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VIBRATIONS OF DISCRETE-CONTINUOUS MODELS OF LOW STRUCTURES WITH A NONLINEAR SOFT SPRING

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In the paper discrete-continuous models with a local nonlinearity are proposed for the dynamic investigations of low structures subject to kinematic excitations caused by transverse waves. The models consist of rigid bodies and of elastic elements which only undergo shear deformations, while the local nonlinearity is represented by a damper and a nonlinear spring. It is assumed that the nonlinear characteristic of the spring is of a soft type. In the paper this characteristic is described by four nonlinear functions. In the discussion a wave approach is used. Numerical calculations are performed for the models with two, three and four rigid bodies for a harmonic kinematic excitation. They focus on the investigation of the effect of the local nonlinearity expressed by various functions on displacements of selected cross-sections of the elastic elements in the considered models, and on the determination of the application ranges of these functions.

Key words: nonlinear dynamics, discrete-continuous models, waves

1. Introduction

The paper deals with the dynamic analysis of low structures subject to kinematic excitations caused by transversal waves using discrete-continuous models with a local nonlinearity. The discrete-continuous models consist of rigid bodies and elastic elements, while the local nonlinearity is represented by a damper and a nonlinear spring. It is assumed that ponderable elastic elements in these models only undergo shear deformations and their motion can be described by the classical wave equation.

Linear discrete-continuous models of low structures subject to shear deformations were studied in the paper by Pielorz (1996). The aim of the present paper is to generalise the results obtained by Pielorz (1996) by including the local nonlinearity in the appropriate discrete-continuous models. The inclusion of that type of nonlinearities is justified by many engineering solutions for low structures, see e.g. Humar (1990), Mengi and Dündar (1988), Okamoto (1973), Sackman and Kelly (1979), Su et al. (1989).

In the present paper the approach used by Pielorz (1996) for linear models of structures undergoing shear deformations is adopted to the discussion of discrete-continuous models with the local nonlinearity represented by a spring having a nonlinear characteristic. This characteristic can be of a hard as well as of a soft type. The case of the hard characteristic was considered by Pielorz (1998). Here, the discussion is focused on the case when the spring characteristic is of a soft type. In the paper such a nonlinearity is described by four nonlinear functions including a polynomial of the third degree. In numerical calculations a harmonic kinematic excitation is assumed and the effect of the parameters of the local nonlinearity described by these nonlinear functions on the amplitude-frequency curves for selected multi-mass systems is considered. Also the application ranges of these functions are discussed and determined for selected parameters of the studied systems. The numerical analysis is done for two-mass, three-mass and four-mass systems undergoing shear deformations.

2. Forces in the nonlinear spring

In the discrete-continuous systems considered below, the local nonlinearity can be introduced in an arbitrary cross-section. Such a nonlinearity can be generally described by an arbitrary nonlinear function. In the discussion of the dynamics of nonlinear discrete systems the polynomial of the third degree is exploited most widely for the description of the considered nonlinearities, cf Hagedorn (1981), Mickens (1981), Szemplińska-Stupnicka (1990). This polynomial was used by Pielorz (1998) in the case of the local nonlinearities having a hard type characteristic. In the present paper it is assumed that the nonlinear characteristic of the spring is soft, and now the polynomial of the third degree can be also used. Analogous to the nonlinearities in discrete systems, the force in the nonlinear spring with a symmetric characteristic could be described by the following function

$$F(t) = k_1 y_i + k_3 y_i^3 \tag{2.1}$$

where y_i is the displacement in the appropriate cross-section of the ith elastic element, and the constants k_1 and k_3 represent linear and nonlinear terms in function (2.1), respectively. Polynomial (2.1) includes the linear case for

$k_3 = 0$, the hard characteristic for $k_3 > 0$ and the soft characteristic for $k_3 < 0$.

In many nonlinear discrete systems, where function (2.1) with $k_3 < 0$ (soft characteristic) is used, phenomena such as escapes from potential wells may occur, cf Steward et al. (1995). In order to avoid the divergence of the numerical solutions to infinity in the case of the soft characteristic, apart from polynomial (2.1) with $k_3 < 0$, the following three functions are proposed for the descriptions of forces in the nonlinear spring

$$F(t) = A\sin(By_i) \tag{2.2}$$

$$F(t) = A\tanh(By_i) \tag{2.3}$$

$$F(t) = \begin{cases} A[-1+\exp(By_i)] & \text{for } \ y_i \leqslant 0 \\ A[1-\exp(-By_i)] & \text{for } \ y_i \geqslant 0 \end{cases} \tag{2.4}$$

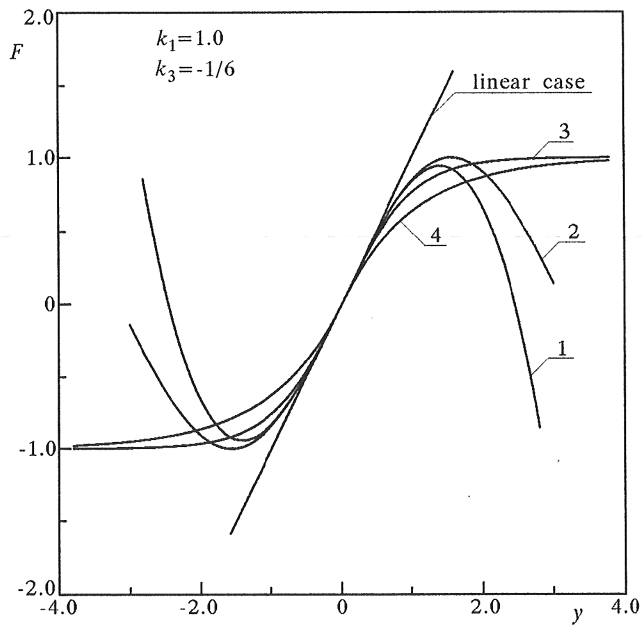

Fig. 1. Diagrams of nonlinear functions (2.1) ÷ (2.4)

The constants A and B are selected in such a way that the expansion of functions (2.1) ÷ (2.4) in series gives the same linear case and that polynomial (2.1) and functions (2.2) ÷ (2.4) have maximum values close to each other.

Then

$$AB = k_1 \qquad AB^3 = -6k_3 \tag{2.5}$$

Below, for convenience nonlinear functions (2.1) ÷ (2.4), will sometimes be called functions (1) ÷ (4), correspondingly.

