dots, e$. Let EA_K and EJ_K represent the axial and flexural stiffnesses of the K th bar. The bar-wise constant stiffness functions are denoted by $EA(x)$ and $EJ(x)$. Assume that the frame is geometrically invariant and bar-wise subjected to axial and transverse loadings of intensity p_x and p_z , respectively. Here z represents an axis perpendicular to x . The unknown axial force \overline{N} and the moment \overline{M} are interrelated with the axial deformation $\varepsilon(\overline{u})$ and a change of the curvature $\kappa(\overline{w})$ by

$$\overline{N} = EA\varepsilon(\overline{u}) \quad \overline{M} = EJ\kappa(\overline{w}) \quad (2.1)$$

Here $\varepsilon(\cdot)$ and $\kappa(\cdot)$ are treated as differential operators

$$\varepsilon(u) = \frac{du}{dx} \quad \kappa(w) = -\frac{d^2w}{dx^2} \quad (2.2)$$

The overbar $\overline{(\cdot)}$ distinguishes the field that is unknown from any admissible trial fields. For the sake of simplicity the displacement-type boundary conditions are assumed to be homogeneous. Then the set V of kinematically admissible fields (u, w) becomes a function space. This space consists of all fields (u, w) preserving all given connections in the joints. The equilibrium equation has the form

$$\int [\overline{N}\varepsilon(u) + \overline{M}\kappa(w)] dx = \int (p_x u + p_z w) dx \quad \forall (u, w) \in V \quad (2.3)$$

and, if augmented with (2.1), provides us with the problem formulation with \overline{u} , \overline{w} as unknowns. This formulation of the equilibrium problem will be called continuous. A rigorous definition of V is omitted here. In fact, formulation (2.3) is flexible and admits a variety of definitions of the space V . For the present purposes it is sufficient to require that both the integrals in (2.3) are finite. To pass to an algebraic formulation at least two methods can be applied:

- (i) the rigid trial fields method
- (ii) the elastic trial fields method

In method (i) the trial fields (u, w) are associated with rigid body motions of a mechanism formed from the given frame by introducing hinges around rigid joints. For such trial fields the second derivative in $(2.2)_2$ should be understood in a generalized meaning. Introduction of such piece-wise linear functions into (2.3) gives algebraic equations involving the nodal quantities. This method can be equally well applied in the case of incompressible bars, or, in the presence of the constraints: $\varepsilon(u) = 0$, see Nowacki (1960). In method (ii) the space V is composed of all admissible fields (u, w) caused by trial forces applied to the nodes of the frame and associated with the degrees of freedom: $q_1, q_2, \dots, q_j, \dots, q_s$. Thus V becomes an s -dimensional space V_s , s being the number of degrees of freedom or displacements and angles of rotations of nodes: $\mathbf{q} = (q_1, \dots, q_s)$. In the case of hinges, the adjacent angles of rotation should be included in (q_j) . Just this method will be discussed here.

Note first, that two numerations are necessary: one concerning the degrees of freedom (q_j) , $j = 1, \dots, s$ and the other concerning the bars $K = 1, \dots, e$. The trial degrees of freedom are denoted by (q_j) while the unknown degrees of freedom are denoted by (\bar{q}_j) . The local coordinate systems (x, z) are attached to each bar thus defining its left and right ends and the direction of a positive deflection w . The quantities with the index $^*(\cdot)$ are referred to the left end, while the sign $(\cdot)^*$ indicates the right end of the bar. Let us stress once again that the nodes are not numbered. The displacement fields (u, w) for the K th bar are denoted by $u_K(\xi)$, $w_K(\xi)$, $\xi = x/l_K$, and are determined by the left-end displacements $(^*u_K, ^*w_K, ^*\varphi_K)$ and right-end displacements $(u_K^*, w_K^*, \varphi_K^*)$, since, by assumption, the trial fields are associated with the displacements of nodes (the span loading is absent by definition of V_s). The shape functions are well-known polynomials of first and third order. The left and right end displacements of the K th bar can be expressed in terms of (q_j) as follows

$$\begin{aligned}
 ^*u_K &= \sum_{j=1}^s {}^*A_{Kj}^{(u)} q_j & u_K^* &= \sum_{j=1}^s A_{Kj}^{*(u)} q_j \\
 ^*w_K &= \sum_{j=1}^s {}^*A_{Kj}^{(w)} q_j & w_K^* &= \sum_{j=1}^s A_{Kj}^{*(w)} q_j \\
 ^*\varphi_K &= \sum_{j=1}^s {}^*A_{Kj}^{(\varphi)} q_j & \varphi_K^* &= \sum_{j=1}^s A_{Kj}^{*(\varphi)} q_j
 \end{aligned} \tag{2.4}$$

