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ANALYSIS OF FREE VIBRATIONS OF A CONTINUOUS-DISCRETE SYSTEM WITH DAMPING

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In this paper a generalized exact method of solving the problem of beam free vibrations with a discrete rigid body attached to the beam by means of viscoelastic constraints was presented. The vibrations of such a discrete-continuous system were described by a set of coupled partial and ordinary differential equations. Separation of the variables and the obtained results of the boundary-value problem, as well as the proved generalized orthogonality of complex eigenmodes, were used to the analysis of free vibrations of the system with arbitrary initial conditions.

Key words: vibrations, discrete-continuous, damping, operator, boundary-value problem

1. Introduction

Mechanical systems consist of various structural elements. In conventional terminology, these systems are divided into discrete and continuous ones. In practice, however, the combined systems, i.e. discrete-continuous are met as well. Such systems are much more complex in terms of mathematical expression and dynamical analysis than separated systems, i.e. only discrete or continuous. The difficulties lead most often to aproximation methods in the dynamical analysis of discrete-continuous systems. The essence of the aproximation method, i.e. the general one implied by the finite element technique is the discretization of the continuous subsystem that belongs to the combined system. Typical examples of application of the classic aproximation methods were presented by Beer and Johnson (1977), Inman (1994) as well as in White's considerations (1985) that were based on Galerkin's method. Modern formulation of the classic aproximation method in the research of dynamic discrete-continuous systems were presented in the papers by Kruszewski

(1975), Nadolski (1994), Pielorz (1992). It should be mentioned, that Pielorz's description (1992) contains wide bibliography survey of the subject. Although the classic aproximation methods are popular and have advantages, some of the inconveniences resulting from this method should be discussed here. Some authors become discouraged from making use of the classic aproximation methods. First of all, the obtained results give discontinuous image of physical phenomena that occur in mechanical systems. Thus, continuous interpretation of the phenomena is disabled. These methods are time-consuming and have limited usability for analytical optimization of a given design as well as involve significant errors. In principle, such inconveniences do not occur in analytical methods and the continuous results can be achieved in that case. Thus, apart from the classic aproximation methods, new analytical methods are also developed. However, on their way one can meet two basic obstacles. The first obstacle comes from the existence of discrete elements in the mechanical system and the second one is related with the presence of damping phenomenon. Kasprzyk and Dan-Tinh (1979) presented the exact method of solving boundary and initial problems that are related to conservative vibration of a discrete-continuous system. Moreover, Nizioł and Snamina (1990) described the exact method of solving the free vibration problem in a discrete-continuous system with damping. This method combines Kasprzyk and Dan-Tinh's (1979) method and Tse, Morse and Hinkler (1978) conception, which was formulated for the analysis of the discrete system with damping. Kasprzyk (1996) proposed the analytical method of solving the free vibration problem in a discrete-continuous system with damping that considerably varies from the systems described by Kasprzyk and Dan-Tinh (1979) as well as by Nizioł and Snamina (1990). However, Kasprzyk's method is accurate exclusively for the cases where the stiffness operator is similar to the damping operator. In other cases, the obtained results are only approximate.

The classic analysis of vibration of discrete-continuous systems modelling more sophisticated structures is presented in papers by Andreev (1970), Mandryka and Monogarov (1975).

Moreover the investigations, which were performed by Lee et al. (1988) are worth to be mentioned, however, they are beyond the scope of this paper.

The monograph by Kukla (1999) and the quoted there extensive bibliography contains a number of exact solutions to boundary-value problems concerning conservative linear mechanical systems with the particular use of Green's function.

To sum up the bibliography survey, one can mention the efforts to formulate generalized principles applicable to vibration analysis of any non-conservative linear element separated from the given mechanical system (see Cabański, 1993, 1994, 1999).

The main purpose of this paper is the introducing of a generalized and unified exact method for solving the free vibration problem of some non-conservative combined linear mechanical systems.

2. Formulation of the problem

2.1. Physical model

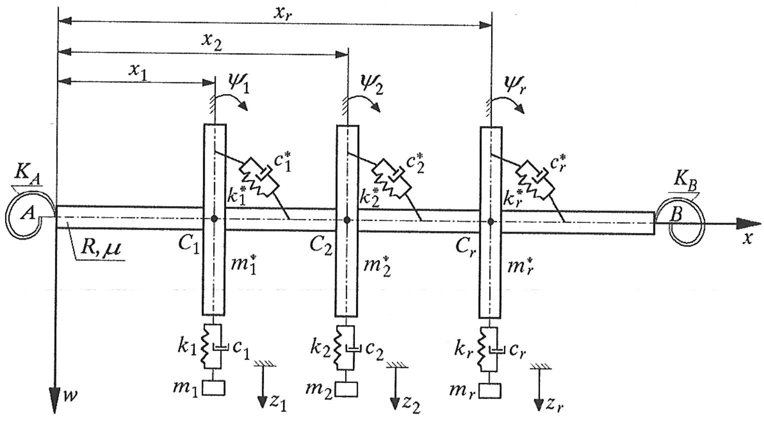

Fig. 1. Physical model of the mechanical continuous-discrete system

The basis of the research that have been carried out in this paper is model of the mechanical system shown in Fig.1. The continuous subsystem in this physical model is a Bernoulli-Euler's beam with optional supports of its ends. The existing constraints at the ends of the beam show, in general, all the possible support cases. Properties of the assumed model depend on the stiffness of spiral springs and kind of the joints used for mounting them to the beam. The discrete subsystem consists of a set of translational and rotational oscillators. Each oscillator comprises a discrete rigid body introducing translational and rotational inertia as well as viscoelastic constraints according to Voigt-Kelvin's model. Mutual exclusion of the interaction between the translatory and rotary motion of the oscillator implies that each discrete body can be presented as two independent rigid bodies, i.e. a particle and thin disc (see Fig.1).

