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A CONTRIBUTION TO MODELLING OF ANISOTROPIC BEHAVIOUR OF BONE AND BONE REMODELLING

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Anisotropic behaviour of bones in the elastic and plastic ranges is discussed. The adaptive elasticity with evolving structure is examined from the point of view of tensor functions. The equations of adaptive piezoelectricity are formulated. A general framework for bone remodelling combined with homogenization is proposed. It is suggested that the bone adaptation to variable loads may be viewed as a shakedown problem. A possibility of studying bone remodelling via optimal design is considered.

Key words: bone, remodelling, piezoelectricity, homogenization

1. Introduction

From a mechanical standpoint, bone is an inhomogeneous and anisotropic composite material with solid and fluid phases. At the macroscopic (phenomenological) level, there are two major forms of the bone tissue: cortical (compact) and cancellous or trabecular (spongious). Both of them are anisotropic and inhomogeneous. The bone structure is nicely depicted by Cowin (1989), Currey (1984), Gibson and Ashby (1988), Lowet et al. (1997), Martin and Burr (1989) Odgaard and Weinans (1995).

The aim of the present contribution is to propose general phenomenological models enabling one to study bone anisotropy and its remodelling. Our approach exploits tensor functions, homogenization and relaxation of functionals.

2. Elastic and plastic anisotropy of bone

The fabric tensor of cancellous bone is defined as the converse square root of the mean intercept length tensor $\mathbf{M}$, cf Cowin (1985, 1989), Jemioło and Telega (1997b, 1998)

$$\mathbf{H} = \frac{1}{\sqrt{\mathbf{M}}} \tag{2.1}$$

The tensor $\mathbf{H}$ is positive definite. The following measure of anisotropy (orthotropy) degree of $\mathbf{H}$ is convenient in applications (Jemioło and Telega, 1998; Rychlewski and Zhang, 1989)

$$\delta(\mathbf{H}) = \frac{\sqrt{2}}{2}\frac{H_1 - H_3}{\|\mathbf{H}\|} \tag{2.2}$$

where H_i, $i = 1,2,3$ are the ordered eigenvalues of $\mathbf{H}$. If $\mathbf{H}$ is an isotropic tensor the above measure is equal to zero. For transversely isotropic material two of the eigenvalues of $\mathbf{H}$ coincide.

Let $\mathbf{T}$ and $\mathbf{e}$ denote the stress tensor and the small strain tensor, respectively. In Jemioło and Telega (1998) the anisotropic elastic constitutive equation of the following form has been studied

$$\mathbf{T} = \alpha_1\mathbf{I} + \alpha_2\mathbf{H} + \alpha_3\mathbf{H}^2 + 2\alpha_4\mathbf{e} + \alpha_5(\mathbf{eH} + \mathbf{He}) + \alpha_6(\mathbf{eH}^2 + \mathbf{H}^2\mathbf{e}) + 3\alpha_7\mathbf{e}^2 \tag{2.3}$$

where

$$\alpha_m = \frac{\partial f}{\partial I_m} \qquad \frac{\partial \alpha_m}{\partial I_n} = \frac{\partial \alpha_n}{\partial I_m} \qquad m,n = 1,...,7 \tag{2.4}$$

and, in turn

$$W(\mathbf{e}) = f\Big(I_m(\mathbf{e})\Big) = f(\operatorname{tr}\mathbf{e},\ \operatorname{tr}\mathbf{eH},\ \operatorname{tr}\mathbf{eH}^2,\ \operatorname{tr}\mathbf{e}^2,\ \operatorname{tr}\mathbf{e}^2\mathbf{H},\ \operatorname{tr}\mathbf{e}^2\mathbf{H}^2,\ \operatorname{tr}\mathbf{e}^3) \tag{2.5}$$

We assume that for $\mathbf{e} = \mathbf{0}$, $\mathbf{T} = \alpha_1\mathbf{I} + \alpha_2\mathbf{H} + \alpha_3\mathbf{H}^2 = \mathbf{0}$. Here the structural tensor $\mathbf{H}$ is not an argument of the elastic potential $W(\mathbf{e})$. This tensor describes only the microstructure of the material. Experimental data validate the assumption of small elastic deformations in bones. The fabric tensor $\mathbf{H}$ could be treated as an argument of the elastic potential W provided that elastic deformations would lead to a significant change of this tensor, see the next section.