The strains (2.2) associated with the trial fields u_K, w_K are

$$\varepsilon_K = \frac{\Delta_K}{l_K} \quad \kappa_K = \frac{{}^*\chi_K}{l_K} {}^*f(\xi) + \frac{\chi_K^*}{l_K} f^*(\xi) \quad (2.5)$$

where

$${}^*f(\xi) = 4 - 6\xi \quad f^*(\xi) = 2 - 6\xi \quad (2.6)$$

and the following quantities represent the deformations of the frame

$$\Delta_K = u_K^* - {}^*u_K \quad {}^*\chi_K = {}^*\varphi_K - \psi_K \quad \chi_K^* = \varphi_K^* - \psi_K \quad (2.7)$$

and $\psi_K = (w_K^* - {}^*w_K)/l_K$ represents the slope of the K th bar. The linear relations between the deformations (2.7) and the degrees of freedom (q_j) are represented by

$$\Delta_K = \sum_{j=1}^s B_{Kj} q_j \quad {}^*\chi_K = \sum_{j=1}^s {}^*\beta_{Kj} q_j \quad \chi_K^* = \sum_{j=1}^s \beta_{Kj}^* q_j \quad (2.8)$$

or

$$\Delta = \mathbf{B} \mathbf{q} \quad {}^*\chi = {}^*\beta \mathbf{q} \quad \chi^* = \beta^* \mathbf{q} \quad (2.9)$$

The geometrical matrices $\mathbf{B}, {}^*\beta, \beta^*$ are determined by the allocation matrices involved in (2.4). Method (ii) consists in taking $(u, w) \in V_s$, which makes the left-hand side of (2.3) algebraic with respect to the effective internal forces in the bars

$$\int [\bar{N}\varepsilon(u) + \bar{M}\kappa(w)] dx = \sum_{K=1}^e (\bar{N}_K \Delta_K + {}^*\bar{M}_K {}^*\chi_K + \bar{M}_K^* \chi_K^*) \quad (2.10)$$

where

$$\bar{N}_K = \int_0^1 \bar{N}_K(\xi) d\xi \quad {}^*\bar{M}_K = \int_0^1 {}^*f(\xi) \bar{M}_K(\xi) d\xi \quad (2.11)$$

$$\bar{M}_K^* = \int_0^1 f^*(\xi) \bar{M}_K(\xi) d\xi$$

Substitution of (2.8) into (2.10) gives

$$\int [\bar{N}\varepsilon(u) + \bar{M}\kappa(w)] dx = \mathbf{q}^\top [\mathbf{B}^\top \bar{\mathbf{N}} + ({}^*\beta)^\top {}^*\bar{\mathbf{M}} + (\beta^*)^\top \bar{\mathbf{M}}^*] \quad (2.12)$$

where $\overline{\mathbf{N}} = (\overline{N}_K)$, ${}^*\overline{\mathbf{M}} = ({}^*\overline{M}_K)$, $\overline{\mathbf{M}}^* = (\overline{M}_K^*)$. By expressing (u, w) in terms of $({}^*u_K, u_K^*, {}^*w_K, w_K^*, \varphi_K, \varphi_K^*)$ and, by (2.4), in terms of (q_j) , one rearranges the linear form at the right-hand side of (2.3) to the form $\mathbf{q}^\top \mathbf{Q}$. The effective loads can be defined rigorously, but we neglect these definitions, since each Q_j has a clear physical meaning: Q_j represents the work of (p_x, p_z) on the deformation mode from V_s such that $q_i = \delta_{ij}$.

By equating the right-hand side of (2.12) to $\mathbf{q}^\top \mathbf{Q}$ and taking \mathbf{q} as the basis vectors of V_s one finds s equations that can then be put in the following matrix form

$$\mathbf{B}^\top \overline{\mathbf{N}} + ({}^*\beta)^\top {}^*\overline{\mathbf{M}} + (\beta^*)^\top \overline{\mathbf{M}}^* = \mathbf{Q} \quad (2.13)$$

