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OPTIMAL PLASTIC SHAPE DESIGN OF ROTATIONALLY SYMMETRIC ELEMENTS WITH CONICAL BEARING SURFACES

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The paper presents the boundary perturbation method applied to optimal plastic shape design. Perfect plasticity is assumed. The procedure consists of two steps: the class of fully plastic solutions in the limit state is first determined, and then the optimal shape is chosen from among these solutions. Rotationally symmetric elements with conical bearing surfaces are considered. The optimal angle of inclination of such surfaces is also evaluated. The final results are verified by means of the ADINA program.

Key words: optimisation, perfect plasticity, heads of tension members

1. Introduction

Heads of rotationally symmetric tension members (circular cylinders, e.g. rivets or screws) are usually formed as hemispheres supported on bearing planes perpendicular to the axis of the member. The shape of a hemisphere is not optimal from the point of view of minimal volume; optimal plastic design of such heads was considered by Życzkowski and Egner (1995). The boundary perturbation method (BPM) was used and a thick-walled sphere under pressure served as the basic solution. Spherical coordinates r, ψ, θ were employed for the head, and cylindrical coordinates r, θ, z for the cylindrical part.

The present paper is an extension of Życzkowski and Egner (1995), namely the bearing plane will be generalized to a conical bearing surface (Fig.1). Such an extension will bring an additional design variable, namely the angle ψ_0 between the generatrix of the bearing surface and the z axis, and consequently an additional profit, it means lower volume of the head. It is assumed that a conical head is admissible from the structural point of view. A similar problem for plane heads (under the assumption of the plane-strain state) was considered by Egner (1999). However, the present problem is more difficult, since governing equations in spherical coordinates are more complicated making it necessary to consider both stresses and displacements.

Many problems of optimal plastic shape design of structural elements were solved by Szczepiński and Szlagowski (1990), who used statically admissible stress fields, uniform inside appropriately chosen subdomains. A more exact approach, based on slip-line fields, was used by Zowczak (1981, 1989) in plane strain problems, and also in a rotationally symmetric problem (Zowczak [8], to be published). Similar to that considered in the present paper. A comparison with Zowczak's results will be given.

Application of the BPM to optimal plastic shape design, initiated by Bochenek et al. (1983) was studied in detail by Egner et al. (1994), where also numerous references to the BPM are quoted. The method consists of two steps: first a family of shapes subject to full plastification at the stage of collapse is established by the BPM (Kordas and Życzkowski 1970; many subsequent solutions were reviewed by Życzkowski 1981), and then optimal design from among this family is chosen. Of course, full plastification is not always possible; in such cases yielding of possibly large subdomains is required. The method can be applied if the basic solution (zeroth approximation) is known and has a relatively simple form. It was used, for example, to optimization of plane heads of tension members (Egner, 1996), and of yoke elements (Egner et al., 1993).

The present paper is based on the following assumptions:

- The element under consideration consists of a cylindrical tension member and of a conical head with an unknown rotationally symmetric free boundary $b = b(\psi)$ serving as a functional design variable, with the angle ψ_0 regarded as an additional design variable (parameter).
- The cylindrical tension member is in a uniform stress state corresponding to full plastification at the stage of collapse.
- The zeroth approximation for the head is assumed as for a thick-walled sphere $a_0 \leqslant r \leqslant b_0$ under internal (negative) pressure equal to the stress in the cylindrical part. The subdomain between the cylinder and the thick-walled sphere is not subject to plastification.
- The material is perfectly plastic, incompressible, subject to the Huber-Mises-Hencky (HMH) yield condition and the Hencky-Ilyushin or Levy-Mises constitutive equations.

- Strains and displacements (or velocities) are small.
- Minimal volume of the head is the design objective.

The general view of the structure is shown in Fig.1.

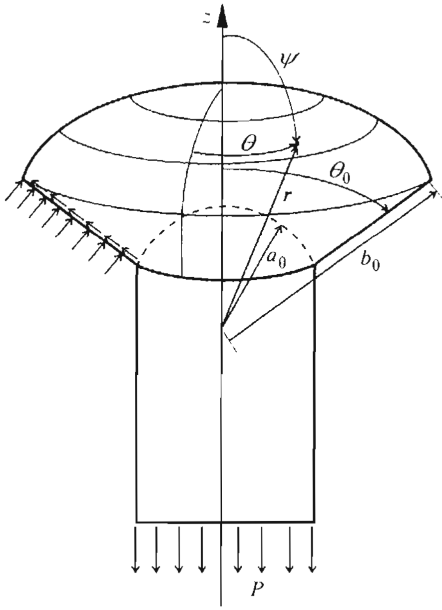


Fig. 1.

2. Zeroth approximation

As the zeroth approximation we assume the solution for a perfectly plastic thick-walled sphere $a_0 \leqslant r \leqslant b_0$ under the internal pressure $p_a = -\sigma_0$ (Życzkowski and Egner, 1995)

$$\begin{aligned} \sigma_{r0} &= -2\sigma_0 \ln \frac{r}{b_0} & \sigma_{\psi 0} &= \sigma_{\theta 0} = -2\sigma_0 \ln \frac{r}{b_0} - \sigma_0 \\ \frac{b_0}{a_0} &= \sqrt{\mathrm{e}} & u_{r0} &= -\frac{C}{r^2} \\ \varepsilon_{r0} &= \frac{2C}{r^3} & \varepsilon_{\psi 0} &= \varepsilon_{\theta 0} = -\frac{C}{r^3} \end{aligned} \tag{2.1}$$

where σ_0 denotes the yield point stress in tension and C denotes an arbitrary positive constant. From the physical equation we can calculate the plastic modulus φ

$$\begin{aligned} \varepsilon_{r0} - \varepsilon_{\psi 0} &= \varphi_0(\sigma_{r0} - \sigma_{\psi 0}) \\ \varphi_0 &= \frac{3C}{\sigma_0 r^3} \end{aligned} \tag{2.2}$$