The denotations of the quantities applied for the description of the physical and mathematical model are following:

l	–	length of the beam
t	–	time
x	–	axis in the Cartesian coordinate system
x_j	–	coordinate of location of jth-oscillator
ρ	–	mass density of the beam material
E	–	Young's modulus of the beam material
m_j	–	mass of jth-oscillator corresponding to particle
m_j^*	–	mass moment of inertia of disc in jth-oscillator with respect to its axis of rotation
k_j, k_j^*	–	stiffness coefficients of translational and rotational constraints of jth oscillator, respectively
c_j, c_j^*	–	damping coefficient of translational and rotational constraints of jth oscillator, respectively
K_A, K_B	–	general stiffness coefficients of the beam constraints at its ends, respectively
w	–	deflection of the beam, $w = w(x,t)$
φ	–	slope of the beam, $\varphi = \varphi(x,t)$
z_j	–	displacement of the particle of jth-oscillator, $z_j = z_j(t)$
ψ_j	–	angular displacement of the disc of jth-oscillator, $\psi_j = \psi_j(t)$
F	–	cross-sectional area of the beam, $F = F(x)$
I	–	axial moment of inertia of the beam cross-section, $I = I(x)$
q	–	distributed force acting on the beam, $q = q(x,t)$
q^*	–	distributed moment acting on the beam, $q^* = q^*(x,t)$
P_j	–	concentrated force acting on jth-oscillator, $P_j = P_j(t)$
P_j^*	–	concentrated moment acting on jth-oscillator, $P_j^* = P_j^*(t)$.

2.2. Mathematical model

Vibrations of the discrete-continuous system, presented in Fig.1 are described by the following set of one partial and two ordinary differential equations

$$\frac{\partial^2}{\partial x^2}\Big(R(x)\frac{\partial^2 w}{\partial x^2}\Big) + \mu\frac{\partial^2 w}{\partial t^2} + \sum_{j=1}^{r}\Big(k_j + c_j\frac{\partial}{\partial t}\Big)(w_j - z_j)\delta(x - x_j) +$$
$$-\sum_{j=1}^{r}\Big(k_j^* + c_j^*\frac{\partial}{\partial t}\Big)(\varphi_j - \psi_j)\delta'(x - x_j) = q - \frac{\partial q^*}{\partial x}$$
$$m_j\frac{d^2 z_j}{dt^2} - \Big(k_j + c_j\frac{d}{dt}\Big)(w_j - z_j) = P_j \tag{2.1}$$
$$m_j^*\frac{d^2 \psi_j}{dt^2} - \Big(k_j^* + c_j^*\frac{d}{dt}\Big)(\varphi_j - \psi_j) = P_j^* \qquad j = 1,2,...,r$$

with the boundary conditions

$$\gamma_s(a,t) = 0 \qquad s = 1,2 \quad a = 0,l \tag{2.2}$$

and the initial conditions

$$\begin{array}{lll} w_0 = w(x,0) & z_{0j} = z_j(0) & \psi_{0j} = \psi_j(0) \\ \dot{w}_0 = \dfrac{\partial w}{\partial t}\Big|_{t=0} & \dot{z}_{0j} = \dfrac{\partial z_j}{\partial t}\Big|_{t=0} & \dot{\psi}_{0j} = \dfrac{\partial \psi_j}{\partial t}\Big|_{t=0} \\ x \in (0,l) & t \in [0,\infty) & j = 1,2,...,r \end{array} \tag{2.3}$$

where $R = EI$ denotes the flexural rigidity of the beam, $\mu = \rho F$ is the mass of the beam per unit length, $\delta(\cdot)$ and $\delta'(\cdot)$ presents the Dirac delta function and its derivative, respectively. Moreover, some of the quantities appearing in Eqs (2.1) are defined as follows

$$\begin{array}{ll} w_j = w(x_j,t) & z_j = z_j(x_j,t) \\ \varphi_j = \dfrac{\partial w}{\partial x}\Big|_{x=x_j} & \psi_j = \dfrac{\partial z}{\partial x}\Big|_{x=x_j} \end{array} \tag{2.4}$$

where $z(x,t)$ denotes fictitious function which is filterd by Dirac's delta function at the points x_j and simultaneously suppressed outside these points.