The linearized form of Eq (2.3) was studied in Jemioło and Telega (1998). There we have concluded that, approximately, human cortical bone is transversely isotropic whilst human cancellous bone is rather an orthotropic material.

Let us denote by $\dot{\mathbf{e}}_e$, $\dot{\mathbf{e}}_p$ the elastic and plastic part of the strain rate tensor. As usual, we assume that

$$\dot{\mathbf{e}} = \dot{\mathbf{e}}_e + \dot{\mathbf{e}}_p \tag{2.6}$$

and construct constitutive relationships for elastic perfectly-plastic materials. The elastic behaviour is described by the linearized form of Eq (2.3). The associated flow rule assumes the form

$$\dot{\mathbf{e}}_p = \lambda\frac{\partial F}{\partial \mathbf{T}} \qquad \lambda \geqslant 0 \tag{2.7}$$

The following general form of the yield function is assumed

$$F(\mathbf{T}) = \widetilde{F}(\operatorname{tr}\mathbf{T},\ \operatorname{tr}\mathbf{TH},\ \operatorname{tr}\mathbf{TH}^2,\ \operatorname{tr}\mathbf{T}^2,\ \operatorname{tr}\mathbf{T}^2\mathbf{H},\ \operatorname{tr}\mathbf{T}^2\mathbf{H}^2,\ \operatorname{tr}\mathbf{T}^3) \tag{2.8}$$

whilst the yield condition is given by

$$F(\mathbf{T}) - 1 = 0 \tag{2.9}$$

The bone tissue reveals different plastic behaviour in tension and compression: cf Cowin (1989), Gibson and Ashby (1988), Jemioło and Kowalczyk (1997). *Therefore, in Jemioło and Kowalczyk (1997) and Jemioło and Telega (1998)* the invariant form of Hoffman criterion yield has been proposed

$$\begin{aligned} F(\mathbf{T}) &= c_1(K_2 - K_3)^2 + c_2(K_3 - K_1)^2 + c_3(K_1 - K_3)^2 + 2c_4K_6 + \\ &+2c_5K_5 + 2c_6K_4 + c_7K_1 + c_8K_2 + c_9K_3 - 1 = 0 \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} c_1 &= \frac{1}{2}\Big(\frac{1}{Y_{t2}Y_{c2}} + \frac{1}{Y_{t3}Y_{c3}} - \frac{1}{Y_{t1}Y_{c1}}\Big) \qquad c_2 = \frac{1}{2}\Big(\frac{1}{Y_{t3}Y_{c3}} + \frac{1}{Y_{t1}Y_{c1}} - \frac{1}{Y_{t2}Y_{c2}}\Big) \\ c_3 &= \frac{1}{2}\Big(\frac{1}{Y_{t1}Y_{c1}} + \frac{1}{Y_{t2}Y_{c2}} - \frac{1}{Y_{t3}Y_{c3}}\Big) \\ 2c_4 &= \frac{1}{k_{23}^2} \qquad 2c_5 = \frac{1}{k_{13}^2} \qquad 2c_6 = \frac{1}{k_{12}^2} \\ c_7 &= \frac{Y_{c1} - Y_{t1}}{Y_{c1}Y_{t1}} \qquad c_8 = \frac{Y_{c2} - Y_{t2}}{Y_{c2}Y_{t2}} \qquad c_9 = \frac{Y_{c3} - Y_{t3}}{Y_{c3}Y_{t3}} \end{aligned} \tag{2.11}$$