Since the K th bar can be loaded within its span, the axial force and the bending moment are decomposed as follows

$$\begin{aligned} \overline{N}_K(\xi) &= \frac{EA_K}{l_K} \overline{\Delta}_K + N_K^o(\xi) \\ \overline{M}_K(\xi) &= \frac{EJ_K}{l_K} [{}^*f(\xi) {}^*\overline{\chi}_K + f^*(\xi) \overline{\chi}_K^*] + M_K^o(\xi) \end{aligned} \quad (2.14)$$

where N_K^o, M_K^o are the internal forces caused by the loading applied to the K th bar and measured in the frame in which all degrees of freedom are kept zero $q_j = 0$. This frame is called kinematically determined or KD frame. Note that

$$\begin{aligned} \int_0^1 N_K^o(\xi) d\xi &= 0 & \int_0^1 {}^*f(\xi) M_K^o(\xi) d\xi &= 0 \\ \int_0^1 f^*(\xi) M_K^o(\xi) d\xi &= 0 \end{aligned} \quad (2.15)$$

The first of these equations means that the elongation of the K th bar of the KD frame vanishes. The equations (2.15)_{2,3} mean that the angles of rotations of both the ends of this bar in the KD frame vanish. Substitution of (2.14) into (2.11) gives the constitutive equations for the frame

$$\begin{aligned} \overline{N}_K &= \frac{EA_K}{l_K} \overline{\Delta}_K & {}^*\overline{M}_K &= \frac{2EJ_K}{l_K} (2{}^*\overline{\chi}_K + \overline{\chi}_K^*) \\ \overline{M}_K^* &= \frac{2EJ_K}{l_K} ({}^*\overline{\chi}_K + 2\overline{\chi}_K^*) \end{aligned} \quad (2.16)$$

We note that ${}^* \overline{M}_K$ and \overline{M}_K^* represent the left- and right-end moments of the K th bar subjected to the given displacements at its ends. Thus, the well-known slope deflection equations are recovered. Let us stress once again that the loading applied to the bars has no influence on the result of (2.12) and (2.16). The notion of "primary moments" is redundant here.

Let \mathbf{D} and \mathbf{E} be $e \times e$ diagonal matrices with diagonal components equal $(2EJ_K/l_K)$ and (EA_K/l_K) , respectively. Substitution of (2.8) into (2.16) and – further – into (2.13) gives

$$\mathbf{K}\overline{\mathbf{q}} = \mathbf{Q} \tag{2.17}$$

with the stiffness matrix given by

$$\mathbf{K} = \mathbf{B}^\top \mathbf{E} \mathbf{B} + 2 \left\{ ({}^* \boldsymbol{\beta})^\top \mathbf{D} {}^* \boldsymbol{\beta} + \frac{1}{2} \left[({}^* \boldsymbol{\beta})^\top \mathbf{D} \boldsymbol{\beta}^* + (({}^* \boldsymbol{\beta})^\top \mathbf{D} \boldsymbol{\beta}^*)^\top \right] + (\boldsymbol{\beta}^*)^\top \mathbf{D} \boldsymbol{\beta}^* \right\} \tag{2.18}$$

A frame is statically determined, if $\overline{\mathbf{N}}$, ${}^* \overline{\mathbf{M}}$, $\overline{\mathbf{M}}^*$ can be found from (2.13). This is possible if $3e = s$. Then also $\overline{\mathbf{q}}$ can be determined from (2.8) knowing $(\overline{\boldsymbol{\Delta}}, {}^* \overline{\boldsymbol{\chi}}, \overline{\boldsymbol{\chi}}^*)$.

If $s > 3e$ a frame is kinematically variant. The statical quantities can be found provided that \mathbf{Q} is orthogonal to the admissible rigid body motions. Theory of such frames is exposed in Kuznetsov (1991).

3. Plane trusses

The equations for trusses follow by neglecting the moments and stiffnesses EJ/l . We have

$$\mathbf{B}^\top \overline{\mathbf{N}} = \mathbf{Q} \qquad \overline{\mathbf{N}} = \mathbf{E} \overline{\boldsymbol{\Delta}} \qquad \overline{\boldsymbol{\Delta}} = \mathbf{B} \overline{\mathbf{q}} \tag{3.1}$$

and $\mathbf{K} = \mathbf{B}^\top \mathbf{E} \mathbf{B}$. A detailed discussion of (3.1) can be found in Borkowski (1985). The equations of space trusses look the same.