In order to express boundary conditions (2.2) in the explicit form, the following equations of constraints at the beam ends can be applied (see Pietrowski, 1967)

$$\gamma_s(x,t) = \sum_{p=1}^{2}\frac{\partial^k}{\partial x^4}\Big[\alpha_{sp}^{(a)}w(x,t) + \beta_{sp}^{(a)}\frac{\partial^2 w(x,t)}{\partial x^2}\Big] \qquad s = 1,2 \tag{2.5}$$

where $\alpha_{sp}^{(a)}$, $\beta_{sp}^{(a)}$ are the coefficients describing the kind of the beam supports and $k = \delta_{2p}$, here δ_{2p} is Kronecker's delta. After differentation of the right-hand side of equation (2.5), and replacing x with $a = 0, l$ we get this formula in fully explicit form.

It will be convenient for further investigations to rewrite Eqs (2.1) in the following vectorial form

$$\mathsf{M}\frac{\partial^2 \boldsymbol{u}}{\partial t^2} + \mathsf{L}\frac{\partial \boldsymbol{u}}{\partial t} + \mathsf{K}\boldsymbol{u} = \boldsymbol{F} \tag{2.6}$$

where

$$\boldsymbol{u} = \begin{bmatrix} w(x,t) \\ z(x,t) \end{bmatrix} \qquad \boldsymbol{F} = \begin{bmatrix} g(x,t) \\ p(x,t) \end{bmatrix} \tag{2.7}$$

are the vectors of displacements and loads of the system (Fig.1), respectively, besides

$$\mathsf{M} = \begin{bmatrix} \mu(x) & 0 \\ 0 & M_j \end{bmatrix} \qquad \mathsf{K} = \begin{bmatrix} R_j & -K_j \\ -K_j & K_j \end{bmatrix} \qquad \mathsf{L} = \begin{bmatrix} L_j & -L_j \\ -L_j & L_j \end{bmatrix} \tag{2.8}$$

are the global, linear operators of inertia, stiffness and damping, respectively.

Whereas

$$\begin{aligned} R_j &= \frac{\partial^2}{\partial x^2}\Big(R(x)\frac{\partial^2}{\partial x^2}\Big) + K_j \\ M_j &= \sum_{j=1}^{r}\Big(m_j - m_j^*\frac{d}{dx}\frac{\partial}{\partial x}\Big)\delta(x - x_j) \\ K_j &= \sum_{j=1}^{r}\Big(k_j - k_j^*\frac{d}{dx}\frac{\partial}{\partial x}\Big)\delta(x - x_j) \\ L_j &= \sum_{j=1}^{r}\Big(c_j - c_j^*\frac{d}{dx}\frac{\partial}{\partial x}\Big)\delta(x - x_j) \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} g(x,t) &= q - \frac{\partial q^*}{\partial x} \\ p(x,t) &= \sum_{j=1}^{r}\Big(P_j(t) - P_j^*(t)\frac{d}{dx}\Big)\delta(x - x_j) \end{aligned} \tag{2.10}$$

It should be noticed, that the operators K and L are self-adjoint because of their symmetry (see Banach, 1932), and only in particular cases they can be homothetic.

3. Boundary-value problem

3.1. Separation of variables

In the case of free vibrations, i.e. when $F \equiv 0$ Eq (2.6) reduces to the form

$$\mathsf{M}\frac{\partial^2 \boldsymbol{u}}{\partial t^2} + \mathsf{L}\frac{\partial \boldsymbol{u}}{\partial t} + \mathsf{K}\boldsymbol{u} = \boldsymbol{0} \tag{3.1}$$

The classic Fourier method of separation of the variables in Eq (3.1) can be applied in the case, when $\mathsf{L} \equiv \boldsymbol{0}$, i.e for the undemped system or, if the operators K and L are homothetic, i.e. when the vectors appearing in Eq (3.1) are colinear (see Kasprzyk, 1996). In general, these vectors are coplanar and separation of the variables in Eq (3.1) can be achieved only when the problem is extended to the complex Hilbert space (see Maurin, 1959). Such extension makes if possible to separate the variables like in the classic case (see Cabański, 1999)

$$\boldsymbol{u} = \boldsymbol{U}^\top \tag{3.2}$$

where

$$\boldsymbol{U}(x) = \begin{bmatrix} W \\ Z \end{bmatrix} \tag{3.3}$$

is the vector of the complex modes of vibrations, $T = T(t)$ denotes a scalar function of motion, besides $W = W(x)$ and $Z = Z(x)$.

Substituting Eq (3.2) into Eq (3.1) we obtain the ordinary differential equation of motion

$$\dot{T} - \mathrm{i}\nu T = 0 \tag{3.4}$$

and the vectorial equation of mechanical impedance

$$(\nu^2\mathsf{M} - \mathsf{K} - \mathrm{i}\nu\mathsf{L})\boldsymbol{U} = \boldsymbol{0} \tag{3.5}$$

where $\nu = \mathrm{i}\eta + \omega$ denotes the complex vibration frequency, in wich ω and η stand for the angular frequency and damping, respectively.