Here Y_{ci}, Y_{ti} and k_{ij} are the yield limit in compression and tension in the directions of orthotropy and the yield limit in shear in the principal planes of orthotropy, respectively. The invariants K_p $(p = 1, ..., 6)$ are given by

$$K_1 = \operatorname{tr}\mathbf{M}_1\mathbf{T} \qquad K_2 = \operatorname{tr}\mathbf{M}_2\mathbf{T} \qquad K_3 = \operatorname{tr}\mathbf{M}_3\mathbf{T}$$

$$K_4 = \frac{1}{2}\Big[(\operatorname{tr}\mathbf{M}_3\mathbf{T})^2 - (\operatorname{tr}\mathbf{M}_1\mathbf{T})^2 - (\operatorname{tr}\mathbf{M}_2\mathbf{T})^2 - \operatorname{tr}\mathbf{M}_3\mathbf{T}^2 + \operatorname{tr}\mathbf{M}_1\mathbf{T}^2 + \operatorname{tr}\mathbf{M}_2\mathbf{T}^2\Big] \tag{2.12}$$

$$K_5 = \frac{1}{2}\Big[(\operatorname{tr}\mathbf{M}_2\mathbf{T})^2 - (\operatorname{tr}\mathbf{M}_1\mathbf{T})^2 - (\operatorname{tr}\mathbf{M}_3\mathbf{T})^2 - \operatorname{tr}\mathbf{M}_2\mathbf{T}^2 + \operatorname{tr}\mathbf{M}_1\mathbf{T}^2 + \operatorname{tr}\mathbf{M}_3\mathbf{T}^2\Big]$$

$$K_6 = \frac{1}{2}\Big[(\operatorname{tr}\mathbf{M}_1\mathbf{T})^2 - (\operatorname{tr}\mathbf{M}_2\mathbf{T})^2 - (\operatorname{tr}\mathbf{M}_3\mathbf{T})^2 - \operatorname{tr}\mathbf{M}_1\mathbf{T}^2 + \operatorname{tr}\mathbf{M}_2\mathbf{T}^2 + \operatorname{tr}\mathbf{M}_3\mathbf{T}^2\Big]$$

The tensors $\mathbf{M}_j = \boldsymbol{i}_j \otimes \boldsymbol{i}_j$ (no summation over j) are the eigentensors of $\mathbf{H}$. By using the following relation

$$\begin{bmatrix} \operatorname{tr}\mathbf{T}^\alpha \\ \operatorname{tr}\mathbf{H}\mathbf{T}^\alpha \\ \operatorname{tr}\mathbf{H}^2\mathbf{T}^\alpha \end{bmatrix} = \mathbf{h} \begin{bmatrix} \operatorname{tr}\mathbf{M}_1\mathbf{T}^\alpha \\ \operatorname{tr}\mathbf{M}_2\mathbf{T}^\alpha \\ \operatorname{tr}\mathbf{M}_3\mathbf{T}^\alpha \end{bmatrix} \qquad \alpha = 1, 2 \tag{2.13}$$

where

$$\mathbf{h} = \begin{bmatrix} 1 & 1 & 1 \\ H_1 & H_2 & H_3 \\ H_1^2 & H_2^2 & H_3^2 \end{bmatrix} \tag{2.14}$$

the criterion (2.10) can be written in the form (2.8). By using Eq (2.8) and transformation formula of tensor components under orthogonal transformations, one can derive the formulae for determination of sample strength in the case of compression and tension, in the direction defined by an angle ϕ, in each of the principal orthotropy planes, cf Jemioło and Kowalczyk (1997), Jemioło and Telega (1997a).

For $Y_{ci} = Y_{ti}$ the criterion (2.10) reduces to Hill's criterion, which has also been applied in the bone mechanics, cf Rokotomanana et al. (1991).

Remark 2.1

- The Hoffman condition (2.10) may be also viewed as a strength criterion limiting the applicability of nonlinear Eq (2.3) in the range of small deformations, cf Jemioło and Kowalczyk (1997), Cowin (1979). This statement pertains also to linear behaviour in the elastic range. Accordingly, an evaluation of stress concentrations in bones should rather be performed by applying the criterion (2.10) and not, as is usually done in finite element codes, the principal or Mises stresses.