Let us define the P th row of \mathbf{B} by \mathbf{b}_P . Since the matrix \mathbf{E} is diagonal the components of the stiffness matrix has a special form

$$K_{ij} = \sum_{P=1}^e \frac{EA_P}{l_P} (\mathbf{b}_P)_i (\mathbf{b}_P)_j \tag{3.2}$$

Let us recall the dyadic product $\mathbf{a} \otimes \mathbf{b}$ of two vectors: \mathbf{a} and \mathbf{b} ; its components are given by $\mathbf{a} \otimes \mathbf{b} = \{a_i b_j\}$. Hence

$$\mathbf{K} = \sum_{P=1}^e \mathbf{K}^{(P)} \qquad \mathbf{K}^{(P)} = \frac{EA_P}{l_P} \mathbf{b}_P \otimes \mathbf{b}_P \tag{3.3}$$

Due to the dyadic form of the expression for \mathbf{K} for trusses some theorems on the optimum design assume a special form, see Achtziger (1997).

4. Plane frames made of incompressible bars

The constraints $\varepsilon(u) = 0$ reduce the space V to a subspace U and rearrange variational equilibrium equation (2.3) to the form

$$\int \overline{M}\kappa(w) dx = \int (p_x u + p_z w) dx \quad \forall (u, w) \in U \quad (4.1)$$

Once again both methods (i) and (ii), the Section 2, can be applied. Method (i) is taught to our students although its theory requires generalized functions. Here u and w are taken from the space larger than $C^0 \times C^1$, see Nowacki (1960) and Kączkowski (1988). Consequently, not the theory itself but rather its algorithm is taught. Method (ii) is much more lucid, but is not taught at all, since it is usually dwarfed and squeezed by purely computational algorithms. Method (ii) consists in shrinking the space U to an s -dimensional subspace U_s composed of the fields (u, w) associated with moving the nodes under the constraints $\varepsilon(u) = 0$. The possible and independent degrees of freedom are still denoted by (q_j) . Some of them represent sways of the frame, so they are not connected with the nodes but are global kinematical characteristics. Here ${}^*u_K = u_K^* = u_K$ with $\mathbf{u} = A^{(u)}\mathbf{q}$. Thus two equations (2.4)_{1,2} are reduced to one. Relation (2.5)₂ holds. Equation (2.10) assumes the form

$$\int \overline{M}\kappa(w) dx = \mathbf{q}^\top [({}^*\boldsymbol{\beta})^\top {}^*\overline{\mathbf{M}} + (\boldsymbol{\beta}^*)^\top \overline{\mathbf{M}}^*] \quad (4.2)$$

and the equilibrium equation (2.13) reduces to

$$({}^*\boldsymbol{\beta})^\top {}^*\overline{\mathbf{M}} + (\boldsymbol{\beta}^*)^\top \overline{\mathbf{M}}^* = \mathbf{Q} \quad (4.3)$$

Here (Q_j) assume a new meaning, associated with the new meaning of (q_j) . The constitutive equations

$${}^*\overline{\mathbf{M}} = \mathbf{D}(2{}^*\overline{\boldsymbol{\chi}} + \overline{\boldsymbol{\chi}}^*) \quad \overline{\mathbf{M}}^* = \mathbf{D}({}^*\overline{\boldsymbol{\chi}} + 2\overline{\boldsymbol{\chi}}^*) \quad (4.4)$$

remain homogeneous, the "primary moments" do not occur. Together with (2.9) specified for the unknown quantities, equations (4.3), (4.4) form a complete system of algebraic equations. It leads to (2.17). The stiffness matrix is given by (2.18) with the first term omitted.

A frame is statically determined if ${}^* \overline{M}_K$ and \overline{M}_K^* can be found by solving Eq (4.3). To recover the axial forces \overline{N}_K one should recall Eq (2.13) and try to solve it with respect to \overline{N} .

Equations (4.3), (4.4), (2.9)_{2,3} can be easily programmed by using any symbolic computation package. The sparse matrices ${}^* \beta, \beta^*$ are formed automatically upon writing the equations linking the deformations with displacements, cf (2.9)_{2,3}.

5. Grillages of rigid joints

Let GC_K represent the torsional stiffness of the K th bar and $\phi(x)$ mean an angle of rotation of a cross-section around the bar axis. The moment, the change of the curvature, the deflection and the bending stiffness of the K th bar are still denoted by M, κ, w, EJ , respectively. The joints are assumed to be rigid, capable of resisting both bending and torsion. Some bars can be simply supported on other bars, but at least one connection is assumed to be rigid. The loading is vertical. The torsional moment is denoted by \mathfrak{m} and the torsional deformation by $\tau(\phi)$.