Equation (3.5) can be written in the scalar form expressed by a set of one ordinary differential equation and two linear algebraic ones

$$\begin{aligned}&\frac{d^2}{dx^2}\Big(R(x)\frac{d^2W}{dx^2}\Big) - \nu^2\mu W + \sum_{j=1}^{r}\kappa_j(W_j - Z_j)\delta(x - x_j) + \\ &-\sum_{j=1}^{r}\kappa_j^*(\Phi_J - \Psi_j)\delta'(x - x_j) = 0\end{aligned} \tag{3.6}$$

$$\nu^2 m_j Z_j + \kappa_j (W_j - Z_j) = 0 \qquad j = 1, 2, ..., r$$

$$\nu^2 m_j^* \Psi_j + \kappa_j^* (\Phi_j - \Psi_j) = 0$$

where

$$\kappa_j = k_j + \mathrm{i}\nu c_j \qquad \kappa_j^* = k_j^* + \mathrm{i}\nu c_j^* \tag{3.7}$$

are the complex stiffnesses corresponding to the translational and rotational components of complex constraints of the jth oscillator, respectively.

Moreover, some of the quantities appearing in Eqs (3.6) we define as follows

$$\begin{aligned} W_j &= W(x_j) & Z_j &= Z(x_j) \\ \Phi_j &= \left.\frac{dW}{dx}\right|_{x=x_j} & \Psi_j &= \left.\frac{dZ}{dx}\right|_{x=x_j} \end{aligned} \tag{3.8}$$

3.2. Transformation of the equation set

The two algeraic equations occuring in Eqs (3.6) can be expressed in the simplified form

$$Z_j = -\frac{\kappa_{Ij}}{\kappa_{IIj}} W_j \qquad \Psi_j = -\frac{\kappa_{Ij}^*}{\kappa_{IIj}^*} \Phi_j \tag{3.9}$$

where

$$\kappa_{Ij} = \frac{\kappa_{IIj}\kappa_j}{\kappa_{IIj} - \kappa_j} \qquad \kappa_{Ij}^* = \frac{\kappa_{IIj}^*\kappa_j^*}{\kappa_{IIj}^* - \kappa_j^*} \tag{3.10}$$

and

$$\kappa_{IIj} = \nu^2 m_j \qquad \kappa_{IIj}^* = \nu^2 m_j^* \tag{3.11}$$

Eqs (3.9) and (3.10) imply existence of the motionless nodes O_j and O_j^* at the translational and rotational components of the complex constraints of the jth oscillator, as shown in Fig.2a and Fig.2b, respectively.

The above-mentioned "nodes" split this mechanical system but yet do not exclude a dynamical interaction between the discrete rigid bodies and the continuous subsystem at the same time.

With the help of Eqs (3.9) ÷ (3.11) the differential equation, which appears in Eqs (3.6), is reduced to the goemetrically isolated form

$$\begin{aligned} &\frac{d^2}{dx^2}\Big(R(x)\frac{d^2W}{dx^2}\Big) - \nu^2\mu(x)W + \sum_{j=1}^{r} \kappa_{Ij} W(x_j)\delta(x - x_j) + \\ &- \sum_{j=1}^{r} \kappa_{Ij}^* \left.\frac{dW}{dx}\right|_{x=x_j} \delta'(x - x_j) = 0 \end{aligned} \tag{3.12}$$

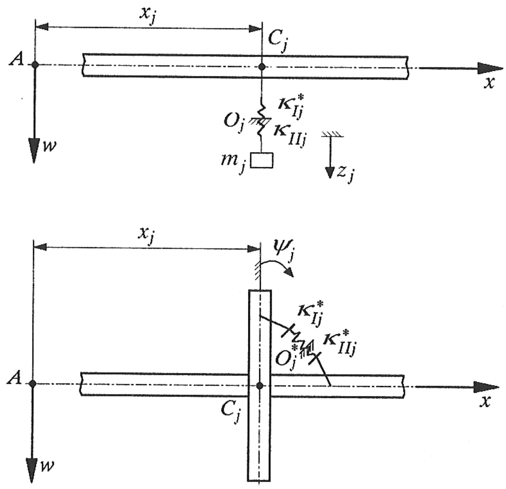

Fig. 2. Scheme of formation of the motionless nodes O_j and O_j^* on jth constraint: (a) translational constrain, (b) rotational constraint

3.3. Solution to the differential equation

The general solution to Eq (3.12) has the form

$$W(x) = \sum_{k=1}^{4} D_k f_k(x) - \sum_{j=1}^{r} \kappa_{Ij} W(x_j) G_j + \sum_{j=1}^{r} \kappa_{Ij}^* \frac{dW}{dx}\Big|_{x=x_j} \frac{dG_j}{dx} \tag{3.13}$$

where $f_k(x)$ $(k = 1,2,3,4)$ are the linearly independent particular solutions to the following differential equation

$$\frac{d^2}{dx^2}\Big(R(x)\frac{d^2W}{dx^2}\Big) - \nu^2\mu(x)W = 0 \tag{3.14}$$

where D_k $(k = 1,2,3,4)$ are arbitrary integration constants and $G_j = G(x, x_j)$ is the solution to the following differential equation

$$\frac{d^2}{dx^2}\Big(R(x)\frac{d^2G_j}{dx^2}\Big) - \nu^2\mu(x)G_j = \delta(x - x_j) \tag{3.15}$$

with the zero boundary conditions. This solution is well-known as the Green function.

