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# ON INTEGRATION OF THE GREEN FUNCTION IN A DISCONTINUOUS TEMPERATURE FIELD IN A 2D DOMAIN

Grzegorz Jemielita

*Institute of Structural Mechanics, Warsaw University of Technology*

*e-mail: g.jemielita@il.pw.edu.pl*

The paper discusses the problem of Green's functions application to stress calculations in the case of a discontinuous temperature field acting in a 2D domain. It is proved that the double integral of the Green function for the stresses does not exist. A new method of finding iterrated integrals of Green's functions which enables obtaining correct functions for the stress $\sigma_{\alpha\beta}$ is presented as well.

Key words: Green's functions, discontinuous temperature field

## 1. Introduction

Consider an infinite space with a discontinuous temperature field

$$\theta(x_1,x_2,x_3) = \begin{cases} \theta_0(x_1,x_2) & \text{for} \quad (x_1,x_2,x_3) \in \varOmega \times (-\infty,+\infty) \\ 0 & \text{for} \quad (x_1,x_2,x_3) \notin \varOmega \times (-\infty,+\infty) \end{cases} \tag{1.1}$$

where $\varOmega$ is the temperature domain, $\varOmega =< -a, a > \times < -b, b >$ (Fig.1).

We seek for displacement and stress fields in the space. Solution to the above problem can be reduced to solving the Poisson equation in the form given by Nowacki (1986)

$$\nabla^2 \varPhi = \begin{cases} m\theta_0 & \text{for} \quad (x_1,x_2) \in \varOmega \\ 0 & \text{for} \quad (x_1,x_2) \notin \varOmega \end{cases} \tag{1.2}$$

where $\varPhi$ represents the potential of the thermoelastic displacement, while the displacements can be represented as follows

$$u_\alpha = \varPhi_{,\alpha} \tag{1.3}$$

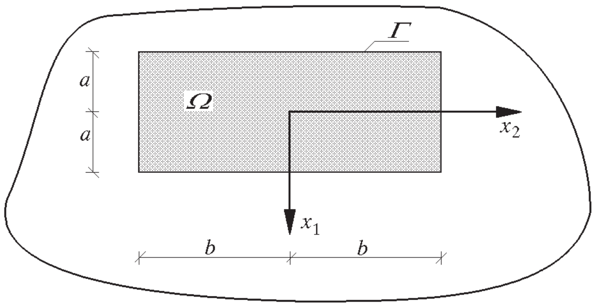


Fig. 1.

and

$$m = \frac{\nu}{2\mu + \lambda} \tag{1.4}$$

where

| | | |
|---|---|---|
| $\mu, \lambda$ | – | Lame's constants |
| $\nu$ | – | constant to be determined by thermal and mechanical properties of the material |
| $\nabla^2$ | – | 2D Laplace's operator. |

In Eq (1.3) the coma stands for differentiation with respect to $x_\alpha$ $(\alpha = 1, 2)$.

The stresses can be found from the formulae

$$\begin{aligned} \sigma_{\alpha\beta} &= 2\mu(\Phi_{,\alpha\beta} - \nabla^2\Phi\delta_{\alpha\beta}) \\ \sigma_{33} &= \begin{cases} -2\mu m\theta_0 & \text{for} \quad (x_\alpha) \in \Omega \\ 0 & \text{for} \quad (x_\alpha) \notin \Omega \end{cases} \end{aligned} \tag{1.5}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta.

The approach presented below, based on the application of Green's functions, is widely used in the literature (cf Nowacki, 1970a,b, 1986).