The above solution gives the upper bound to the volume and this volume will be the reference solution. The volume equals

$$V = V_0 = \frac{2}{3}\pi b_0^3(1 - \cos\psi_0) = \frac{2}{3}\pi \mathrm{e}^{3/2} a_0^3(1 - \cos\psi_0) \tag{2.3}$$

where ψ_0 is an angle between the bearing surface and the z axis (symmetry axis). For $\psi_0 = 60°$ the volume is equal

$$V_0 = 1.047 b_0^3 = 4.693 a_0^3 \tag{2.4}$$

Withn the inside zone $a_0 \cos\psi_0/\cos\psi < r < a_0$ it is then assumed $\sigma_r = \sigma_\psi = \sigma_\theta = \sigma_0$, $\sigma_{red} = 0$. Fig.2 shows the distribution of the reduced stress σ_{red} in the spherical head $(b_0/a_0 = \sqrt{\mathrm{e}})$ obtained using the ADINA program. It is seen that, in a relatively large subdomain, σ_{red} is much smaller than σ_0, hence the design is far from the optimal.

3. Perturbation analysis

In order to optimize the shape determined by $b = b(\psi)$ we introduce rotationally symmetric perturbations of solutions (2.1) and $(2.2)_2$ and of the external contour, writing

$$\begin{aligned} \boldsymbol{X} &= \sum_{p=0}^{\infty} X_p \alpha^p \\ \boldsymbol{X} &= [\sigma_{kl}(r,\psi), \varepsilon_{kl}(r,\psi), u_k(r,\psi), \varphi(r,\psi), b(\psi)]^\top \end{aligned} \tag{3.1}$$

From among 11 equations for subsequent perturbations, determining the 4 stress components σ_r, σ_ψ, σ_θ, $\tau_{r\psi}$, the 4 strain components ε_r, ε_ψ, ε_θ, $\gamma_{r\psi}$, the 2 displacements u_r, u_ψ and the plastic modulus φ, 7 equations are linear

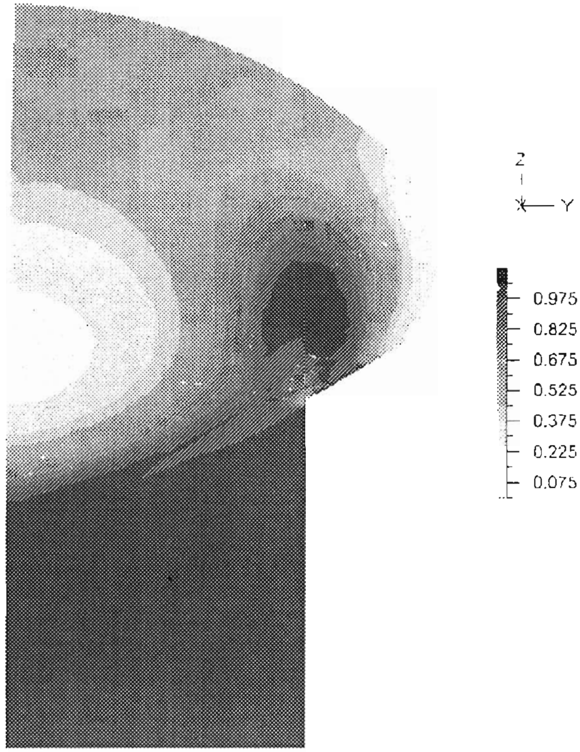


Fig. 2.

and retain their original form for each perturbation. They are: two equilibrium equations

$$\begin{aligned}
&\frac{\partial\sigma_{r_i}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\psi r_i}}{\partial\psi} + \frac{1}{r}[2\sigma_{r_i} - \sigma_{\psi_i} - \sigma_{\theta_i} + \tau_{\psi r_i}\cot\psi] = 0 \\
&\frac{\partial\tau_{\psi r_i}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\psi_i}}{\partial\psi} + \frac{1}{r}[3\tau_{\psi r_i} + (\sigma_{\psi_i} - \sigma_{\theta_i})\cot\psi] = 0
\end{aligned} \tag{3.2}$$

Strain-displacement relations

$$\begin{aligned}
\varepsilon_{r_i} &= \frac{\partial u_{r_i}}{\partial r} & \varepsilon_{\psi_i} &= \frac{1}{r}\frac{\partial u_{\psi_i}}{\partial\psi} + \frac{u_{r_i}}{r} \\
\varepsilon_{\theta_i} &= \frac{u_{\psi_i}}{r}\cot\psi + \frac{u_{r_i}}{r} & \gamma_{\psi r_i} & \frac{\partial u_{\psi_i}}{\partial r} - \frac{u_{\psi_i}}{r} + \frac{1}{r}\frac{\partial u_{r_i}}{\partial\psi}
\end{aligned} \tag{3.3}$$

and the incompressibility condition

$$\varepsilon_{r_i} + \varepsilon_{\psi_i} + \varepsilon_{\theta_i} = 0 \tag{3.4}$$