The variational equation of equilibrium has the form

$$\int [\overline{\mathfrak{m}}\tau(\phi) + \overline{M}\kappa(w)] dx = \int p_z w dx \quad \forall (\phi, w) \in W \tag{5.1}$$

where W represents the space of kinematically admissible angles of torsion ϕ and deflections w . Displacements of supports are excluded. The constitutive relations read

$$\overline{\mathfrak{m}} = GC\tau(\overline{\phi}) \quad \overline{M} = EJ\kappa(\overline{w}) \tag{5.2}$$

where $\tau(\phi) = d\phi/dx$; note that $\tau(\phi) = \varepsilon(\phi)$. To make problems (5.1) and (5.2) algebraic we apply method (ii) of Section 2. Thus W is shrunked to W_s , an s dimensional subspace generated by admissible, virtual loads concentrated in the joints. The loads are admissible if they can be equilibrated by the grillage. Thus, the field ϕ is bar-wise linear and w is bar-wise a polynomial of third order. The torsional deformation τ is bar-wise constant

$$\tau_K = \frac{\theta_K}{l_K} \quad \theta_K = \phi_K^* - {}^* \phi_K \tag{5.3}$$

Consequently

$$\int \overline{\mathfrak{m}}\tau(\phi) dx = \sum_{K=1}^e \overline{\mathfrak{m}}_K \theta_K \tag{5.4}$$

and

$$\bar{\mathbf{m}}_K = \frac{GC_K}{l_K} \bar{\theta}_K \quad (5.5)$$

There exist coefficients (β_{Kj}) such that

$$\theta_K = \sum_{j=1}^s \beta_{Kj} q_j \quad (5.6)$$

which rearranges (5.4) to the form

$$\int \bar{\mathbf{m}} \tau(\phi) dx = \mathbf{q}^\top (\boldsymbol{\beta}^\top \bar{\mathbf{m}}) \quad (5.7)$$

and, by a relation similar to (4.2), one finds the equilibrium equation in the matrix form

$$\boldsymbol{\beta}^\top \bar{\mathbf{m}} + (*\boldsymbol{\beta})^\top * \bar{\mathbf{M}} + (\boldsymbol{\beta}^*)^\top \bar{\mathbf{M}}^* = \mathbf{Q} \quad (5.8)$$

Let \mathbf{G} be a diagonal matrix of the components (GC_K/l_K) . The constitutive equations can be put in the form

$$\bar{\mathbf{m}} = \mathbf{G} \bar{\theta} \quad * \bar{\mathbf{M}} = \mathbf{D}(2* \bar{\chi} + \bar{\chi}^*) \quad \bar{\mathbf{M}}^* = \mathbf{D}(* \bar{\chi} + 2 \bar{\chi}^*) \quad (5.9)$$

The equation for \mathbf{q} has the form of Eq (2.17) with the stiffness matrix \mathbf{K} given by (2.18), where \mathbf{E} and \mathbf{B} should be replaced by \mathbf{G} and $\boldsymbol{\beta}$, respectively.

6. Pin jointed grillages

Let us assume that the only interaction between the bars of a grillage is vertical and the only common degree of freedom of the bars connected in a joint is the deflection of this joint. Consequently, bending does not cause torsion. Since the load is vertical as before, the torsion is absent. By neglecting the torsion in the equations (5.8),(5.9) one arrives at the algebraic system of the same form as that describing the bending of plane frames made of incompressible bars (Eqs (4.3), (4.4), (2.9)_{2,3}). We omit here obvious consequences of this analogy.

7. Space frames

We omit the continuum formulation of the general frame equilibrium problem. Let us pass to the algebraic formulation within method (ii) of Section 2.

The local coordinate system (x, y, z) is attached to all bars, x representing their neutral axes. The quantities¹

$${}^*M_K, M_K^*, {}^*\chi_K, \chi_K^*, {}^*\beta_{Kj}, \beta_{Kj}^*, EJ_K^{(M)}, \mathbf{D}_M \quad (7.1)$$

concern the bending in the plane (x, z) , while their counterparts for the bending in the (x, y) plane are denoted by

$${}^*L_K, L_K^*, {}^*\varrho_K, \varrho_K^*, {}^*\eta_{Kj}, \eta_{Kj}^*, EJ_K^{(L)}, \mathbf{D}_L \quad (7.2)$$

The equilibrium equations comprise the elastic deformations in both the planes

$$\mathbf{B}^\top \mathbf{N} + \mathfrak{B}^\top \mathbf{m} + ({}^*\beta)^\top {}^*M + (\beta^*)^\top M^* + ({}^*\eta)^\top {}^*L + (\eta^*)^\top L^* = \mathbf{Q} \quad (7.3)$$