Consider the infinite space with the unit temperature kernel acting along the line $\{L:\ x_1 = \zeta_1, x_2 = \zeta_2, x_3 \in R\}$, i.e.

$$\theta = \delta(x_1 - \zeta_1)\delta(x_2 - \zeta_2) \tag{1.6}$$

A solution to the differential equation

$$\nabla^2\Phi_0 = m\delta(x_1 - \zeta_1)\delta(x_2 - \zeta_2) \tag{1.7}$$

has the following Green's function form

$$G(x_\alpha;\zeta_\beta) = \varPhi_0 = \frac{m}{2\pi}\ln R \tag{1.8}$$

where

$$R^2 = (x_1-\zeta_1)^2 + (x_2-\zeta_2)^2 \tag{1.9}$$

Employing Eq (1.8), we can represent the potential of the thermoelastic displacement (according to the theory of potential it is called the logarithmic potential) as follows

$$\varPhi(x_\alpha) = \frac{m}{2\pi}\int\limits_{\varOmega} \theta(\zeta_\beta)\ln R\, d\varOmega_\zeta \tag{1.10}$$

The following theorems are to be proved when making use of the potential theory approach (see Giunter, 1957):

**I.** *If the function $\theta(\zeta_\alpha)$ is integrable in the domain (region) $\varOmega$, then the logarithmic potential $\varPhi(x_\alpha)$ is of $C^1$ class in the entire space, and its partial derivatives result directly from differentiation of the integrand, i.e.*

$$\varPhi_{,\alpha} = \frac{m}{2\pi}\int\limits_{\varOmega} \theta(\zeta_\beta)(\ln R)_{,\alpha}\, d\varOmega_\zeta = \frac{m}{2\pi}\int\limits_{\varOmega} \theta(\zeta_\beta)\frac{x_\alpha-\zeta_\alpha}{R^2}\, d\varOmega_\zeta \tag{1.11}$$

**II.** *Poisson theorem*

*If the function $\theta(\zeta_\alpha)$ integrable in the domain $\overline{\varOmega}$ is of $C^1$ class in a certain neighbourhood of the point $(x_\alpha)\in\varOmega$, then the potential $\varPhi(x_\alpha)$ is of $C^2$ class at the point $(x_\alpha)$ and Eq (1.2) is true.*

It should be noted, however, that the second order derivatives of the potential $\varPhi$, i.e. $\varPhi_{,\alpha\beta}$ cannot be calculated in the way used for the first order derivative $\varPhi_{,\alpha}$, i.e. by differentiating the integrand.

By virtue of Eqs (1.5) and (1.8) Green's functions for the stresses $\widehat{\sigma}_{\alpha\beta}$ can be written in the form

$$\begin{aligned}
\widehat{\sigma}_{11}(x_\alpha;\zeta_\beta) &= -G_{,22} = -\frac{\mu m}{\pi}\frac{(x_1-\zeta_1)^2-(x_2-\zeta_2)^2}{R^4}\\
\widehat{\sigma}_{22} &= -\widehat{\sigma}_{11}\\
\widehat{\sigma}_{12}(x_\alpha;\zeta_\beta) &= -G_{,12} = -\frac{2\mu m}{\pi}\frac{(x_1-\zeta_1)(x_2-\zeta_2)}{R^4}
\end{aligned} \tag{1.12}$$

If the temperature domain is represented by Eq (1.1) the stresses $\sigma_{\alpha\beta}$ have the following form (see Nowacki, 1970a,b, 1986)

$$\sigma_{\alpha\beta}(x_\alpha) = \int\limits_{\Omega} \theta(\zeta_\beta)\widehat{\sigma}_{\alpha\beta}(x_\alpha;\zeta_\gamma)\, d\Omega_\zeta \tag{1.13}$$

When one uses the above approach the following questions arise:

- Does double integral (1.13) exist for every $(x_\alpha)$?
- For which $(x_\alpha)$ is Eq (1.13) true?
- What way should $\sigma_{\alpha\beta}$ be calculated from Eq (1.13) for the results to be correct?

The present paper aims at answering the above questions. We will consider the case of a constant temperature field acting upon the thermally insulated domain $\Omega$.

## 2. Displacements

Consider a 2D problem in a space with the temperature field

$$\theta = \begin{cases} \theta_0 & \text{for} \quad (x_1,x_2) \in \Omega \\ 0 & \text{for} \quad (x_1,x_2) \notin \Omega \end{cases} \tag{2.1}$$

where $\theta_0 = \text{const}$.