The remaining equations are nonlinear and change their form in perturbations. The physical equations (Hencky-Ilyushin's or Levy-Mises's) take the form

$$\varepsilon_{kl_i} = \sum_{j=0}^{i} \varphi_j [\sigma_{kl_{(i-j)}} - \delta_{kl}\sigma_{m_{(i-j)}}] \qquad i = 1, 2, ... \tag{3.5}$$

where $\sigma_m = (\sigma_r + \sigma_\psi + \sigma_\theta)/3$, and δ_{kl} stand for Kronecker's symbol. In the case under consideration three of the above equations are independent. Finally, the HMH yield condition for subsequent perturbations takes the form

$$\sigma_{r_i} - \sigma_{m_i} = f_i(\sigma_{r_0}, \sigma_{\psi_0}, ..., \tau_{r\psi_{i-1}}) \qquad i = 1, 2, ... \tag{3.6}$$

where

$$f_1 = 0 \qquad f_2 = -\frac{1}{\sigma_0}[(\sigma_{r_1} - \sigma_{\psi_1})^2 + \tau_{r\psi_1}^2], ... \tag{3.7}$$

4. First perturbation

The system of 11 equations mentioned in the previous section may easily be reduced to 4 equations. Four strain components are directly expressed in terms of displacements in spherical coordinates. Further, from yield condition (3.6) we eliminate σ_{θ_1}

$$\sigma_{\theta_1} = 2\sigma_{r_1} - \sigma_{\psi_1} \tag{4.1}$$

From physical equations (3.5) we can calculate the plastic modulus φ and the second normal stress σ_{ψ_1}

$$\varphi_1 = \frac{3\varepsilon_{r_1}}{2\sigma_0} \qquad \sigma_{\psi_1} = \sigma_{r_1} + \frac{\sigma_0 r^2}{6C}\Big(\frac{\partial u_{\psi_1}}{\partial \psi} - u_{\psi_1}\cot\psi\Big) \tag{4.2}$$

Finally, substituting these expresions into the two equilibrium equations, the incompressibility condition and the last independent physical equations we obtain the following system of 4 governing equations

$$\begin{aligned}
&\frac{\partial t}{\partial \psi} + r\frac{\partial s}{\partial r} + t\cot\psi = 0\\
&\frac{\partial s}{\partial \psi} + 3t + 2W - 2r\frac{\partial W}{\partial r} + r^2\frac{\partial^2 W}{\partial r^2} = 0\\
&\frac{\partial U}{\partial \psi} - t - 3W + r\frac{\partial W}{\partial r} = 0\\
&\frac{\partial W}{\partial \psi} + r\frac{\partial U}{\partial r} + W\cot\psi = 0
\end{aligned} \tag{4.3}$$

where the following dimensionless quantities are introduced

$$s = \frac{\sigma_{r_1}}{\sigma_0} \qquad t = \frac{\tau_{r\psi_1}}{\sigma_0} \qquad U = \frac{r^2}{6C} u_{r_1} \qquad W = \frac{r^2}{6C} u_{\psi_1} \tag{4.4}$$

5. Solution of the equations and boundary conditions

It can be seen that Eqs (4.3) are of Euler's type with respect to the variable r, hence their solution may be presented as power functions of this variable. We assume the exponents of r to be complex, $n_j + im_j$, and substitute into (4.3) polynomials with J terms

$$\begin{aligned}
s &= \sum_{j=1}^{J} r^{n_j} \Big[f_{s_j}(\psi) \sin\Big(m_j \ln \frac{r}{b_0}\Big) + \overline{f}_{s_j}(\psi) \cos\Big(m_j \ln \frac{r}{b_0}\Big)\Big] \\
t &= \sum_{j=1}^{J} r^{n_j} \Big[f_{t_j}(\psi) \sin\Big(m_j \ln \frac{r}{b_0}\Big) + \overline{f}_{t_j}(\psi) \cos\Big(m_j \ln \frac{r}{b_0}\Big)\Big] \\
U &= \sum_{j=1}^{J} r^{n_j} \Big[f_{U_j}(\psi) \sin\Big(m_j \ln \frac{r}{b_0}\Big) + \overline{f}_{U_j}(\psi) \cos\Big(m_j \ln \frac{r}{b_0}\Big)\Big] \\
W &= \sum_{j=1}^{J} r^{n_j} \Big[f_{W_j}(\psi) \sin\Big(m_j \ln \frac{r}{b_0}\Big) + \overline{f}_{W_j}(\psi) \cos\Big(m_j \ln \frac{r}{b_0}\Big)\Big]
\end{aligned} \tag{5.1}$$

Substitution of the above terms into (4.3) results in systems of $4J$ equations. First of them is presented below

$$\begin{aligned}
& n_j(f_{s_j} s_j + \overline{f}_{s_j} c_j) + m_j(f_{s_j} c_j - \overline{f}_{s_j} s_j) + (f'_{t_j} s_j + \overline{f}'_{t_j} c_j) + \\
& + (f_{t_j} s_j + \overline{f}_{t_j} c_j) \cot\psi = 0 \\
& \vdots \\
& j = 1, 2, ..., J
\end{aligned} \tag{5.2}$$

where for the sake of brevity the following notations have been introduced

$$s_j = \sin\Big(m_j \ln \frac{r}{b_0}\Big) \qquad c_j = \cos\Big(m_j \ln \frac{r}{b_0}\Big) \tag{5.3}$$

It can be seen that fulfilment of the above equations is possible if the following system of linear ordinary differential equations of the first order for each j,