3.4. Solution to the boundary-value problem

By analog with Eq (3.2), the left-hand side of Eq (2.2) can be written in the form

$$\gamma_s(a,t) = \Gamma_s(a)T(t) \tag{3.16}$$

Equating to zero the right-hand side of Eq (3.16) for $T(t) \neq 0$, one obtains homogeneous boundary conditions of Sturm's type (see Pietrowski, 1967)

$$\varGamma_s(a) = 0 \qquad s = 1,2 \qquad a = 0,l \tag{3.17}$$

Like it was done in Eqs (3.16) and (3.2) one can write

$$\gamma_s(x,t) = \varGamma_s(x)T(t) \qquad w(x,t) = W(x)T(t) \tag{3.18}$$

Then, substituting Eq (3.18) into Eq (2.5) we obtain the formula presenting the explicit form of boundary conditions of Sturm,s type (Pietrowski, 1967)

$$\varGamma_s(x) = \sum_{p=1}^{2} \frac{d^k}{dx^k}\Big(\alpha_{sp}^{(a)} W(x) + \beta_{sp}^{(a)} \frac{d^2W}{dx^2}\Big) \qquad s = 1,2 \tag{3.19}$$

The coefficients $\alpha_{sp}^{(a)}$ and $\beta_{sp}^{(a)}$ $(s,p = 1,2)$ are determined from the consistency conditions of generalized internal forces and displacements of the beam and constraints at its ends.

The requirements concerning Eq (3.19) are the same as in an instance of terms Eq (2.5). The aforementioned coefficients should satisfy the following equality (see Pietrowski, 1967)

$$\begin{vmatrix} \alpha_{11}^{(a)} & \beta_{12}^{(a)} \\ \alpha_{21}^{(a)} & \beta_{22}^{(a)} \end{vmatrix} = \begin{vmatrix} \alpha_{12}^{(a)} & \beta_{11}^{(a)} \\ \alpha_{22}^{(a)} & \beta_{21}^{(a)} \end{vmatrix} \qquad a = 0,l \tag{3.20}$$

By applying solution (3.13) to formula (3.19), then substituting the obtained results into boundary conditions (3.17) the homogeneous system of simultaneous linear algebraic equations can be constituted, which in a matrix notation takes the following form

$$\mathbf{A}\boldsymbol{X} = \mathbf{0} \tag{3.21}$$

where $\mathbf{A}$ is the coefficient matrix with respect to ν and $\boldsymbol{X}$ is the vector of unknowns of the system.

The system of equations has a nontrivial solution, provided that matrix $\mathbf{A}$ is singular, i.e. the determinant of this matrix is equal to zero. Hence, the transcendental, complex frequency equation can be written in the symbolic form as

$$\det \mathbf{A} = 0 \tag{3.22}$$

The solution to Eq (3.22) can be presented as an infinite sequence of the complex eigenfrequencies $\{\nu_n\}$, where $\nu_n = \mathrm{i}\eta_n + \omega_n$ $(n = 1,2,...)$.

With regard to the singularity of the matrix $\mathsf{A}(\nu_n)$, the solution to Eqs (3.21) gives an infinite sequence of the complex vectors $\{\boldsymbol{X}_n\}$, $\boldsymbol{X}_n = [D_{1n}, D_{2n}, D_{3n}, D_{4n}]^\top$, corresponding to ν_n. By substituting ν_n and $\boldsymbol{X}_n$; $n = 1, 2, ...$, succesively into Eq (3.13) and using the obtained results in Eqs (3.9), we obtain an infinite sequence of the complex eigenvectors $\{\boldsymbol{U}_n\}$, $\boldsymbol{U}_n = [W_n, Z_{1n}, ..., Z_{rn}, \Psi_{1n}, ..., \Psi_{rn}]^\top$, of the boundary-value problem corresponding to ν_n and satisfying Eqs (3.6) and (3.17).

According to notation (3.3), the eigenvector $\boldsymbol{U}_n$ can be written in the compact form

$$\boldsymbol{U}_n = \begin{bmatrix} W_n \\ Z_n \end{bmatrix} \tag{3.23}$$

where $W_n = W_n(x)$ and $Z_n = Z_n(x)$ are the complex eigenmodes corresponding to the continuous subsystem and discrete oscillators, respectively, but yet we must remember, that relations (3.8) are still obligatory.

The above presented solution to the boundary-value problem and the fundamental principle (see Cabański, 1999)

$$\Big([(\nu_n + \nu_m)\mathrm{i}\mathsf{M} + \mathsf{L}]\boldsymbol{U}_n, \boldsymbol{U}_m\Big) = N_n\delta_{nm} \tag{3.24}$$

defining the generalized orthogonality condition of the eigenvectors $\boldsymbol{U}$ are the base for solving the free and forced vibration problems (δ_{nm} denotes Kronecker's delta).