From Eqs (1.3), (1.11) the following formulae for the displacements $u_\alpha$ yield

$$\begin{aligned} u_1(x_\alpha) = {} & \frac{m\theta_0}{2\pi}\Big\{(x_1-a)\Big(\arctan\frac{x_2-b}{x_1-a} - \arctan\frac{x_2+b}{x_1-a}\Big) - \\ & -(x_1+a)\Big(\arctan\frac{x_2-b}{x_1+a} - \arctan\frac{x_2+b}{x_1+a}\Big) + (x_2-b)\ln\frac{r_1}{r_2} - (x_2+b)\ln\frac{r_3}{r_4}\Big\} \end{aligned} \tag{2.2}$$

$$\begin{aligned} u_2(x_\alpha) = {} & \frac{m\theta_0}{2\pi}\Big\{(x_2-b)\Big(\arctan\frac{x_1-a}{x_2-b} - \arctan\frac{x_1+a}{x_2-b}\Big) - \\ & -(x_2+b)\Big(\arctan\frac{x_1-a}{x_2+b} - \arctan\frac{x_1+a}{x_2+b}\Big) + (x_1-a)\ln\frac{r_1}{r_2} - (x_1+a)\ln\frac{r_2}{r_4}\Big\} \end{aligned}$$

where

$$
\begin{aligned}
r_1^2 &= (x_1-a)^2+(x_2-b)^2 \qquad & r_2^2 &= (x_1+a)^2+(x_2-b)^2 \\
r_3^2 &= (x_1-a)^2+(x_2+b)^2 \qquad & r_4^2 &= (x_1+a)^2+(x_2+b)^2
\end{aligned}
\tag{2.3}
$$

Eqs (2.2) are true at each point of the space and it can be easily shown that the functions $u_\alpha$ are continuous in the entire space.

## 3. Stresses

We can rewrite the formula for the thermoelastic potential $\Phi$ in the following, more suitable form

$$
\Phi(x_\alpha) = \frac{m}{4\pi}\int\limits_{\Omega} \theta(\zeta_\beta)\ln R^2 \, d\Omega_\zeta
\tag{3.1}
$$

In can be noted that the integrand in Eq (1.11) for $(x_\alpha)\notin\Omega$ does not reveal any singular points and the second order derivatives $\Phi_{,\alpha\beta}$ can be calculated directly by differentiation of the integrand

$$
\Phi_{,\alpha\beta} = \frac{m}{4\pi}\int\limits_{\Omega} \theta(\zeta_\gamma)(\ln R^2)_{,\alpha\beta} \, d\Omega_\zeta
\tag{3.2}
$$

Slight obstacles can be encountered, however, when one wants to prove that for $(x_\alpha)\in\Omega$ the second order derivatives $\Phi_{,\alpha\beta}$ can be determined from the formula

$$
\Phi_{,\alpha\beta} = -\frac{m}{4\pi}\Big[\int\limits_{\Gamma} \theta(\zeta_\gamma)(\ln R^2)_{,\beta}\cos(n,\zeta_\alpha)\, d\Gamma_\zeta - \int\limits_{\Omega} \theta(\zeta_\gamma)_{,\alpha}(\ln R^2)_{,\beta}\, d\Omega_\zeta\Big]
\tag{3.3}
$$

where $\Gamma=\partial\Omega$ denotes the boundary of $\Omega$.

Substituting Eq (3.3) into Eq (1.5), the following formulae for the stresses $\sigma_{\alpha\beta}$ in $\Omega$ yield

$$
\begin{aligned}
\sigma_{11}(x_\alpha) &= \\
&= -\frac{m\mu\theta_0}{\pi}\Big(\arctan\frac{x_1-a}{x_2-b}+\arctan\frac{x_1+a}{x_2+b}-\arctan\frac{x_1-a}{x_2+b}-\arctan\frac{x_1+a}{x_2-b}\Big) \\
\sigma_{22}(x_\alpha) &= \\
&= -\frac{m\mu\theta_0}{\pi}\Big(\arctan\frac{x_2-b}{x_1-a}+\arctan\frac{x_2+b}{x_1+a}-\arctan\frac{x_2+b}{x_1-a}-\arctan\frac{x_2-b}{x_1+a}\Big) \\
\sigma_{12}(x_\alpha) &= \frac{m\mu\theta_0}{2\pi}\ln\frac{r_1^2r_4^2}{r_2^2r_3^2}
\end{aligned}
\tag{3.4}
$$

It can be seen that the stresses $\sigma_{\alpha\beta}$ calculated from Eq (3.2) for $(x_\alpha) \notin \Omega$ are the same as those represented by Eqs (3.4), therefore the following conclusion can be drawn: the formulae for stresses (3.4) are true at each point of the space.