$j = 1, 2, ..., J$ with the unknowns $f_{s_j}, ..., \overline{f}_{W_j}$ is satisfied (it is one of $8J$ sets of equations)

$$
\begin{aligned}
&f'_{t_j} + f_{t_j} \cot\psi + n_j f_{s_j} - m_j \overline{f}_{s_j} = 0 \\
&\overline{f}'_{t_j} + \overline{f}_{t_j} \cot\psi + n_j \overline{f}_{s_j} - m_j \overline{f}_{s_j} = 0 \\
&f'_{s_j} + 3f_{t_j} + (n_j^2 - 3n_j + 2 - m_j^2) f_{W_j} - (2n_j - 3) m_j \overline{f}_{W_j} = 0 \\
&\overline{f}'_{s_j} + 3\overline{f}_{t_j} + (n_j^2 - 3n_j + 2 - m_j^2) \overline{f}_{W_j} - (2n_j - 3) m_j f_{W_j} = 0 \\
&f'_{U_j} - f_{t_j} + (n_j - 3) f_{W_j} - m_j \overline{f}_{W_j} = 0 \\
&\overline{f}'_{U_j} - 3\overline{f}_{t_j} + (n_j - 3) \overline{f}_{W_j} + m_j f_{W_j} = 0 \\
&f'_{W_j} + f_{W_j} \cot\psi + n_j f_{U_j} - m_j \overline{f}_{U_j} = 0 \\
&\overline{f}'_{W_j} + \overline{f}_{W_j} \cot\psi + n_j \overline{f}_{U_j} - m_j \overline{f}_{U_j} = 0
\end{aligned}
\tag{5.4}
$$

At the pole $\psi = 0$ these systems are singular because of $\cot\psi$, hence the following expansions are used (based on symmetry or antisymmetry conditions)

$$
\begin{aligned}
f_{r_j} &= A_{r_j} + B_{r_j}\psi^2 + C_{r_j}\psi^4 + ... & \overline{f}_{r_j} &= \overline{A}_{r_j} + \overline{B}_{r_j}\psi^2 + \overline{C}_{r_j}\psi^4 + ... \\
f_{t_j} &= A_{t_j}\psi + B_{t_j}\psi^3 + ... & \overline{f}_{t_j} &= \overline{A}_{t_j}\psi + \overline{B}_{t_j}\psi^3 + ... \\
f_{U_j} &= A_{U_j} + B_{U_j}\psi^2 + C_{U_j}\psi^4 + ... & \overline{f}_{U_j} &= \overline{A}_{U_j} + \overline{B}_{U_j}\psi^2 + \overline{C}_{U_j}\psi^4 + ... \\
f_{W_j} &= A_{W_j}\psi + B_{W_j}\psi^3 + ... & \overline{f}_{W_j} &= \overline{A}_{W_j}\psi + \overline{B}_{W_j}\psi^3 + ...
\end{aligned}
\tag{5.5}
$$

Substituting (5.5) into (5.4), we obtain the following dependencies between the coefficients of the series given by (5.5)

$$
\begin{aligned}
A_{t_j} &= \tfrac{1}{2} m_j \overline{A}_{r_j} - \tfrac{1}{2} n_j A_{r_j} & \overline{A}_{t_j} &= -\tfrac{1}{2} n_j \overline{A}_{r_j} - \tfrac{1}{2} m_j A_{r_j} \\
A_{W_j} &= \tfrac{1}{2} m_j \overline{A}_{U_j} - \tfrac{1}{2} n_j A_{U_j} & \overline{A}_{W_j} &= -\tfrac{1}{2} n_j \overline{A}_{U_j} - \tfrac{1}{2} m_j A_{U_j}
\end{aligned}
\tag{5.6}
$$

The subsequent coefficients B, C, ... can be expressed in terms of A. These relations are very complicated so that we present below only two of them

$$
\begin{aligned}
B_{r_j} \;=\;& \frac{3}{4}(n_j A_{r_j} - m_j \overline{A}_{r_j}) + \frac{1}{4}[n_j^2(n_j - 3) + 2n_j - 3m_j^2(n_j - 1)] A_{U_j} + \\
-\;& \frac{1}{4} m_j [3n_j(n_j - 2) + m_j(2 - m_j^2)] \overline{A}_{U_j}
\end{aligned}
\tag{5.7}
$$

$$
\begin{aligned}
\overline{C}_{r_j} &= -\frac{1}{64}[n_j^4 - 3n_j^3 - 7n_j^2 - 2n_j + m_j^2(-6n_j^2 + 9n_j + 7) + m_j^4]\overline{A}_{r_j} + \\
&+ \frac{m_j}{64}[-4n_j^3 + 9n_j^2 + 14n_j + 2 + m_j^2(4n_j - 3)]A_{r_j} + \\
&+ \frac{1}{192}[3n_j^5 - 9n_j^4 + 8n_j^3 - 6n_j^2 + 4n_j + \\
&+ m_j^2(-30n_j^3 + 54n_j^2 - 24n_j + 6) + m_j^4(-9 + 15n_j)]\overline{A}_{U_j} + \\
&+ \frac{m_j}{192}[15n_j^4 - 36n_j^3 + 24n_j^2 - 12n_j + 4 + \\
&+ m_j^2(-30n_j^2 + 36n_j - 8) + 3m_j^4]A_{U_j}
\end{aligned}
$$