Exemplary diagrams of nonlinear functions (2.1) ÷ (2.4) are shown in Fig.1 in the interval $y \in < -3.9, 3.9 >$ for $k_1 = 1$, $k_3 = -1/6$ together with the linear case obtained for $k_3 = 0$. It can be seen that all the functions are linear in the neighbourhood of the origin of coordinates and have similar maxima.

3. Assumptions and governing equations

The paper concerns dynamic investigations of low structures subject to kinematic excitation caused by transverse waves. The kinematic excitations can be of a seismic type or can be caused by highway traffic, surface and subsurface railways, and by machinery in a nearby location. In the literature, engineering structures subject to various kinematic excitations are discussed using discrete as well as continuous models, cf Okamoto (1973), Sackman and Kelly (1979), Gasparini et al. (1981) and Mengi and Dündar (1988).

The elastic elements of the structures considered in the present paper have the transverse dimension, alongside of which shear forces act, close to the length of the element, i.e. they have a low slenderness ratio. To such structures belong, e.g., machine supports, bridge piers and low columns in buildings. Many structure elements subject to transverse excitation can be modelled by means of a Timoshenko's beam. In the work by Pielorz (1996) it is shown that in the case of short beams, in which shear forces are predominant, the Timoshenko equations can be replaced by the classical wave equation.

The studied model consists of n elastic elements connected by rigid bodies, see Fig.2. When subject to external excitations all cross-sections of the elastic elements remain flat and parallel to the cross-sections where rigid bodies are located. The elastic elements only undergo shear deformations. They may have different mechanical properties, however for simplicity it is assumed that all the elements are characterised by the shear modulus G, cross-sectional area A, shear coefficient k, density ρ and the length l. A discrete element with a nonlinear spring can be attached to the rigid body m_0. Such an element may represent various parts of the considered structures, which ought to be described by local nonlinearities. For example, it may represent an elastic

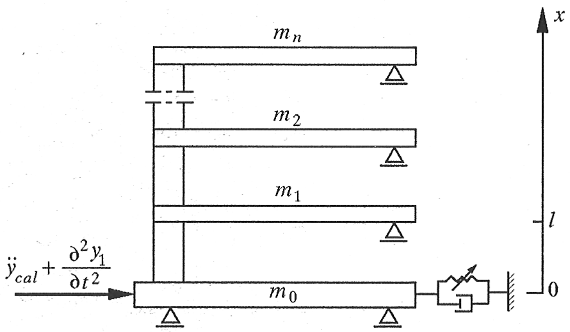

Fig. 2. Nonlinear discrete-continuous model

segment of an isolation type, cf Su et al. (1989), Humar (1990). The characteristic of the nonlinear spring is assumed to be of a soft type, and the spring force is expressed by nonlinear functions (2.1) $\div$ (2.4) with $k_3 < 0$, correspondingly. Now, the displacement y_i in formulas (2.1) $\div$ (2.4) is replaced by the displacement y_1 of the first elastic element in the cross-section $x = 0$.

The rigid body m_0 is subject to the absolute acceleration $\partial^2[y_1(0,t) + y_{cal}(t)]/\partial t^2$, where $y_1(0,t)$ is the displacement of the rigid body m_0 in relation to the ground and $y_{cal}(t)$ is the ground displacement in relation to the fixed spatial system.

In the model damping is described by means of an equivalent external and internal damping expressed by

$$R_{di} = d_i y_{i,t} \qquad R_{Vi} = D_i y_{i,xt} \qquad i = 0,1,...,n \tag{3.1}$$

where the constants d_i and D_i are coefficients of external and internal damping, respectively, and the comma denotes partial differentiation. The equivalent damping is taken into account in boundary conditions. It is assumed that the x-axis direction is normal to the direction of the displacements y_i, its origin coincides with the location of the rigid body m_0 in an undisturbed state, and that the velocities and displacements of the cross-sections of all the elastic elements are equal to zero at the time instant $t = 0$.

The problem of determining the displacements, strains and velocities in the cross-sections of the elastic elements for the analysed model is reduced to solving n classical wave equations

$$y_{i,tt} - c^2 y_{i,xx} = 0 \qquad \text{for} \quad i = 1,2,...,n \tag{3.2}$$

with the following initial conditions

$$y_i(x,0) = y_{i,t}(x,0) = 0 \qquad \text{for} \quad i = 1,2,...,n \tag{3.3}$$

and the nonlinear boundary conditions:
— for $x=0$

$$-m_0[\ddot{y}_{cal}(t)+y_{1,tt}]-d_0y_{1,t}+AkG(D_1y_{1,xt}+y_{1,x})-F(t)=0 \tag{3.4}$$

— for $x=il$, $i=1,2,...,n-1$

$$y_i=y_{i+1} \tag{3.5}$$

— for $x=il$, $i=1,2,...,n-1$

$$-AkG(D_iy_{i,xt}+y_{i,x})+AkG(D_{i+1}y_{i+1,xt}+y_{i+1,x})-m_iy_{i+1,tt}-d_iy_{i+1,t}=0 \tag{3.6}$$

— for $x=nl$

$$-AkG(D_ny_{n,xt}+y_{n,x})-m_ny_{n,tt}-d_ny_{n,t}=0 \tag{3.7}$$

where $c^2 = kG/\rho$. Relations (3.4) ÷ (3.7) are the conditions for displacements and forces acting in the contacting cross-sections of neighbouring elastic elements of the considered model, and y_{cal} is a given time function representing the external excitation which can be either irregular (cf Okamoto, 1973; Sackman and Kelly, 1979; Mengi and Dündar, 1988) or regular. Relations (3.4) ÷ (3.7) differ from the appropriate boundary conditions for discrete-continuous models discussed by Pielorz (1998) by the nonlinear functions $F(t)$. When function (2.1) is taken into account they coincide with the appropriate relations in the paper by Pielorz (1998). However, now the nonlinear characteristic of the spring is of a soft type and the coefficient k_3 in (2.1) is negative.