Upon analysing the stresses in the vicinity of $\Gamma$ it can be easily seen that the stress $\sigma_{11}$ reveals discontinuity for $x_2 = \pm b$, $\sigma_{22}$ reveals discontinuity for $x_1 = \pm a$, while the stress $\sigma_{12} = \sigma_{21}$ tends to infinity when approaching the corners of $\Omega$ (Fig. 1). The same conclusion for the first time was formulated by Goodier (1937). He also determined the jump, which in our case equals (see Fig. 2)

$$\Delta\sigma = \sigma_{ss}^{(i)} - \sigma_{ss}^{(e)} = -2\mu m(\theta^{(i)} - \theta^{(e)}) = -2\mu m\theta_0 \tag{3.5}$$

while

$$\sigma_{nn}^{(i)} = \sigma_{nn}^{(e)} \tag{3.6}$$

where

$\sigma_{ss}^{(i)}, \sigma_{nn}^{(i)}$ – stresses on the domain boundary, when approaching the discontinuity from the inside of $\Omega$

$\sigma_{ss}^{(e)}, \sigma_{nn}^{(e)}$ – stresses on the domain boundary, when approaching the discontinuity from the outside.

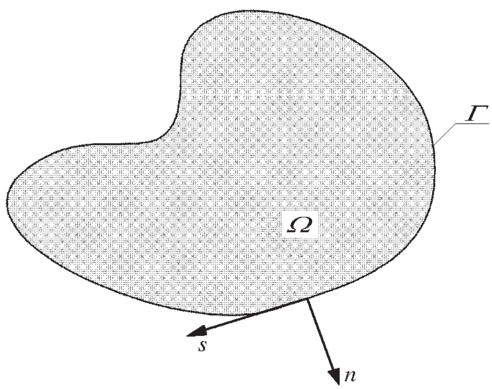


Fig. 2.

Let $(x_\alpha) \in \Omega$, we can calculate then the following iterated integrals

$$
\begin{aligned}
A_{11}(x_\alpha) &= \theta_0 \int\limits_{-b}^{b} \Big( \int\limits_{-a}^{a} \widehat{\sigma}_{11}\; d\zeta_1 \Big)\; d\zeta_2 = -2\mu\theta_0 \int\limits_{-b}^{b} \Big( \int\limits_{-a}^{a} G_{,22}\; d\zeta_1 \Big)\; d\zeta_2 = \\
&= -\frac{m\mu\theta_0}{\pi} \Big( \arctan\frac{x_2-b}{x_1-a} + \arctan\frac{x_2+b}{x_1+a} - \arctan\frac{x_2+b}{x_1-a} - \arctan\frac{x_2-b}{x_1+a} \Big)
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
B_{11}(x_\alpha) &= \theta_0 \int\limits_{-a}^{a} \Big( \int\limits_{-b}^{b} \widehat{\sigma}_{11}\; d\zeta_2 \Big)\; d\zeta_1 = -2\mu\theta_0 \int\limits_{-a}^{a} \Big( \int\limits_{-b}^{b} G_{,22}\; d\zeta_2 \Big)\; d\zeta_1 = \\
&= -\frac{m\mu\theta_0}{\pi} \Big( \arctan\frac{x_1-a}{x_2-b} + \arctan\frac{x_1+a}{x_2+b} - \arctan\frac{x_1-a}{x_2+b} + \arctan\frac{x_1+a}{x_2-b} \Big)
\end{aligned}
$$

It can be seen from the above formulae that for $(x_\alpha) \in \Omega$, $A(x_\alpha) \neq B(x_\alpha)$, on the grounds of the Fubini theorem of the double integral existence, the double integral does not exist $\forall\, x_\alpha \in \Omega$. Therefore, Eqs (1.13) are not true.