The boundary conditions at the outer, free edge $b = b(\psi)$ in view of rotational symmetry may be reduced to the two following equations

$$
\begin{aligned}
\sigma_r \cos(n,r) + \tau_{r\psi} \cos(n,\psi) &= 0 \\
\tau_{r\psi} \cos(n,r) + \sigma_\psi \cos(n,\psi) &= 0
\end{aligned}
\tag{5.8}
$$

After expressing the cosines in terms of the function $b = b(\psi)$, and expanding the stresses and shape function $b = b(\psi)$ into power series and equating the corresponding coefficients of α we obtain the following two equations

$$
\begin{aligned}
\sigma_{r_1}(b_0,\psi)b_0 - 2\sigma_0 b_1(\psi) &= 0 \\
\tau_{r\psi_1}(b_0,\psi)b_0 + \sigma_0 b_1'(\psi) &= 0
\end{aligned}
\tag{5.9}
$$

The function $b_1 = b_1(\psi)$ appears in both equations; they might be satisfied simultaneously if

$$
\frac{\partial \sigma_{r_1}(b_0,\psi)}{\partial \psi} + 2\tau_{r\psi_1}(b_0,\psi) = 0 \tag{5.10}
$$

In general, such a relation does not hold, though in the plane strain case it was possible to satisfy a corresponding condition (Egner et al., 1993, 1994). In the spatial case full plastification even of a layer is not always possible. Namely, expressing the stresses by means of the earlier obtained functions f

$$
\begin{aligned}
\sigma_{r_1} &= \sigma_0 \sum_{j=1}^{J} r^{n_j} \Big[f_{s_j}(\psi) \sin\Big(m_j \ln \frac{r}{b_0}\Big) + \overline{f}_{s_j}(\psi) \cos\Big(m_j \ln \frac{r}{b_0}\Big)\Big] \\
\tau_{r\psi_1} &= \sigma_0 \sum_{j=1}^{J} r^{n_j} \Big[f_{t_j}(\psi) \sin\Big(m_j \ln \frac{r}{b_0}\Big) + \overline{f}_{t_j}(\psi) \cos\Big(m_j \ln \frac{r}{b_0}\Big)\Big]
\end{aligned}
\tag{5.11}
$$

and substituting into equation (5.10) we obtain

$$\sum_{j=1}^{J}(\overline{f}'_{s_j} + 2\overline{f}_{t_j}) = 0 \tag{5.12}$$

It is an additional differential equation which with the system of equation (5.4) gives a set of $9J$ equations with $8J$ unknowns. Physically, it means that in a limited state the structure is not fully plastified. Applying series (5.5) to (5.10) we obtain additional relations between coefficients of the series

$$\sum_{j=1}^{J}(\overline{A}_{t_j} + \overline{B}_{s_j}) = 0 \qquad \sum_{j=1}^{J}(\overline{B}_{t_j} + 2\overline{C}_{s_j}) = 0 \tag{5.13}$$

It turned out that not all of the above equations can be satisfied. The number of equations which can be taken into account depends on the number of terms J in series (5.1) which we retain for further consideration.

In order to have a possibly small error we calculate first the correction of the outer edge b_1 as an arithmetic mean from both equations (5.9)

$$b_1(\psi) = \frac{b_0}{4\sigma_0}\Big[\sigma_{r_1}(b_0,\psi) + \sigma_{r_1}(b_0,0) - 2\int_0^{\psi} \tau_{r\psi_1}(b_0,\overline{\psi})\, d\overline{\psi}\Big] \tag{5.14}$$

where $\overline{\psi}$ is the variable of the integration.

6. Optimization problem for a head

As an example, optimal design of a rivet or screw head is considered. The optimization problem is formulated as follows: we look for the minimal volume of the head

$$V = \frac{2}{3}\pi\int_0^{\psi_0} b^3(\psi)\sin\psi\, d\psi = \frac{2}{3}\pi\Big[b_o^3 + 3b_0^2\alpha\int_0^{\psi_0} b_1(\psi)\sin\psi\, d\psi + ...\Big] \tag{6.1}$$

under the constraint of a constant force transmitted, equal to the maximal force transmitted by the cylindrical tension member (rivet or screw shank)

$$P = -2\pi\int_{a_0}^{b(\psi_0)}\Big[\sigma_\psi(r,\psi_0) + \tau_{r\psi}(r,\psi_0)\cot\psi_0\Big]r\, dr = \pi a_0^2\sigma_0 \tag{6.2}$$

The above equation gives us J relations which have a quite complicated form

$$\sum_{j=1}^{j} F_{st_j}[m_j^2+(n_j+2)^2] = \sum_{j=1}^{j}[X_jK_{1_j}+Y_jK_{2_j}+(f_{t_j}K_{1_j}+\overline{f}_{t_j}K_{2_j})\cot\psi_0] \qquad (6.3)$$

where the following notations are introduced

$$\begin{aligned}
F_{st_j} &= \frac{1}{2}\Big[\frac{1}{2}\overline{f}_{s_j}(\psi_0)+\frac{1}{2}\overline{f}_{s_j}(0)-\int\limits_0^{\psi_0}\overline{f}_{t_j}\,d\psi\Big]\\
X_j &= f_{s_j}(\psi_0)-n_jf_{U_j}(\psi_0)+m_j\overline{f}_{U_j}(\psi_0)\\
Y_j &= \overline{f}_{s_j}(\psi_0)-n_j\overline{f}_{U_j}(\psi_0)-m_jf_{U_j}(\psi_0) \qquad (6.4)\\
K_{1_j} &= -m_j-\Big(\frac{a_0}{b_0}\Big)^{n_j+2}\Big[(n_j+2)\sin\Big(m_j\ln\frac{a_0}{b_0}\Big)-m_j\cos\Big(m_j\ln\frac{a_0}{b_0}\Big)\Big]\\
K_{2_j} &= n_j+2-\Big(\frac{a_0}{b_0}\Big)^{n_j+2}\Big[(n_j+2)\cos\Big(m_j\ln\frac{a_0}{b_0}\Big)+m_j\sin\Big(m_j\ln\frac{a_0}{b_0}\Big)\Big]
\end{aligned}$$