Upon the introduction of the nondimensional quantities

$$\begin{aligned}
&\overline{x}=\frac{x}{l} && \overline{t}=\frac{ct}{l} && K_r=\frac{A\rho l}{m_r}\\
&\overline{D}_i=\frac{D_ic}{l} && \overline{d}_i=\frac{d_il}{m_rc} && \overline{y}_i=\frac{y_i}{y_r}\\
&R_i=\frac{m_i}{m_r} && \overline{F}=\frac{Fl^2}{c^2y_rm_r}
\end{aligned} \tag{3.8}$$

where m_r and y_r are the fixed mass and displacement, respectively, relations (3.2) ÷ (3.7) are as follows:
— for $i=1,2,...,n$

$$\begin{aligned}
&y_{i,tt}-y_{i,xx}=0\\
&y_i(x,0)=y_{i,t}(x,0)=0
\end{aligned} \tag{3.9}$$

— for $x = 0$

$$R_0 \ddot{y}_{cal}(t) + R_0 y_{1,tt} + d_0 y_{1,t} - K_r(D_1 y_{1,xt} + y_{1,x}) + F(t) = 0 \tag{3.10}$$

— for $x = i$, $i = 1, 2, ..., n-1$

$$y_i = y_{i+1} \tag{3.11}$$

— for $x = i$, $i = 1, 2, ..., n-1$

$$K_r(D_i y_{i,xt} + y_{i,x}) - K_r(D_{i+1} y_{i+1,xt} + y_{i+1,x}) + R_i y_{i+1,tt} + d_i y_{i+1,t} = 0 \tag{3.12}$$

— for $x = n$

$$K_r(D_n y_{n,xt} + y_{n,x}) + R_n y_{n,tt} + d_n y_{n,t} = 0 \tag{3.13}$$

In equations (3.9) $\div$ (3.13) the bars denoting nondimensional quantities are omitted for convenience.

The solution of equations $(3.9)_1$, taking into account initial conditions $(3.9)_2$, are sought in the form

$$y_i(x,t) = f_i(t-x) + g_i[t + x - 2(i-1)] \qquad i = 1, 2, ..., n \tag{3.14}$$

where the unknown functions f_i and g_i represent the waves, caused by the kinematic excitation, propagating in the ith elastic element of the discrete-continuous model in a direction consistent and opposite to the x-axis direction, respectively. In sought solution (3.14) it is taken into account that the first disturbance occurs in the ith element at the time $t = i-1$ in the cross-section $x = i-1$ for $i = 1, 2, ..., n$. The functions f_i and g_i are continuous and identical to zero for negative arguments.

Upon substituting solution (3.14) into boundary conditions (3.10) $\div$ (3.13) and denoting the largest argument of the functions appearing in each equality by z, the following nonlinear equations are obtained for the functions f_i and g_i

$$\begin{aligned}
&g_i(z) = f_{i+1}(z-2) + g_{i+1}(z-2) - f_i(z-2) \qquad i = 1, 2, ..., n-1\\
&r_{n+1,1} g_n''(z) + r_{n+1,2} g_n'(z) = r_{n+1,3} f_n''(z-2) + r_{n+1,4} f_n'(z-2)\\
&r_{11} f_1''(z) = -R_0 \ddot{y}_{cal}(z) + r_{12} g_1''(z) + r_{13} f_1'(z) + r_{14} g_1'(z) - F(z)\\
&r_{i1} f_i''(z) + r_{i2} f_i'(z) = r_{i3} g_i''(z) + r_{14} g_i'(z) + r_{i5} f_{i-1}''(z) +\\
&+ r_{i6} f_{i-1}'(z) \qquad i = 2, 3, ..., n
\end{aligned} \tag{3.15}$$

where

$$
\begin{array}{ll}
r_{11} = K_r D_1 + R_0 & r_{12} = K_r D_1 - R_0 \\
r_{13} = -K_r - d_0 & r_{14} = K_r - d_0 \\
r_{i1} = K_r D_i + K_r D_{i-1} + R_{i-1} & r_{i2} = 2K_r + d_{i-1} \\
r_{i3} = K_r D_i - K_r D_{i-1} - R_{i-1} & r_{i4} = -d_{i-1} \\
r_{i5} = 2K_r D_{i-1} & r_{i6} = 2K_r \qquad i = 2, 3, ..., n \\
r_{n+1,1} = K_r D_n + R_0 & r_{n+1,2} = K_r + d_n \\
r_{n+1,3} = K_r D_n - R_0 & r_{n+1,4} = K_r - d_n
\end{array}
\tag{3.16}
$$

Equations (3.15) are nonlinear differential equations with a retarded argument. Although appropriate equations for linear models can be solved analytically or numerically by means of the finite difference method, cf Nadolski and Pielorz (1980) and (1992), Pielorz (1996), nonlinear equations (3.15) can be solved only numerically using e.g. the Runge-Kutta method. Having obtained from (3.15) the functions $f_i(z)$ and $g_i(z)$ and their derivatives, one can determine displacements, strains and velocities in an arbitrary cross-section of the elastic elements in the considered model at an arbitrary time instant. The solution can be obtained in transient as well as in steady states.

4. Numerical results

The numerical analysis is performed for the model presented in Fig.1 when $n = 1, 2, 3$ with two, three or four rigid bodies. The function of the external excitation $y_{cal}(t)$ is arbitrary: irregular or regular, periodic or nonperiodic. In the paper by analogy with the nonlinear discrete problems it is assumed in the form

$$\ddot{y}_{cal}(t) = a_0 \sin(pt) \tag{4.1}$$

and the considerations focus on the determination of displacements in the steady states. By means of function (4.1) various direct and indirect external excitations can be described, where p is the nondimensional frequency of the external excitation.

The considered discrete-continuous systems represent low structures and are described by the nondimensional parameters R_i, K_r, see (3.8). These parameters can have various values. The constants R_i are the ratios of the masses m_i and the mass of the foundation m_0 while the constant K_r is the ratio of the mass of the columns and m_0. For real structures such parameters are usually smaller than 1. In the presented calculations they are assumed to be equal 0.5 and 0.3, similarly as in the paper by Pielorz (1998) where the nonlinear models with a hard characteristic were studied.

In the numerical calculations in the present paper we concentrate on the presentation of the influence of the local nonlinearity with a soft characteristic on the displacements in selected cross-sections.

The local nonlinearity is described by four functions (2.1)÷(2.4). Functions (2.2) ÷ (2.4) are connected with the parameters k_1 and k_3 representing the linear and nonlinear terms in polynomial (2.1) by relation (2.5). In the numerical analysis the nondimensional parameter k_1 is fixed and is equal to 0.3 while the nondimensional parameters k_3 and a_0 can be changed. The parameter k_3 is connected directly with the local nonlinearity, and a_0 is the amplitude of external excitation (4.1). The remaining nondimensional parameters appearing in equations (3.16) are fixed and equal to

$$\begin{array}{llll} R_0 = 1.0 & K_r = 0.3 & k_1 = 0.3 & m_r = m_0 \\ R_i = 0.5 & d_i = D_i = 0.1 & i = 1, 2, ..., n & \end{array} \tag{4.2}$$

The efficiency of the method applied in the paper was shown by Pielorz (1996) in the case of linear discrete-continuous systems for low structures subject to shear deformations. Certain comparable calculations were presented by Pielorz (1998) for a single-mass system with a local nonlinearity having a hard characteristic. This type of the characteristic was also assumed there for multi-mass systems. In the present paper further aspects of the nonlinear problems are investigated. Namely, it is assumed that the local nonlinearity has a soft characteristic. Numerical calculations are concentrated on the presentation of the influence of this type of the local nonlinearity on the amplitude-frequency curves for selected cross-sections of the elastic elements in the considered models using four nonlinear functions (2.1) ÷ (2.4) for the description of the nonlinearity. It is done for harmonic kinematic excitation (4.1).