The constitutive equations read

$$\begin{aligned} \mathbf{N} &= \mathbf{E}\Delta & {}^*M &= \mathbf{D}_M(2{}^*\chi + \chi^*) & {}^*L &= \mathbf{D}_L(2{}^*\varrho + \varrho^*) \\ \mathbf{m} &= \mathbf{G}\theta & M^* &= \mathbf{D}_M({}^*\chi + 2\chi^*) & L^* &= \mathbf{D}_L({}^*\varrho + 2\varrho^*) \end{aligned} \quad (7.4)$$

Let us introduce aggregate quantities

$$\begin{aligned} \boldsymbol{\epsilon} &= \text{col}[\Delta, \theta, {}^*\chi, \chi^*, {}^*\varrho, \varrho^*] \\ \mathfrak{N} &= \text{col}[\mathbf{B}, \mathfrak{B}, {}^*\beta, \beta^*, {}^*\eta, \eta^*] \end{aligned} \quad (7.5)$$

The deformations depend on the displacements (q_j) by

$$\boldsymbol{\epsilon} = \mathfrak{N}\mathbf{q} \quad (7.6)$$

Substitution of (7.6) into (7.4) and (7.3) gives (2.7) with the stiffness matrix of the form

$$\begin{aligned} \mathbf{K} &= \mathbf{B}^\top \mathbf{E}\mathbf{B} + \mathfrak{B}^\top \mathbf{G}\mathfrak{B} + \\ &+ 2\left\{ ({}^*\beta)^\top \mathbf{D}_M {}^*\beta + \frac{1}{2} \left[({}^*\beta)^\top \mathbf{D}_M \beta^* + \left(({}^*\beta)^\top \mathbf{D}_M \beta^* \right)^\top \right] + (\beta^*)^\top \mathbf{D}_M \beta^* \right\} + \\ &+ 2\left\{ ({}^*\eta)^\top \mathbf{D}_L {}^*\eta + \frac{1}{2} \left[({}^*\eta)^\top \mathbf{D}_L \eta^* + \left(({}^*\eta)^\top \mathbf{D}_L \eta^* \right)^\top \right] + (\eta^*)^\top \mathbf{D}_L \eta^* \right\} \end{aligned} \quad (7.7)$$

¹The overbars are omitted in this section since the trial fields do not appear

Let us introduce the aggregate matrices

$$\boldsymbol{\varsigma} = \text{col}[N, \mathbf{m}, {}^*M, M^*, {}^*L, L^*] \tag{7.8}$$

$$\boldsymbol{\Xi} = \begin{bmatrix} \mathbf{E} & \dots & \dots & \dots & \dots & \dots \\ \dots & \mathbf{G} & \dots & \dots & \dots & \dots \\ \dots & \dots & 2\mathbf{D}_M & \mathbf{D}_M & \dots & \dots \\ \dots & \dots & \mathbf{D}_M & 2\mathbf{D}_M & \dots & \dots \\ \dots & \dots & \dots & \dots & 2\mathbf{D}_L & \mathbf{D}_L \\ \dots & \dots & \dots & \dots & \mathbf{D}_L & 2\mathbf{D}_L \end{bmatrix}$$

Note that constitutive equations (7.4) are equivalent to $\boldsymbol{\varsigma} = \boldsymbol{\Xi}\boldsymbol{\epsilon}$ and equilibrium equation (7.3) assumes the form $\mathbf{N}^\top \boldsymbol{\varsigma} = \mathbf{Q}$. By (7.6) one finds the stiffness matrix in the form $\mathbf{K} = \mathbf{N}^\top \boldsymbol{\Xi} \mathbf{N}$.

The aggregated description can be rearranged to a form in which the matrix $\boldsymbol{\Xi}$ is diagonal. To this end we decompose the bending deformations and bending moments as follows

$$\begin{aligned} {}^*\chi &= \chi_s + \chi_a & \chi^* &= \chi_s - \chi_a \\ {}^*M &= M_s + M_a & M^* &= M_s - M_a \\ {}^*\varrho &= \varrho_s + \varrho_a & \varrho^* &= \varrho_s - \varrho_a \\ {}^*L &= L_s + L_a & L^* &= L_s - L_a \end{aligned} \tag{7.9}$$

where

$$\begin{aligned} \chi_s &= \beta_s q & \chi_a &= \beta_a q \\ \varrho_s &= \eta_s q & \varrho_a &= \eta_a q \end{aligned} \tag{7.10}$$

and

$$\begin{aligned} {}^*\beta &= \beta_s + \beta_a & \beta^* &= \beta_s - \beta_a \\ {}^*\eta &= \eta_s + \eta_a & \eta^* &= \eta_s - \eta_a \end{aligned} \tag{7.11}$$