Finally, the solution of (4.3) is assumed in the form of (5.1) with $J = 2$ (a binomial form). Equations (5.4) are integrated numerically starting from the series (5.5). The external contour $b(\psi)$ is determined from (5.14) whereas the internal contour is not expanded; the domain $a_0\cos\psi_0/\cos\psi < r < a_0$ is subject to uniform triaxial tension in the basic state and it remains not plastified after the perturbation. Finally, we introduce the boundary conditions at the supporting plane (bearing surface). They may be specified in three forms

Perfectly clamped edge

$$W(\psi_0) = 0 \qquad U(\psi_0) = 0 \qquad (6.5)$$

the edge modelling contact of the two frictionless surfaces

$$W(\psi_0) = 0 \qquad \tau_{r\psi} = 0 \qquad (6.6)$$

the edge modelling contact of the two surfaces with friction

$$W(\psi_0) = 0 \qquad \tau_{r\psi} = k\sigma_\psi \qquad (6.7)$$

The fulfilment of (6.5) in case of $J = 2$, would determine all the free parameters. Thus, the optimization is not possible here, as well as in the case of boundary conditions (6.6). So, the case defined by equations (6.7) was solved approximately. The condition ensuring the vanishing of the transversal displacements was applied, neglecting the stress boundary condition. Then we obtain the solution which describes the contact of the surface with friction. First relation from (6.7) gives the following equations

$$f_{W_1}(\psi_0) = \overline{f}_{W_1}(\psi_0) = f_{W_2}(\psi_0) = \overline{f}_{W_2}(\psi_0) = 0 \tag{6.8}$$

The small parameter α is found from the condition

$$\pi(b^2 - a_0^2)\sigma_0 = \pi a_0^2\sigma_0 \quad \Longrightarrow \quad b(\psi_0) = a_0\sqrt{2} = b_0\sqrt{\frac{2}{\mathrm{e}}} = 0.858b_0 \tag{6.9}$$

This condition means that in the reactive area $\sigma_\psi = \sigma_0$.

From among 12 free parameters m_1, n_1, m_2, n_2, α, and 7 constants A, we eliminate 8 by satisfying Eq (6.8), condition (6.9), two equations (5.13) and one condition (6.2). Finally, the four parameters remain free for the optimization: m_1, m_2, n_1, n_2.

7. Examples

The first example was calculated for the angle $\psi_0 = 60°$. The optimal values of the free parameters are following

$$\begin{aligned} m_1 &= 1.25 & \qquad n_1 &= 3.93 \\ m_2 &= 3.195 & \qquad n_2 &= -1.35 \end{aligned} \tag{7.1}$$

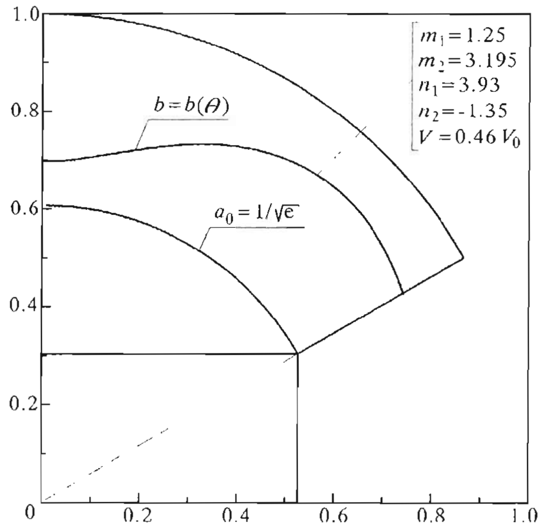
The minimal volume equals

$$V = 0.46V_0 \tag{7.2}$$

where V_0 is the volume of the spherical head for the angle $\psi_0 = \pi/2$. The shape of the head is shown in Fig.3. A remarkable concavity is observed at the top. This concavity is possible in view of the circumferential latitudinal stresses σ_θ.

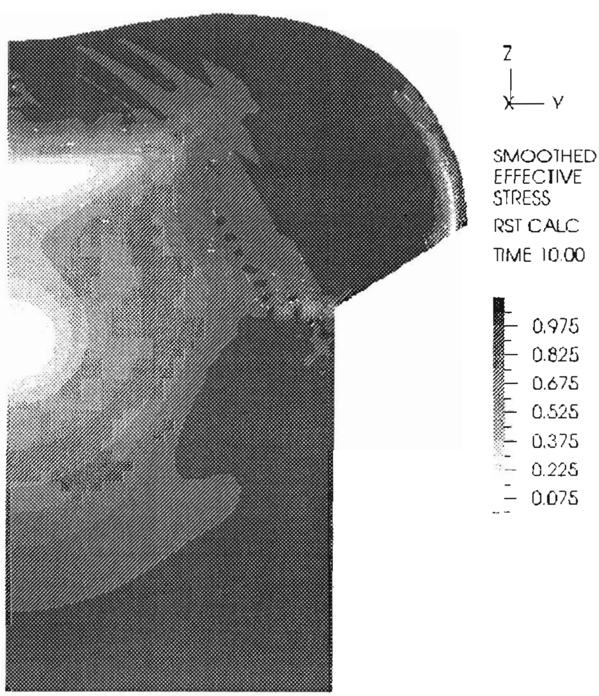
The numerical verification was done using the ADINA program. The distribution of the reduced stresses is shown in Fig.4.