In the paper by Pielorz (1996) it was shown that equations (3.15) enable determination of the numerical solution in an arbitrary cross-section of the discrete-continuous systems. Below, the effect of the local nonlinearity only in the cross-section $x = 0$ is investigated.

4.1. Two-mass system

Numerical calculations for the two-mass system are performed using Eqs (3.15) with parameters (4.2) for $n = 1$. In Fig.3 amplitude-frequency diagrams for the displacement in the cross-section $x = 0$ are plotted for $k_3 = -0.05$ and $a_0 = 0.15,\ 0.25$. Functions (2.1)÷(2.4) are used for the description of the local

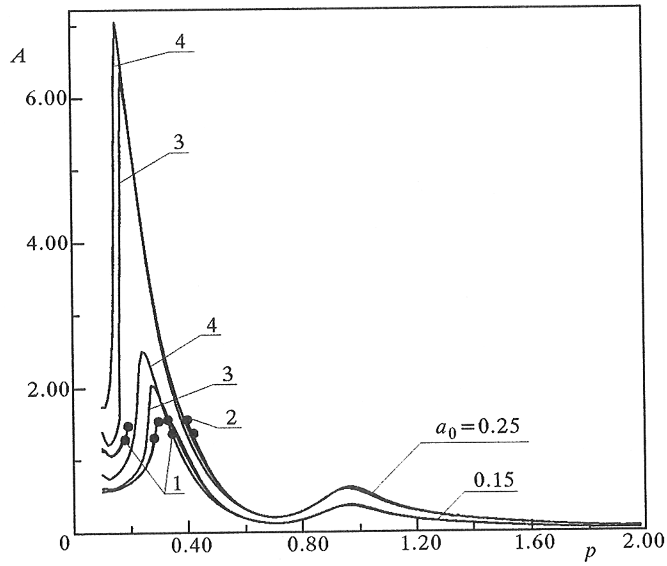

Fig. 3. Amplitude-frequency curves for the displacements in $x=0$ for the two-mass system for $k_3=-0.05$, $a_0=0.15$, 0.25 with nonlinear functions (2.1) $\div$ (2.4)

nonlinearity. They are marked $1 \div 4$, respectively, and they are connected by relation (2.5) with the parameters k_1 and k_3. The diagrams in Fig.3 include 2 resonant regions ($\omega_1=0.379$, $\omega_2=0.975$). In the further resonant regions the results for all the assumed nonlinear functions are practically the same. From Fig.3 it follows that the maximal displacement amplitudes in the first resonant region are obtained for exponential function (2.4), next for hyperbolic tangent function (2.3). The remaining functions for both values of a_0 in the neighbourhood of the first resonant region give sometimes solutions which are not harmonic functions within certain intervals of the frequency p of external excitation (4.1). The maximum values of p, for which the solutions are harmonic functions, are marked by points. The interval for the frequencies p giving irregular solutions in the case of polynomial (2.1) is slightly wider than for sinusoidal function (2.2). One may find the values of a_0 being smaller than 0.15 when the solutions for all nonlinear functions are harmonic functions. The irregular solutions within the marked intervals of p in the case of functions (2.1) tend to infinity while in the case of function (2.2) they stop to behave as a harmonic function. The divergence of the solution to infinity is known in the dynamics of nonlinear discrete systems having a soft characteristic described by a polynomial function as the escape from potential wells, cf Stewart et al.

(1995). From Fig.3 it follows that similar escape phenomena can also occur in nonlinear discrete-continuous systems when using function (2.1). From Fig.3 it also follows that the results in the second resonant region practically coincide with each other for all the nonlinear functions.

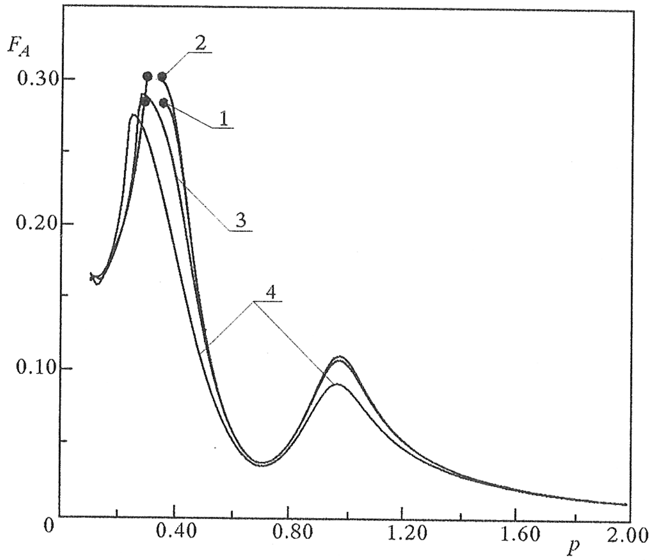

Fig. 4. Amplitudes of the force F for the two-mass system for $k_3 = -0.05$, $a_0 = 0.15$ with nonlinear functions (2.1) $\div$ (2.4)

In Fig.4 the diagrams of the amplitude F_A of the nonlinear forces $F(t)$ applied in the cross-section $x = 0$ for $k_3 = -0.05$ and $a_0 = 0.15$ for the two resonant regions of the considered two-mass system are plotted. Functions (2.1) $\div$ (2.4) are used for the description of these forces. From Fig.4 it follows that in the second resonant region the amplitudes of forces are minimal for exponential function (2.4) while for the remaining functions they are similar. In the first resonant region the amplitudes are smallest also for functions (2.4), next for hyperbolic tangent function (2.3) while for remaining functions (2.1) and (2.2) there exists an interval of p where the solutions are not harmonic functions. In the case of polynomial (2.1) the solution in this interval approaches infinity. The appropriate values of the amplitudes F_A are higher in the case of sinusoidal function (2.2). This is connected with the fact that according to relation (2.5) the maximal values postulated by the assumed parameters are higher for function (2.2). For the parameters $k_1 = 0.3$ and $k_3 = -0.05$ these values are equal 0.3 and 0.2828 for functions (2.2) and (2.1), respectively.