- Inhomogeneity of bone follows from the dependence of the fabric tensor H on position of a point $\boldsymbol{x}$ in the body B identified with the closure of a domain $\Omega \in I\!R^3$, i.e., $\mathsf{H} = \mathsf{H}(\boldsymbol{x})$. Consequently, the experimental data aiming at the determination of H should include full information about this tensor, e.g. its principal values H_i $(i = 1, 2)$ and the eigenvectors determining the principal axes of orthotropy. We observe that the data given by Turner et al. (1990) concern only the principal values H_i of the human femoral cancellous bone. No data concerning the eigenvectors of H were appended. Even a superfluous analysis of microstructure of the human bone indicates that the principal axes of orthotropy depend in an essential manner on $\boldsymbol{x}$. It seems that one can consider the averaged values of H_i over a certain region of cancellous bone (thus also of elasticity moduli). However, the averaging procedure is useless when applied to the principal axes of orthotropy. At the current level of finite element programs, the elastic analysis of bone requires proper determination of its anisotropy and inhomogeneity.

- From the point of view of continuum mechanics the elastic-plastic model proposed by Jemioło and Telega (1998) is different from Cowin's (1985, 1986) model. In the last papers the elastic energy and the strength or plasticity criterion depend explicitly on e and H. In the constitutive relationship (2.3) the fabric tensor H plays only the role of a parameter and intervenes according to the principle of isotropy of the physical space.

- Zysset and Curnier (1996) proposed a model of degradation of bone mechanical properties within the framework of elastoplasticity and continuum damage mechanics. This model involves the fabric tensor (2.1).

3. Adaptive elasticity and piezoelectricity with evolving fabric tensor

The aim of this section is to develop a general model of adaptive piezoelectricity with evolving microstructure. Our approach is different from that by Gjelsvik (1973) and Güzelsu and Saha (1984). In fact, it extends the model developed previously by Cowin et al. (1992).

Let $\boldsymbol{E} = (E_i)$ and $\boldsymbol{D} = (D_i)$ denote the electric field vector and electric displacement vector, respectively. As usual we have $E_i = -\partial\varphi/\partial x_i$, where

φ stands for the electric potential. The elastic potential of a bone with an evolving microstructure is assumed in the following form

$$W = \widetilde{W}\Big(\mathbf{e}, \boldsymbol{D}; r, h(\mathbf{N})\Big) \tag{3.1}$$

where

$$r = r\Big(\boldsymbol{x}(t)\Big) \qquad\qquad \mathbf{N} = \boldsymbol{n} \otimes \boldsymbol{n}$$

$\boldsymbol{n} = (n_i)$ is a unit vector and $h(\mathbf{N})$ stands for the morphological orientation distribution function which may also depend on time. The constitutive equations are given by, cf Telega and Jemioło (1998)

$$\mathbf{T} = \frac{\partial \widetilde{W}}{\partial \mathbf{e}} \qquad\qquad \boldsymbol{E} = \frac{\partial \widetilde{W}}{\partial \boldsymbol{D}} \tag{3.2}$$

It is convenient to assume that, cf Jemioło and Telega (1998)

$$h(\mathbf{N}) = g(\mathbf{N})1 + \mathbf{G} \cdot \mathbf{F}(\mathbf{N}) + \overline{\mathbf{G}} : \overline{\mathbf{F}}(\mathbf{N}) + \ldots \tag{3.3}$$

In the specific case where

$$\begin{aligned} W = \widetilde{W}\Big(\mathbf{e}, \boldsymbol{D}; r, h(\mathbf{N})\Big) &= \frac{1}{2} a_{ijkl}\Big(r, h(\mathbf{N})\Big) e_{ij} e_{kl} + \\ -h_{ijk}\Big(r, h(\mathbf{N})\Big) D_i e_{jk} &+ \frac{1}{2}\kappa_{ij}\Big(r, h(\mathbf{N})\Big) D_i D_j \end{aligned} \tag{3.4}$$

we obtain

$$\begin{aligned} T_{ij} &= a_{ijkl}\Big(r, h(\mathbf{N})\Big) e_{kl} - h_{ijk}\Big(r, h(\mathbf{N})\Big) D_i \\ E_i &= -h_{ijk}\Big(r, h(\mathbf{N})\Big) e_{jk} + \kappa_{ij}\Big(r, h(\mathbf{N})\Big) D_j \end{aligned} \tag{3.5}$$