As can be seen in Fig.4 that the elastic area is quite large in the tension member as well as in the head. In this case the reactive area was modelled as a surface with a clamped edge. The theoretical calculation (boundary perturbation method) shows a non-zero tangent stress at this surface as well as radial displacement. Another extremal case is work of both surfaces as frictionless. Fig.5 shows the distribution of the reduced stresses obtained using the ADINA



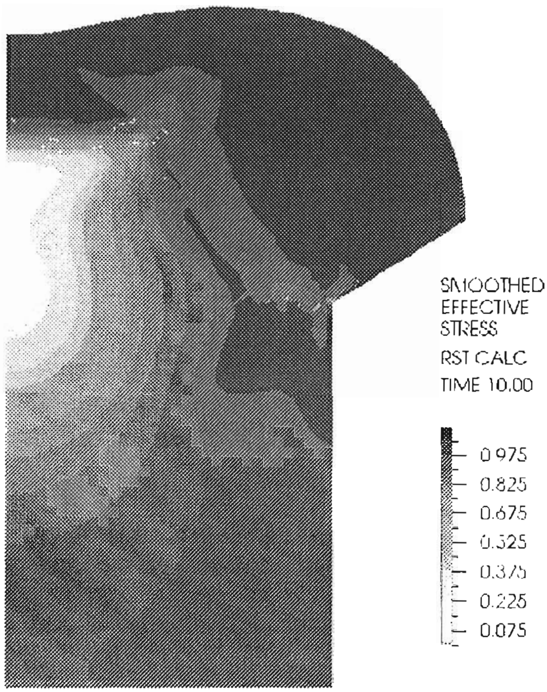
$m_1=1.25$
$m_2=3.195$
$n_1=3.93$
$n_2=-1.35$
$V=0.46\,V_0$
$b=b(\theta)$
$a_0=1/\sqrt{e}$
0
0.2
0.4
0.6
0.8
1.0

Fig. 3.



Z
Y
SMOOTHED
EFFECTIVE
STRESS
RST CALC
TIME 10.00
0.975
0.825
0.675
0.525
0.375
0.225
0.075

Fig. 4.


SMOOTHED
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0.975
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0.675
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Fig. 5.

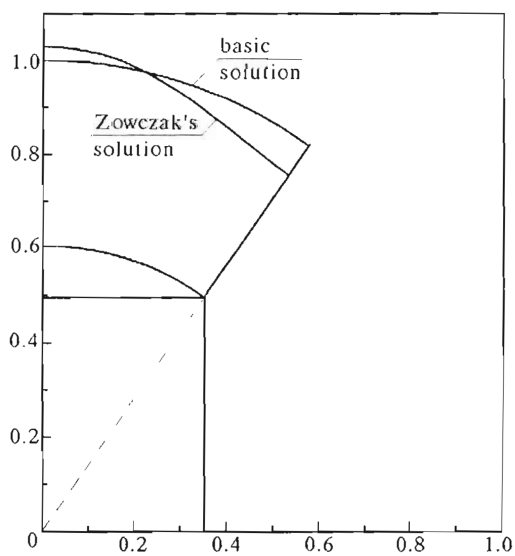
basic
solution
Zowczak's
solution
1.0
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1.0

Fig. 6.

program for the case of frictionless surfaces. It can be seen that the elastic area in the head is smaller here, but the carrying capacity is lower (about 7%).

The second example was calculated for the angle $\psi_0 = 35.3°$. This special value of the angle was chosen in order to compare the solution with the solution obtained by Zowczak [8]. Zowczak used the slip- line method to the optimal design of a head of spherical elements. The minimal volume obtained was equal

$$V = 0.78V_0 \tag{7.3}$$

The optimal shape is presented in Fig.6.

The distribution of the reduced stress obtained using the ADINA program is shown in Fig.7.

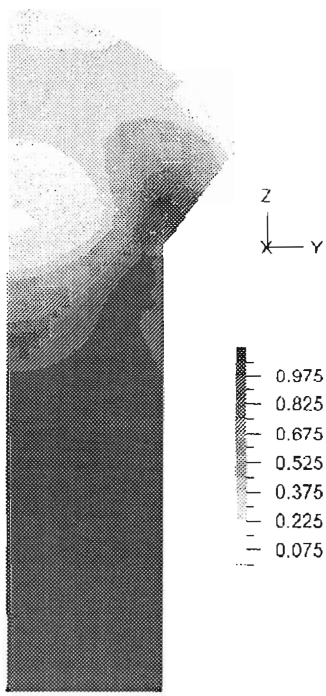

Fig. 7.

It can be seen that this shape is far from the optimal, because of the existence of large elastic domains. Using the boundary perturbation method for the angle $\psi_0 = 35.3°$ the following solution was obtained

$$\begin{aligned} m_1 &= 4.4 \qquad & n_1 &= 2.55 \\ m_2 &= 2.7 \qquad & n_2 &= -1.35 \end{aligned} \tag{7.4}$$

and the minimal volume

$$V = 0.54V_0 \tag{7.5}$$

The shape is presented in Fig.8.