To find the reason why for $(x_\alpha) \in \Omega$ Eq (1.13) does not hold let us notice that by virtue of Eqs (1.5) and (1.13) we have

$$
\sigma_{11}(x_\alpha) = \int\limits_{\Omega} \theta(\zeta_\beta)\widehat{\sigma}(x_\alpha;\zeta_\gamma)\; d\Omega_\zeta = -2\mu \int\limits_{\Omega} \theta(\zeta_\beta) G(x_\alpha;\zeta_\beta)_{,22}\; d\Omega_\zeta
\tag{3.8}
$$

however, from Eqs (1.5) and (1.10) it results

$$
\sigma_{11}(x_\alpha) = -2\mu\Phi_{,22} = -2\mu \Big( \int\limits_{\Omega} \theta(\zeta_\beta) G(x_\alpha;\zeta_\gamma)\; d\Omega_\zeta \Big)_{,22}
\tag{3.9}
$$

Having two equations at our disposal (Eqs(3.8) and (3.9)) we can conclude that the latter one is true since in Eq (3.8) the second derivative of the thermoelastic potential is calculated by differentiating directly the integrand, which can be done only for $(x_\alpha) \notin \Omega$.

Comparing the stresses $\sigma_{11}$ calculated from Eq $(3.4)_1$ with the stresses $B_{11}$ yields

$$
\sigma_{11}(x_\alpha) = B_{11}(x_\alpha) = \theta_0 \int\limits_{-a}^{a} \Big( \int\limits_{-b}^{b} \widehat{\sigma}_{11}\; d\zeta_2 \Big)\; d\zeta_1 = -2\mu\theta_0 \int\limits_{-a}^{a} \Big( \int\limits_{-b}^{b} G_{,22}\; d\zeta_2 \Big)\; d\zeta_1
\tag{3.10}
$$

In an analogous way

$$
\sigma_{22}(x_\alpha) = A_{11}(x_\alpha) = \theta_0 \int\limits_{-b}^{b} \Big( \int\limits_{-a}^{a} \widehat{\sigma}_{22}\; d\zeta_1 \Big)\; d\zeta_2 = -2\mu\theta_0 \int\limits_{-b}^{b} \Big( \int\limits_{-a}^{a} G_{,11}\; d\zeta_1 \Big)\; d\zeta_2
\tag{3.11}
$$

It can be proved that the following equations are true

$$\sigma_{12}(x_\alpha) = \sigma_{21}(x_\alpha) = 2\mu\theta_0 \int\limits_{-b}^{b} \Big( \int\limits_{-a}^{a} G_{,12}\; d\zeta_1 \Big)\; d\zeta_2 = 2\mu\theta_0 \int\limits_{-a}^{a} \Big( \int\limits_{-b}^{b} G_{,21}\; d\zeta_2 \Big)\; d\zeta_1 \tag{3.12}$$

## 4. Conclusions

The considerations presented in Section 3 supply answers to the questions posed in the introduction. In the considered case of constant temperature we have proved that:

- Double integral (1.13) does not exist for $(x_\alpha) \in \Omega$
- Eq (1.13) yields correct results for $(x_\alpha) \notin \Omega$
- Eq (1.13) can be employed only in the case when the double integral is iterated, with the integration performed over the variable with respect to which the potential $G(x_\alpha;\zeta_\beta)$ is differentiated, and then over the second variable, i.e. if the potential $G(x_\alpha;\zeta_\beta)$ is differentiated with respect to $x_1$ we integrate it over $\zeta_1$, and then over $\zeta_2$.

The question whether all the above conclusions are true for an arbitrary function $\theta(x_\alpha)$ remains unanswered.

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**O całkowaniu funkcji Greena przy działaniu nieciągłego pola temperatury**

Streszczenie

W pracy rozpatrzono problem zastosowania funkcji Greena dla naprężeń w przypadku działania nieciągłego pola temperatury w 2-wymiarowym obszarze $\Omega$. Wykazano, że całka podwójna z funkcji Greena dla naprężeń nie istnieje. Podano sposób obliczania całek iterowanych z funkcji Greena taki, za pomocą którego uzyskujemy poprawne wzory na naprężenia $\sigma_{\alpha\beta}$.

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