Constitutive relations (7.4) assume diagonal forms

$$\begin{aligned} N &= \mathbf{E}\Delta & \mathbf{m} &= \mathbf{G}\theta \\ M_s &= 3\mathbf{D}_M\chi_s & M_a &= \mathbf{D}_M\chi_a \\ L_s &= 3\mathbf{D}_L\varrho_s & L_a &= \mathbf{D}_L\varrho_a \end{aligned} \tag{7.12}$$

The quantities $(M_s)_K, (L_s)_K$ are proportional to shear forces in the K th bar and the quantities $(M_a)_K, (L_a)_K$ represent the moments in the mid-span of the K th bar. Equilibrium equation (7.3) changes its form. It reads now

$$\mathbf{B}^\top N + \mathbf{B}^\top \mathbf{m} + 2\left[(\beta_s)^\top M_s + (\beta_a)^\top M_a + (\eta_s)^\top L_s + (\eta_a)^\top L_a\right] = \mathbf{Q} \tag{7.13}$$

and we see that the factor 2 appears, which makes the above equation less clear than the original one (7.3). Substitution of (7.12) into (7.13) gives the stiffness matrix of the form

$$\begin{aligned} \mathbf{K} &= \mathbf{B}^\top \mathbf{E} \mathbf{B} + \mathbf{\beta}^\top \mathbf{G} \mathbf{\beta} + \\ &+ 2 \left[(\boldsymbol{\beta}_a)^\top \mathbf{D}_M \boldsymbol{\beta}_a + (\boldsymbol{\eta}_a)^\top \mathbf{D}_L \boldsymbol{\eta}_a \right] + 6 \left[(\boldsymbol{\beta}_s)^\top \mathbf{D}_M \boldsymbol{\beta}_s + (\boldsymbol{\eta}_s)^\top \mathbf{D}_L \boldsymbol{\eta}_s \right] \end{aligned} \quad (7.14)$$

composed of six terms describing the energies of: tension/compression, torsion, pure bending in both the planes and slope in both the planes. The coupled terms present in (7.7) disappeared. There is no coupling between the states of bending and sway.

To put governing equations (7.10), (7.12), (7.13) in an aggregate form we introduce new matrices

$$\begin{aligned} \tilde{\boldsymbol{\zeta}} &= \text{col} [N, \mathbf{m}, \sqrt{2} \mathbf{M}_s, \sqrt{2} \mathbf{L}_s, \sqrt{2} \mathbf{M}_a, \sqrt{2} \mathbf{L}_a] \\ \tilde{\boldsymbol{\epsilon}} &= \text{col} [\boldsymbol{\Delta}, \boldsymbol{\theta}, \sqrt{2} \boldsymbol{\chi}_s, \sqrt{2} \boldsymbol{\varrho}_s, \sqrt{2} \boldsymbol{\chi}_a, \sqrt{2} \boldsymbol{\varrho}_a] \\ \tilde{\boldsymbol{\Xi}} &= \text{diag} (\mathbf{E}, \mathbf{G}, 3\mathbf{D}_M, 3\mathbf{D}_L, \mathbf{D}_M, \mathbf{D}_L) \\ \tilde{\boldsymbol{\aleph}} &= \text{col} [\mathbf{B}, \mathbf{\beta}, \sqrt{2} \boldsymbol{\beta}_s, \sqrt{2} \boldsymbol{\eta}_s, \sqrt{2} \boldsymbol{\beta}_a, \sqrt{2} \boldsymbol{\eta}_a] \end{aligned} \quad (7.15)$$

to obtain

$$\tilde{\boldsymbol{\aleph}}^\top \tilde{\boldsymbol{\zeta}} = \mathbf{Q} \quad \tilde{\boldsymbol{\zeta}} = \tilde{\boldsymbol{\Xi}} \tilde{\boldsymbol{\epsilon}} \quad \tilde{\boldsymbol{\epsilon}} = \tilde{\boldsymbol{\aleph}} \mathbf{q} \quad (7.16)$$

with $\mathbf{K} = \tilde{\boldsymbol{\aleph}}^\top \tilde{\boldsymbol{\Xi}} \tilde{\boldsymbol{\aleph}}$. Since $\tilde{\boldsymbol{\Xi}}$ is diagonal system (7.16) is similar to that for trusses, (3.1).