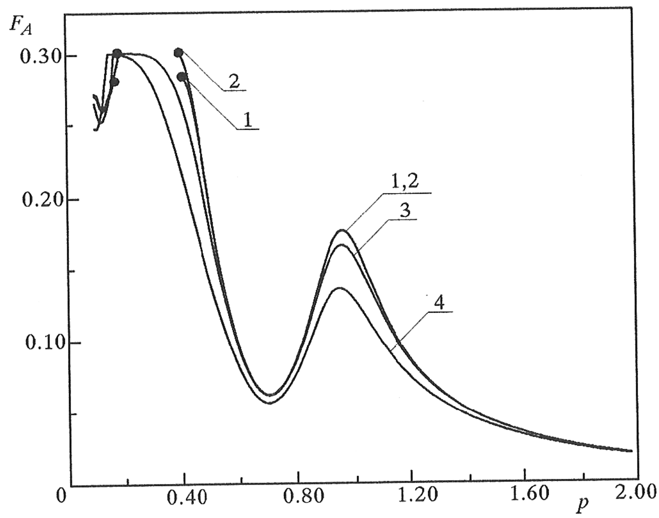

Fig. 5. Amplitudes of the force F for the two-mass system for $k_3 = -0.05$, $a_0 = 0.25$ with nonlinear functions (2.1) ÷ (2.4)

The amplitudes F_A are plotted in Fig.5 for the amplitude of the external excitation $a_0 = 0.25$. From Fig.5 it follows that in the second resonant region the amplitudes for the dynamic forces are the same when functions (2.1) and (2.2) are used. The smallest values of the force amplitudes in this region are obtained for exponential function (2.4). In the first resonant region the results for function (2.4) are usually smaller than for hyperbolic tangent function (2.3). The diagrams of functions (2.1) and (2.2), similarly as in Fig.4, show the cases where the solution is irregular. Moreover, one can notice that for the assumed $a_0 = 0.25$ the diagrams of nonlinear functions (2.3) and (2.4) form a plateau in the neighbourhood of the first resonant region. The plateau is wider in the case of function (2.3).

Fig.3 ÷ Fig.5 show exemplary diagrams for displacements and forces. These diagrams indicate that some nonlinear functions may have restrictions for their application in the discussion of the nonlinear vibrations of discrete-continuous systems with the local nonlinearities having the characteristic of a soft type. This concerns polynomial (2.1) and sinusoidal function (2.2). The application ranges for these functions are investigated for $k_3 = -0.025, -0.05, -0.1$ for the first and second resonant regions. The suitable curves are marked by dashed and continuous lines in Fig.6. These curves determine the amplitudes of external excitation (4.1) below which the numerical solutions behave as harmonic functions with the period equal to the period of the external excitation.

The smallest values of a_0 are acceptable in the neighbourhood of the resonances. From Fig.6 it also follows that for the fixed k_3 the application ranges are slightly wider in the case of sinusoidal function (2.2). It is connected with the fact that function (2.2) has a higher maximum value than function (2.1) for the assumed parameters k_1 and k_3. Besides, there exists an interval of the frequency of excitation (4.1) where the admissible values of a_0 increase in a linear manner when function (2.1) is assumed. This interval occurs between the first and the second resonant regions. No restrictions have been found for the application of nonlinear functions (2.3) and (2.4).

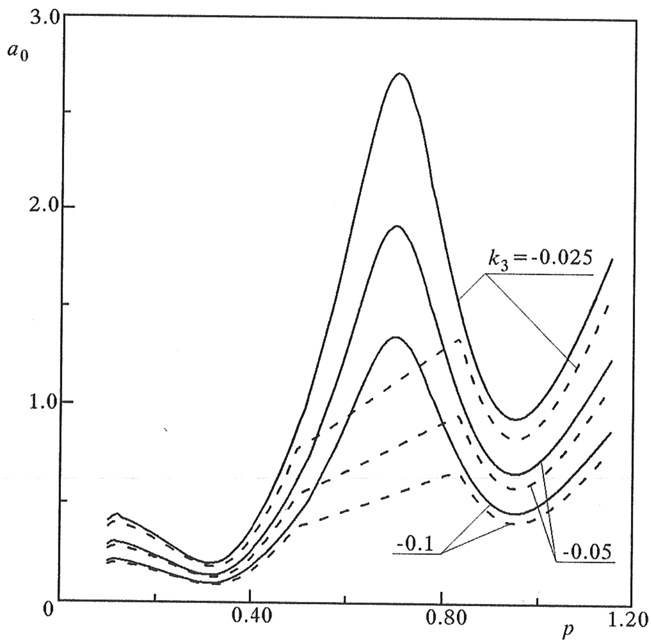

Fig. 6. Application ranges of the sinusoidal function (continuous lines) and polynomial function (dashed lines) for the two-mass system with $k_3 = -0.025$, -0.05, -0.1

4.2. Three-mass system

Diagrams in Fig.7 ÷ Fig.10 concern a three-mass system with parameters (4.2) for $n = 2$. The amplitudes of displacements in the cross-section $x = 0$ are presented in Fig.7 including 3 resonant regions ($\omega_1 = 0.28$, $\omega_2 = 0.727$, $\omega_3 = 1.187$) for $k_3 = -0.05$ and the amplitude of the external excitation $a_0 = 0.2$, 0.3. One can notice that for $a_0 = 0.2$ in the first resonant region the

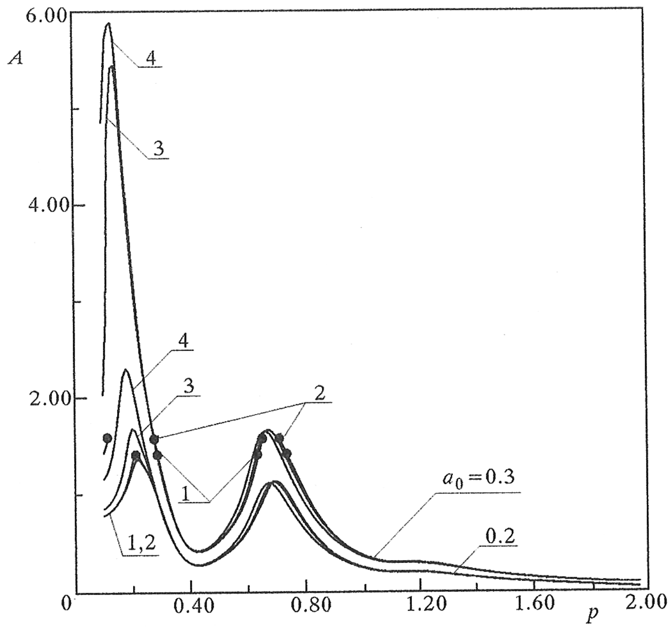

Fig. 7. Amplitude-frequency curves for the displacements in $x = 0$ for the three-mass system for $k_3 = -0.05$, $a_0 = 0.2$, 0.3 with nonlinear functions (2.1) ÷ (2.4)