4. Free vibration

The general solution to Eq (3.1) with homogeneous boundary conditions (2.2) and initial conditions (2.3) is a linear combination of linearly independent particular solutions, as follows

$$\boldsymbol{u} = \sum_{n=1}^{\infty} \boldsymbol{U}_n T_n \tag{4.1}$$

Replacing T_n in Eq (4.1) by the general solution to differential equation (3.4), i.e. $C_n \exp(\mathrm{i}\nu_n t)$ we get

$$\boldsymbol{u} = \sum_{n=1}^{\infty} C_n \boldsymbol{U}_n \exp(\mathrm{i}\nu_n t) \tag{4.2}$$

where C_n are arbitrary integration constants.

The existence of technically possible initial conditions (2.3) implies existence of an expansion (4.2).

In order to find the constants C_n the following formula is used

$$C_n = \frac{\Big([\mathbf{M}(\mathrm{i}\nu_n \boldsymbol{u}_0 + \dot{\boldsymbol{u}}_0) + \mathbf{L}\boldsymbol{u}_0], \boldsymbol{U}_n\Big)}{\Big([2\mathrm{i}\nu_n\mathbf{M} + \mathbf{L}]\boldsymbol{U}_n, \boldsymbol{U}_n\Big)} \tag{4.3}$$

which, owing to orthogonality condition (3.24), was proved by Cabański (1999). In (4.3) $\boldsymbol{u}_0$ and $\dot{\boldsymbol{u}}_0$ are the vectors of initial displacement and velocity, respectively, according to initial conditions (2.3) presented in the scalar form.

Effective calculation of C_n is possible after transforming formula (4.3) into the scalar form

$$C_n = \frac{J_n}{N_n} \tag{4.4}$$

where

$$\begin{aligned} J_n = &\int\limits_0^l \mu(x)(\mathrm{i}\nu_n w_0 + \dot{w}_0)W_n\,dx + \sum_{j=1}^r \Big[m_j(\mathrm{i}\nu_n z_{0j} + \dot{z}_{0j})Z_{jn} + \\ &+ m_j^*(\mathrm{i}\nu_n \psi_{0j} + \dot{\psi}_{0j})\varPsi_{jn} + c_j(w_{0j} - z_{0j})(W_{jn} - Z_{jn}) + \\ &+ c_j^*(\varphi_{0j} - \psi_{0j})(\varPhi_{jn} - \varPsi_{jn})\Big] \\ N_n = &\, 2\mathrm{i}\nu_n \int\limits_0^l \mu(x)W_n^2\,dx + \sum_{j=1}^r \Big[2\mathrm{i}\nu_n(m_j Z_{jn}^2 + m_j^*\varPsi_{jn}^2) + \\ &+ c_j(W_{jn} - Z_{jn})^2 + c_j^*(\varPhi_{jn} - \varPsi_{jn})^2\Big] \end{aligned} \tag{4.5}$$

Now, using the previously obtained results, i.e. the constants C_n and components of the eigenvectors $\boldsymbol{U}_n$ in vectorial from (4.2), we obtain the exact solution to the free vibration problem

$$\left[\begin{array}{c} w \\ z_j \\ \psi_j \end{array}\right] = \sum_{n=1}^{\infty} C_n \left[\begin{array}{c} W_n \\ Z_{jn} \\ \varPsi_{jn} \end{array}\right] \exp(\mathrm{i}\nu_n t) \qquad j = 1, 2, ..., r \tag{4.6}$$

To this end, by expressing the complex components appearing the right-hand side of solution (4.6) in a trigonometrical form and due to the existence of complex conjugated components, solution (4.6) finally takes a more classic and explicit form

$$
\begin{aligned}
w(x,t) &= \sum_{n=1}^{\infty} |C_n||W_n| \mathrm{e}^{-\eta_n t} \cos(\omega_n t + \Theta_n + \vartheta_n) \\
z_j(t) &= \sum_{n=1}^{\infty} |C_n||Z_{jn}| \mathrm{e}^{-\eta_n t} \cos(\omega_n t + \Theta_n + \vartheta_{jn}) \\
\psi_j(t) &= \sum_{n=1}^{\infty} |C_n||\Psi_{jn}| \mathrm{e}^{-\eta_n t} \cos(\omega_n t + \Theta_n + \vartheta^*_{jn})
\end{aligned}
\tag{4.7}
$$

where $\Theta_n = \arg C_n$, $\vartheta_n = \arg W_n$, $\vartheta_{jn} = \arg Z_{jn}$ and $\vartheta^*_{jn} = \arg \Psi_{jn}$.

5. Example

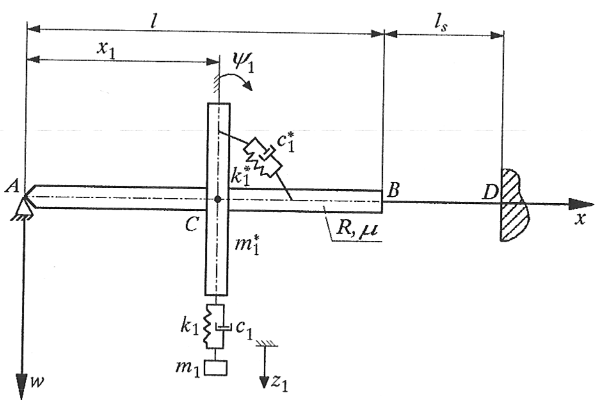

Fig. 3. Exemplary mechanical continuous-discrete system

In order to exercise the aforedescribed method a particular discrete-continuous system is assumed, as shown in Fig.3. The continuous subsystem of this combined system is an elastic, prismatic beam hinged at the end A and fixed to the weightless flexion spring at the end B. The second end of the flexion spring is clamped onto the rigid wall in point D. The flexural rigidity and length of the flexion spring are denoted by R_s and l_s, respectively. The following initial conditions are assumed