Let the P th rows of the matrices $\mathbf{B}, \mathbf{\beta}, \sqrt{2} \boldsymbol{\beta}_s, \sqrt{2} \boldsymbol{\beta}_a, \sqrt{2} \boldsymbol{\eta}_s, \sqrt{2} \boldsymbol{\eta}_a$ be denoted by $\mathbf{b}_P, \mathbf{b}_P, \mathbf{d}_P^{(s)}, \mathbf{d}_P^{(a)}, \mathbf{h}_P^{(s)}, \mathbf{h}_P^{(a)}$, respectively. The matrix \mathbf{K} is decomposed as a sum of e matrices $\mathbf{K}^{(P)}$ for bars, the stiffness matrix for the P th bar being given by the formula

$$\begin{aligned} \mathbf{K}^{(P)} &= \frac{EA_P}{l_P} \mathbf{b}_P \otimes \mathbf{b}_P + \frac{GC_P}{l_P} \mathbf{b}_P \otimes \mathbf{b}_P + \\ &+ \frac{2EJ_P^{(M)}}{l_P} (\mathbf{d}_P^{(a)} \otimes \mathbf{d}_P^{(a)} + 3\mathbf{d}_P^{(s)} \otimes \mathbf{d}_P^{(s)}) + \frac{2EJ_P^{(L)}}{l_P} (\mathbf{h}_P^{(a)} \otimes \mathbf{h}_P^{(a)} + 3\mathbf{h}_P^{(s)} \otimes \mathbf{h}_P^{(s)}) \end{aligned} \quad (7.17)$$

The above form of the stiffness matrix is helpful in computing the sensitivities of spatial frames and in proving the theorems on their optimum design.

8. Final remarks

Method (ii) of Section 2 has the following features:

- It requires two numerations: of the degrees of freedom and of bars. Nodal numeration is redundant.
- The method does not require the approximation of the unknown fields of the displacements and stress resultants.
- The deformations measures of a frame are defined in a natural manner.
- In the case of a static loading the constitutive relations turn out to be homogeneous.
- The effective stress resultants are defined as integrals over the neutral axes of bars, see Eqs (2.11). The conditions of the equilibrium of nodes are satisfied but are not required to form the equilibrium equations. They follow from the global consideration in a manner common to both the cases: of compressible and incompressible bars.
- The algebraic equations derived differ from that reported by Kączkowski (1984) for plane frames. The differences are the following:
 - the static unknowns are grouped differently, which has made it possible here to report the formula for the stiffness matrix in form (2.18)
 - the equilibrium equations are derived here in a variationally consistent manner. No reference to the laws of rigid body mechanics is made. The Clebsch theorem is built-in in the method and does not require explanations
 - a proof is given of the constitutive relations being homogeneous
 - the matrix \mathbf{B} is formed by the method (ii), Section 2, and not by method (i), like in the article by Kączkowski.
- In the case of elementary frame problems method (ii), Section 2 is very easy to algorithmize for both the cases of compressible and incompressible bars. The algorithm is especially short in the case of plane trusses. Any symbolic computation package is very helpful. Note that the matrix \mathbf{B} can be formed from the equation $\mathbf{\Delta} = \mathbf{B}\mathbf{q}$ by using the *genmatrix* command of MAPLE. The analysis of frames and grillages is a little longer. The hitherto educational experiences are encouraging.

- As to the best author's knowledge equations (7.16) with the diagonal stiffness matrix \mathbf{K} (Eq (7.17)) have not been reported in the literature. Note that the equations of Borkowski (1985, Appendix A) have a different form and concern a single bar. Similar equations for trusses are reported by Achtziger (1997), but no generalization to frames can be found.

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O algebraicznych równaniach sprężystych kratownic, ram i rusztów

Streszczenie

Algebraiczne równania ram sprężystych otrzymano na drodze nałożenia naturalnych więzów na próbne pola przemieszczeń. Znaleziony układ równań opisuje odkształcenia kratownic, ram wykonanych z prętów ściśliwych bądź nieściśliwych, rusztów o węzłach sztywnych oraz rusztów przegubowych. Wyprowadzono nową postać równań algebraicznych z diagonalną macierzą konstytutywną. Przedstawione podejście dobrze pasuje do celów dydaktycznych.

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