maximal values justify making use of exponential function (2.4). It appears that the amplitude $a_0 = 0.2$ is chosen in such a way that the solutions are harmonic functions for all the assumed intervals of the frequency of excitation (4.1) when function (2.2) is applied, and in the case of polynomial (2.1) there exists only a very small interval of p where the solution approaches infinity ($p \in < 0.208, 0.217 >$). In the second and third resonant regions all the functions describing the local nonlinearity give similar results for the displacement amplitudes for the considered system. When the amplitude of the external excitation is $a_0 = 0.3$, both functions (2.1) and (2.2) have intervals of p where the solution stops to behave as a harmonic function. Such intervals occur in the first and second resonant region. Similarly as for $a_0 = 0.2$, the maximal displacement amplitude in the first resonant region is now also obtained for function (2.4). In the second resonant region the results for all the applied nonlinear functions are similar, within the application ranges of functions (2.1), (2.2).

In Fig.8 the diagrams of the amplitudes of the forces F described by functions (2.1) ÷ (2.4) for $k_3 = -0.05$ and $a_0 = 0.2$ are plotted. From Fig.8 it

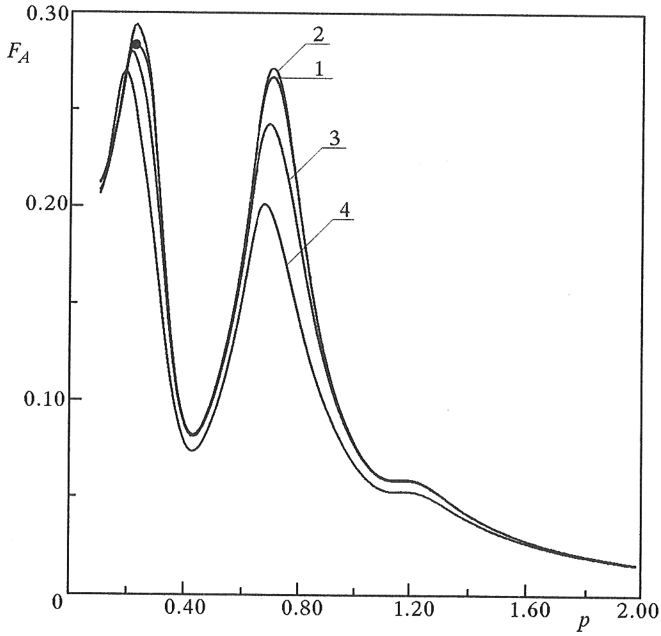

Fig. 8. Amplitudes of the force F for the three-mass system for $k_3 = -0.05$, $a_0 = 0.2$ with nonlinear functions (2.1) ÷ (2.4)

follows that in the third resonant region the results for functions (2.1) ÷ (2.3) are practically the same, and that in the remaining resonant regions the highest amplitudes are obtained for sinusoidal function (2.2) and the smallest ones for exponential function (2.4). One can notice that in Fig.7 as well, in the case of function (2.1) there exists a very small interval of p, marked by the point, where the solution loses its physical meaning.

Diagrams in Fig.9 show the effect of application of four nonlinear functions (2.1) ÷ (2.4) on the behaviour of the amplitudes of the forces F for the three-mass system for $k_3 = -0.05$ and $a_0 = 0.3$. From Fig.9 it follows that the maximal amplitudes occur when the sinusoidal function is used and the minimal ones for exponential function (2.4). It concerns the whole interval of the assumed p. For the curves corresponding nonlinear functions (2.1) and (2.2), in the first and the second resonant regions there exist intervals of p in which the solutions do not behave as harmonic functions. In the first resonant region these intervals are longer. Besides, for polynomial (2.1) these intervals are shorter. In the third resonant region functions (2.1) ÷ (2.3) give the same amplitudes of the force F.

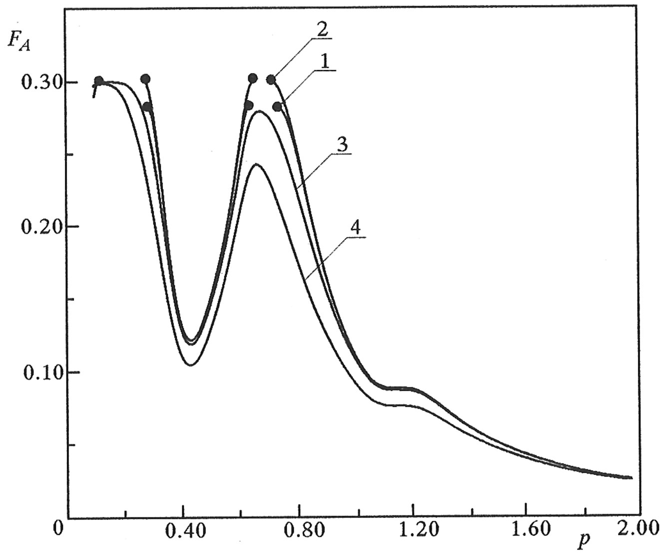

Fig. 9. Amplitudes of the force F for the three-mass system for $k_3 = -0.05$, $a_0 = 0.3$ with nonlinear functions (2.1) $\div$ (2.4)