$$w_0 = a\sin\tfrac{\pi}{b}x \qquad z_0 = \sin\tfrac{\pi l}{2b} \qquad \psi_0 = \tfrac{\pi a}{b}\cos\tfrac{\pi l}{2b}$$

$$\dot{w}_0 = 0 \qquad \dot{z}_0 = 0 \qquad \dot{\psi}_0 = 0$$

The calculations are carried out for the following data

$$R = 3.75\cdot 10^5\mathrm{Nm}^2 \qquad \mu = 40\mathrm{kg\ m}^{-1} \qquad m = 50\mathrm{kg}$$

$$m^* = 5\mathrm{kg\ m}^2 \qquad k = 2.5\cdot 10^5\mathrm{Nm}^{-1} \qquad k^* = 1.0\cdot 10^5\mathrm{Nm}$$

$$c = 12\cdot 10^2\mathrm{Nsm}^{-1} \qquad c^* = 3\cdot 10^2\mathrm{Nsm} \qquad l = 3\mathrm{m}$$

$$x_1 = 1.5\mathrm{m} \qquad a = 0.01\mathrm{m} \qquad b = 4\mathrm{m}$$

In the presented example Eq (3.13) takes the particular form

$$\begin{aligned}W(x) &= D_1\sinh\lambda x + D_2\sin\lambda x + D_3\cosh\lambda x + D_4\cos\lambda x + \\ &-\frac{\kappa_I}{2\lambda^3}W(x_1)[\sinh\lambda(x-x_1) - \sin\lambda(x-x_1)]H(x-x_1) + \\ &+\frac{\kappa_I^*}{2\lambda^2}\frac{dW}{dx}\Big|_{x=x_1}[\cosh\lambda(x-x_1) - \cos\lambda(x-x_1)]H(x-x_1)\end{aligned} \tag{5.1}$$

where

$$\lambda = \sqrt[4]{\frac{\mu\nu^2}{R}} \tag{5.2}$$

and where $H(x-x_1)$ is Heaviside's function.

In Table 1 the coefficients of the contraints of the beam ends are shown (Fig.3).

Table 1

	$\alpha_{sp}^{(0)}$		$\beta_{sp}^{(0)}$		$\alpha_{sp}^{(l)}$		$\beta_{sp}^{(l)}$	
s \ p	1	2	1	2	1	2	1	2
1	1	0	0	0	1	0	$-\dfrac{l_S^2 R}{2R_S}$	$-\dfrac{l_S^3 R}{3R_S}$
2	0	0	1	0	0	1	$\dfrac{l_S R}{R_S}$	$\dfrac{l_S^2 R}{2R_S}$

The further considerations in this example were carried out according to the presented algorithm.