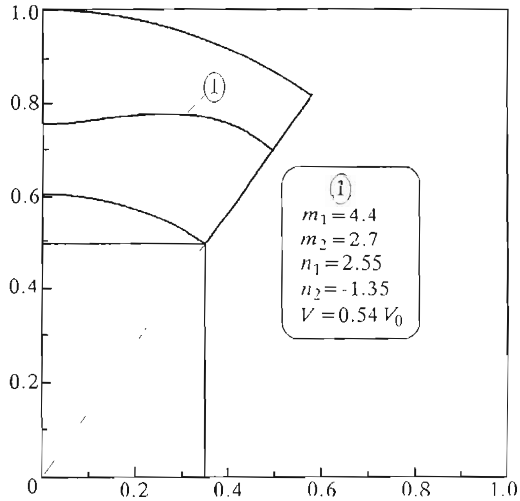

Fig. 8.

The distribution of the reduced stress is shown in Fig.9. The elastic area is here much smaller than in Zowczak's solution, and the volume is 31% smaller.

Basing on the obtained results we can look for the angle for which the volume is the smallest. For the basic solution (sphere) the dimensionless volume of the head for the angle ψ_0 referred to the volume of half of the sphere ($\psi_0 = \pi/2$) is given by

$$\frac{V}{V_0} = \frac{2}{3}\pi \frac{1}{\sin^3 \psi_0}(1 - \cos \psi_0) \tag{7.6}$$

The above function has its minimum $V_{min} = 0.77V_0$ for $\psi_0 = \pi/3$. For the optimal solution obtained using the boundary perturbation method we have the following results

$$\psi_0 = 35.3^\circ \qquad V = 0.54V_0$$

$$\psi_0 = 60^\circ \qquad V = 0.46V_0$$

$$\psi_0 = 90^\circ \qquad V = 0.61V_0$$

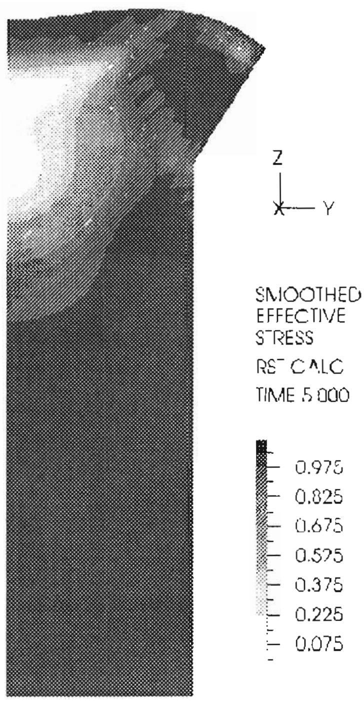
Z
Y
SMOOTHED
EFFECTIVE
STRESS
RST CALC
TIME 5.000
0.975
0.825
0.675
0.525
0.375
0.225
0.075

Fig. 9.

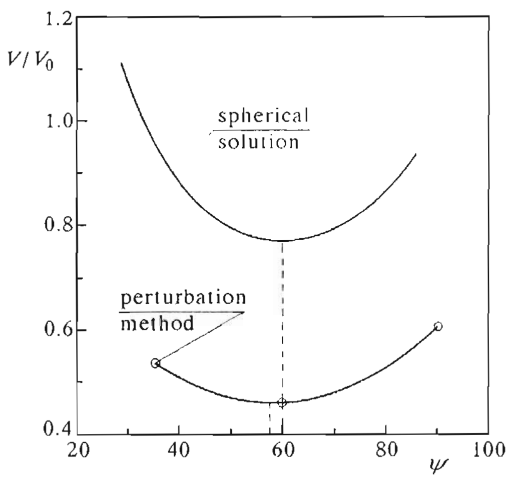
1.2
V/V_0
1.0
0.8
0.6
0.4
20
40
60
80
100
ψ
spherical solution
perturbation method

Fig. 10.

The graphical representation of the dimensionless volume versus the angle for the spherical solutions and the solutions obtained using the boundary perturbation method is shown in Fig.10. The solutions obtained using the boundary perturbation method were interpolated by the use of a parabolic function. The minimum $V_{min} = 0.46V_0$ is reached for the angle $\psi_0 = 58.4°$.

8. Conclusions

- The boundary perturbation method makes it possible to optimize the free boundary of a conical head of a rotationally symmetric tension member. The result is verified by the ADINA program.
- In view of the complicated form of the perturbation equations just the first approximation has been determined.
- The optimal angle of the cone is $\psi_0 = 58.4°$, and the minimal volume is less than a half of the basic volume of the head.
- In contradistinction to the plane-strain solution, Egner (1999), the optimal shape shows concavities, possible here in view of the latitudinal stresses σ_θ.

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Optymalne kształtowanie w zakresie plastycznym elementów obrotowo symetrycznych ze stożkowymi powierzchniami oporowymi

Streszczenie

W pracy przedstawiono zastosowanie metody zaburzenia brzegu do optymalnego kształtowania w zakresie plastycznym. Założono idealną plastyczność. Procedura składa się z dwóch etapów: określenie rodziny rozwiązań wykazujących całkowite uplastycznienie w fazie zniszczenia, a następnie wybranie z nich rozwiązania optymalnego. Rozważane są elementy obrotowo symetryczne ze stożkowymi powierzchniami oporowymi. Obliczono również optymalny kąt nachylenia tych powierzchni. Ostateczne rezultaty zostały zweryfikowane za pomocą programu ADINA.

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