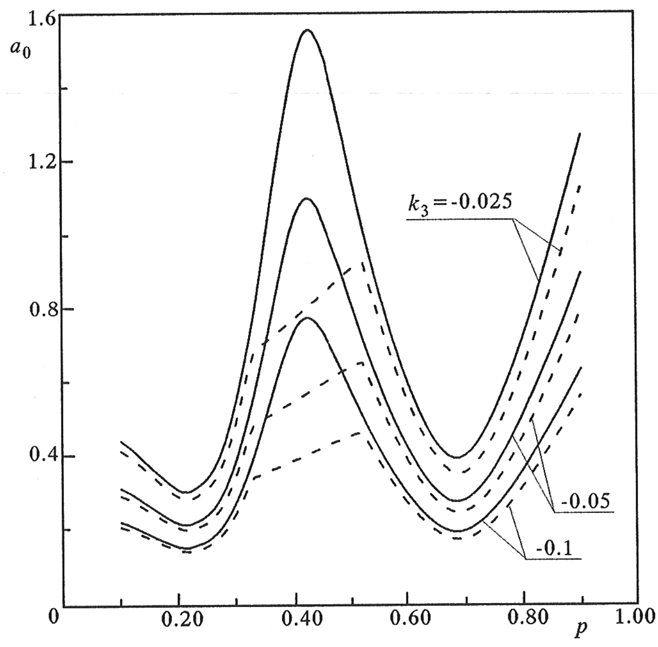

Fig. 10. Application ranges of the sinusoidal function (continuous lines) and polynomial function (dashed lines) for the three-mass system with $k_3 = -0.025, \ -0.05, \ -0.1$

The application ranges of polynomial (2.1) (dashed lines) and the sinusoidal function (continuous lines) are shown in Fig.10 for $k_3 = -0.025$, -0.05, -0.1. They concern two resonant regions. Similarly, as in the case of the two-mass system, the strongest restrictions occur in the neighbourhood of the resonances and the admissible values of a_0 decrease with the decrease of k_3. For the fixed p the acceptable a_0 is higher for the sinusoidal function. In the case of function (2.1), there exists an interval of p between the considered resonant regions where these values increase in a linear manner. Comparing the application ranges of functions (2.1) and (2.2) given in Fig.6 and Fig.10, respectively for the two- and three-mass systems, one can notice that in the case of the two-mass system higher values of the amplitudes of excitation (4.1) are admissible.

4.3. Four-mass system

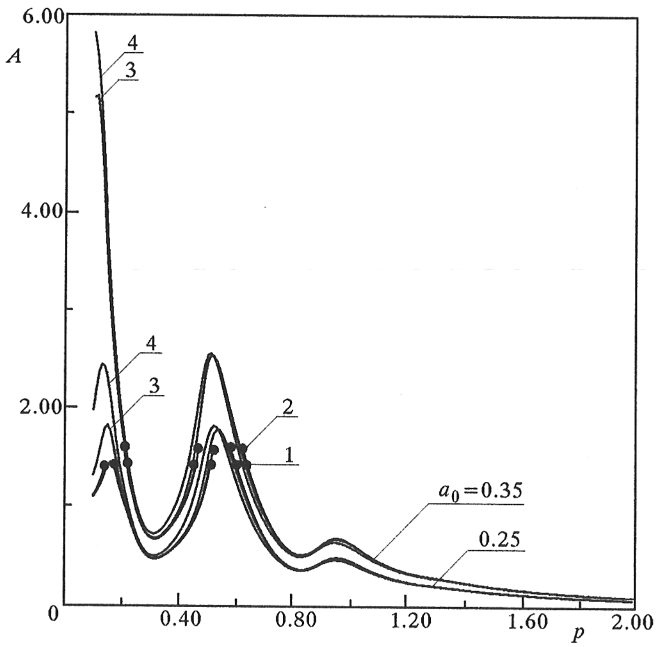

Fig. 11. Amplitude-frequency curves for the displacements in $x = 0$ for the four-mass system for $k_3 = -0.05$, $a_0 = 0.25$, 0.35 with nonlinear functions (2.1) ÷ (2.4)

Numerical calculations for the four-mass system are performed for parameters (4.2) with $n = 3$. The amplitudes of displacements in the cross-section

$x = 0$ are presented in Fig.11 for $k_3 = -0.05$ and $a_0 = 0.25,\ 0.35$. The diagrams concern 4 resonant regions ($\omega_1 = 0.22$, $\omega_2 = 0.594$, $\omega_3 = 0.949$, $\omega_4 = 1.278$). Only three of them are distinct. From Fig.11 it follows that in the first resonant region the maximal displacement amplitudes occur for nonlinear function (2.4) for both values of a_0. In the second resonant region functions (2.3), (2.4) give similar results. In further regions the results for all of the four nonlinear functions are similar. In the case of function (2.1) solutions may approach infinity for $a_0 = 0.25$ as well as for $a_0 = 0.35$. For function (2.2) with $a_0 = 0.25$ one can expect solutions not behaving as a harmonic function only in the second resonance, while with $a_0 = 0.35$ in both of the first resonant regions.

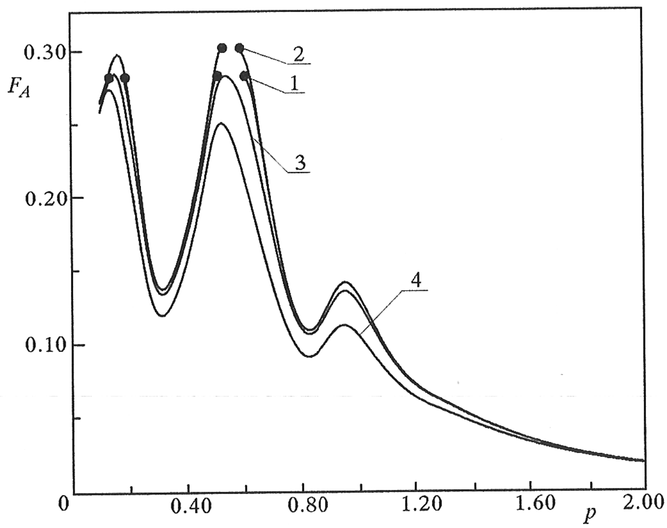

Fig. 12. Amplitudes of the force F for the four-mass system for $k_3 = -0.05$, $a_0 = 0.25$ with nonlinear functions (2.1) ÷ (2.4)

Diagrams of the amplitudes of the force F described by the four functions are given in Fig.12 for $k_3 = -0.05$ and $a_0 = 0.25$. In the first and second resonant regions the maximal amplitudes are obtained for the sinusoidal function, and the smallest ones for exponential function (2.4). After the fourth resonant region results for all assumed functions are difficult to distinguish. In the first resonant region one can notice the interval of p where the solution approaches infinity when the polynomial function is used. In the second resonant region function (2.1) as well as sinusoidal function (2.2) can give solutions losing their physical meaning.