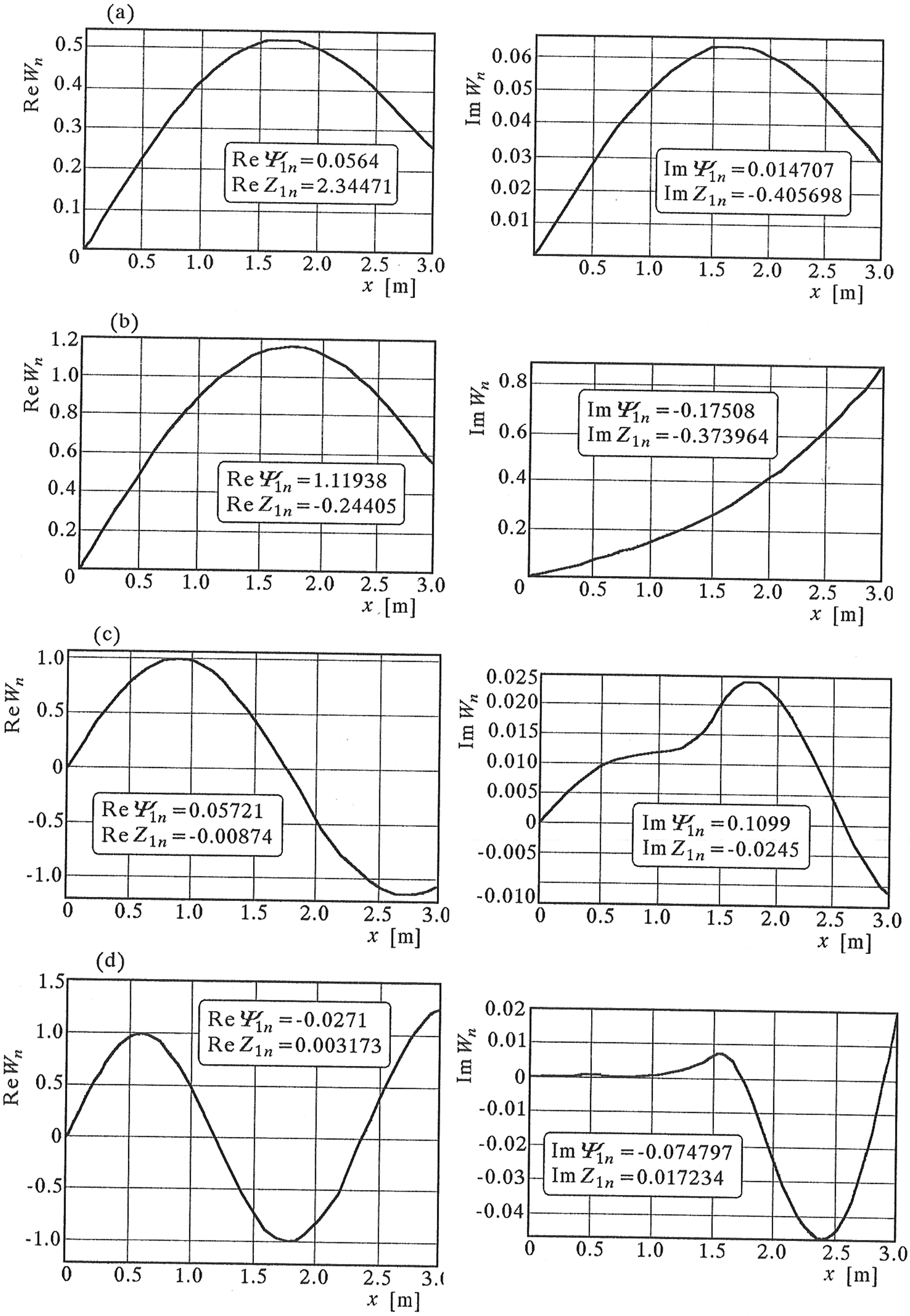

Fig. 4. Complex eigenmodes W_n and Z_n, Ψ_n of the beam and osicillator, respectively, and the corresponding eigenfrequencies ν_n; (a) $n = 1$, (b) $n = 2$, (c) $n = 3$, (d) $n = 4$

6. Conclusions

- The final results presented in the form of Eqs (4.6) and (4.7) can be applied to practical calculations. The form of Eq (4.6) has practical character, whereas the form of Eq (4.7) is rather of cognitive importance. Moreover, the form of Eq (4.7) confirms unambiguously the correctness of this method.

- The calculations of complex eigenfrequencies and eigenmodes presented in the example (see Fig.4), confirm that the described method is useful from the practical point of view.

- From the analytical considerations and calculations, mainly concerning the eigenmodes as shown in Fig.4, one can draw a conclusion that the phenomenon of vibrations possesses dual character. The simultaneous existence of two eigenvectors, i.e. $\mathrm{Re}\boldsymbol{U}_n$ and $\mathrm{Im}\boldsymbol{U}_n$ corresponding to the complex eigenfreqencies $\nu_n = \mathrm{i}\eta_n \pm \omega_n$, supports this binary character. Let us understand that the reason for the binary phenomenon is the so-called biinertia or else apparent interia, i.e. real inertia and damping.

- The oscillators connected with the continuous subsystem often play a role of dynamical dampers or exciters of certain components of the subsystem vibrations. The parameters of these dampers or exciters can be easily determined by means of formulas (3.9), (3.10), (3.11) and (3.7).

- The operational principle describing the generalized orthogonality condition of complex eigenvectors (3.24) and operational formula (4.3) are the invariants of this method. The separation of the variables in Eqs (3.4) and (3.5) as well as the invariants can be used for solving the free vibration problem for any linear visco-elastic system.

- In particular cases, when the oparators K and L are homothetic and the phase angles ϑ_n, ϑ_{jn} and ϑ^*_{jn} invariable, Eqs (4.7) can be reduced to the well-known form by the classical Fourier method (see Osiński, 1980; Kasprzyk, 1996).

- Solution to the steady-state forced vibration problem requires:
  - completing the right-hand side of Eqs (3.6) with amplitudes of forcing loads
  - replacing the complex frequency ν with real the frequencies of the forcing loads ω

– assuming that forcing frequencies are the same and invariable in time.

The problem of vibration forced by arbitrary loads one can solve using principles given by Cabański (1999).

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Analiza drgań swobodnych układu ciągło-dyskretnego z tłumieniem

Streszczenie

W pracy przedstawiono uogólnioną ścisłą metodę rozwiązania zagadnienia drgań swobodnych belki z dołączonymi do niej, za pośrednictwem lepko-sprężystych więzi, skoncentrowanymi sztywnymi ciałami. Drgania układu ciągło-dyskretnego zostały opisane układem sprzężonym, tj. jednym cząstkowym i dwoma podukładami zwyczajnych równań różniczkowych. Rozdzielenie zmiennych i wyniki uzyskane z rozwiązania problemu brzegowego oraz warunek ortogonalności postaci drgań własnych wykorzystano do analizy drgań swobodnych tego układu mechanicznego przy dowolnych warunkach początkowych.

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