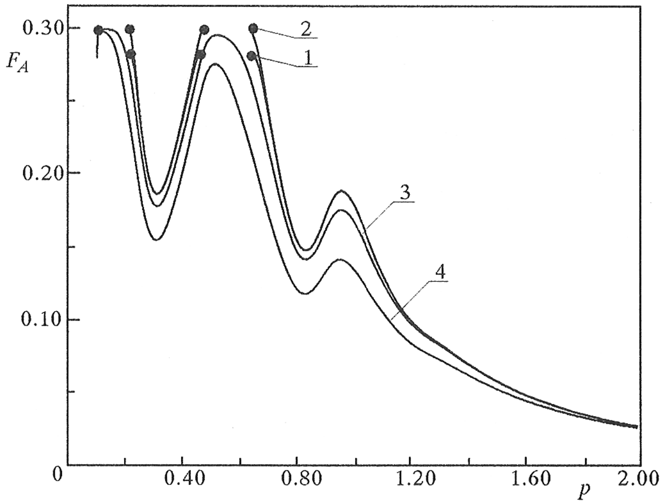

Fig. 13. Amplitudes of the force F for the four-mass system for $k_3 = -0.05$, $a_0 = 0.35$ with nonlinear functions (2.1) ÷ (2.4)

In Fig.13 the diagrams of the force amplitudes for $k_3 = -0.05$ and $a_0 = 0.35$ are plotted. In the third resonant region the highest amplitudes are obtained for functions (2.1) and (2.2) while the smallest ones when the exponential function is applied for the description of the local nonlinearity. In the case of functions (2.1), (2.2) one can observe the intervals of the nonharmonic solutions in the first as well as in the second resonant region. For the remaining functions, the diagrams of the force amplitudes in the first resonant region form a plateau. It means that in this region the solutions corresponding to these functions approach the maximum value postulated by the parameters k_1 and k_3. The plateau is wider for the hyperbolic tangent function.

The application ranges of the polynomial function (dashed lines) and the sinusoidal function (continuous lines) for the four-mass system are shown in Fig.14 for $k_3 = -0.025, -0.05, -0.1$. They include 3 resonant regions. Similarly, as in the case of the two-mass and three-mass systems, one can see, by tracing the appropriate diagrams, how the admissible values of the amplitude a_0 decrease with the decrease of the parameter k_3 representing the local nonlinearity. The strongest restrictions occur in the neighbourhood of the resonances. In the case of the four-mass system the minimal admissible values of a_0 increase with the increase of the frequency p. In this case the application ranges are also slightly wider for sinusoidal function (2.2). Between the first and the second resonant regions the acceptable a_0 for the polynomial function

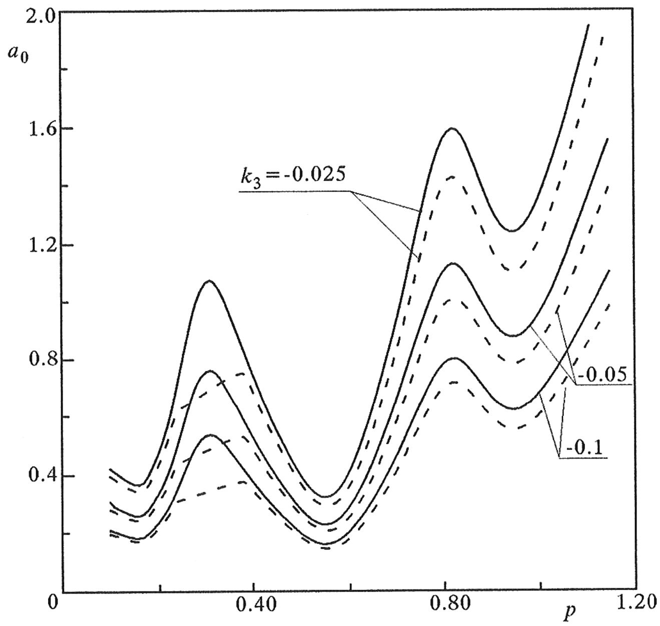

Fig. 14. Application ranges of the sinusoidal function (continuous lines) and polynomial function (dashed lines) for the four-mass system with $k_3 = -0.025,\ -0.05,\ -0.1$

can increase in a linear manner. Besides, one can notice that the differences between the diagrams of function (2.2) and function (2.1) increase slowly with the increase of the frequency of the external excitation.

5. Final remarks

It is shown in the paper that various nonlinear functions can be incorporated in the dynamic analysis of the discrete-continuous models of low structures subject to shear deformations and having a local nonlinearity with a soft characteristic. In the study four nonlinear functions are employed for description of the considered local nonlinearity including a polynomial of the third degree. It is found that the polynomial function can have strong restrictions on its application aiming at determination of numerical solutions of the governing equations for discrete-continuous systems. For this reason, the introduction of other nonlinear functions may have important meaning. It is found that the use of sinusoidal function can also lead to the solutions losing the physical meaning. In exemplary numerical solutions the effect of the introduction of different functions on the amplitude-frequency curves is shown for a harmonic kinematic excitation and the application ranges of the polynomial and sinusoidal functions are determined. From the presented results it follows that increasing number of rigid bodies in the considered systems gives more complicated diagrams. The strongest restrictions imposed on the admissible parameters determining regular numerical solutions are connected with the neighbourhood of the resonances.

Nonlinear systems undergoing shear deformations and having the local nonlinearity with a hard characteristic were studied by Pielorz (1998). It appears that in the case of a hard characteristic the use of the polynomial function gives satisfactory results while in the case of a soft characteristic one may expect certain inconveniences.

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Drgania dyskretno-ciągłych modeli niskich konstrukcji ze sprężyną o nieliniowej charakterystyce typu miękiego

Streszczenie

W pracy przeprowadzono badania dynamiczne niskich obiektów poddanych wymuszeniom kinematycznym wywołanych falami poprzecznymi przy wykorzystaniu dyskretno-ciągłych modeli z lokalną nieliniowością. Modele złożone są z brył sztywnych i elementów sprężystych poddanych tylko odkształceniom ścinającym, natomiast lokalna nieliniowość reprezentowana jest przez dyskretne elementy z nieliniową sprężyną. Założono, że nieliniowa charakterystyka sprężyny jest typu miękkiego. W pracy charakterystykę tę opisano za pomocą czterech funkcji nieliniowych. W dyskusji wykorzystano podejście falowe. Obliczenia numeryczne wykonano dla modeli z dwiema, trzema i czterema bryłami sztywnymi przy wymuszeniu kinematycznym opisanym funkcją harmoniczną. Koncentrują się one na zbadaniu wpływu lokalnej nieliniowości, wyrażonej przez cztery różne funkcje, na przemieszczenia wybranych przekrojów poprzecznych elementów sprężystych w rozpatrywanych modelach, oraz na wyznaczeniu obszarów stosowalności przyjętych funkcji nieliniowych.

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