For a bone we may assume that $r = \rho_s/\rho_0$ and

$$h(\mathbf{N}) = \frac{1}{\sqrt{\mathbf{N} \cdot \mathbf{M}}} = \mathbf{N} \cdot \mathbf{H} \tag{3.6}$$

where $\mathbf{M}$ is the Whitehouse fabric tensor and $\mathbf{H}$ is given by Eq (2.1). Then, to the constitutive equations we must adjoin the evolution equations

$$\dot{r} = \frac{dr}{dt} = \widehat{r}\Big(\mathbf{e}, \boldsymbol{D}; r, h(\mathbf{H})\Big) \qquad\qquad \dot{\mathbf{H}} = \frac{d\mathbf{H}}{dt} = \widehat{\mathbf{H}}\Big(\mathbf{e}, \boldsymbol{D}; r, h(\mathbf{H})\Big) \tag{3.7}$$

Obviously, ρ_s denotes the density of skeleton (in the sense of porous media) whilst ρ_0 is the apparent density of the bone. Eqs (3.2) and (3.7) are equations of *adaptive piezoelectricity with evolving microstructure.*

Consider now the specific case of adaptive elasticity. Then Eq (3.5) reduces to

$$\mathbf{T} = \mathbf{a}\Big(r, h(\mathbf{N})\Big) \cdot \mathbf{e} \tag{3.8}$$

The material symmetry group is

$$S = \Big\{\mathbf{Q} \in O(3) |\ \mathbf{Q} * \mathbf{a} = \mathbf{a}\Big\} \tag{3.9}$$

where

$$\mathbf{Q} * \mathbf{a} = a_{ijkl}(\mathbf{Q}\boldsymbol{i}_i) \otimes (\mathbf{Q}\boldsymbol{i}_j) \otimes (\mathbf{Q}\boldsymbol{i}_k) \otimes (\mathbf{Q}\boldsymbol{i}_l)$$

Here $O(s)$ is the orthogonal group in the three-dimensional case and $\{\boldsymbol{i}_k\}$, $k = 1, 2, 3$, is the orthogonal frame in the space $\mathbb{R}^3$. For instance, $S = S_1 \cap S_2$, where

$$S_1 = \Big\{\mathbf{Q} \in O(3) |\ \mathbf{Q}\mathbf{G}\mathbf{Q}^\top = \mathbf{G}\Big\} \qquad\qquad S_2 = \Big\{\mathbf{Q} \in O(3) |\ \mathbf{Q} * \overline{\mathbf{G}} = \overline{\mathbf{G}}\Big\} \tag{3.10}$$

If $h(\mathbf{N}) = g(\mathbf{N})1 + \mathbf{G} \cdot \mathbf{F}(\mathbf{N})$ then $S = S_1$ (the material is orthotropic, i.e. three eigenvalues of $\mathbf{G}$ are different; transverse isotropy follows provided that two eigenvalues coincide).

Let us briefly discuss the specific case of evolution equations where

$$\begin{aligned} \dot{r} &= a(r) + \mathbf{A}(r) \cdot \mathbf{e} = a(r) + b(r)\,\mathrm{tr}\,\mathbf{e} \\ \dot{\mathbf{H}} &= \mathbf{B}(\mathbf{H}) + \overline{\mathbf{B}}(\mathbf{H}) \cdot \mathbf{e} \end{aligned} \tag{3.11}$$

Here $B_{ijkl} = B_{jikl} = B_{klij}$. The spectral decomposition of the fabric tensor $\mathbf{H}$ yields

$$\mathbf{H} = H_1\mathbf{H}_1 + H_2\mathbf{H}_2 + H_3\mathbf{H}_3 \tag{3.12}$$

where $\mathbf{H}_i = \boldsymbol{h}_i \otimes \boldsymbol{h}_i$ (no summation over i), $\boldsymbol{h}_i$ $(i = 1, 2, 3)$ are the eigenvectors of $\mathbf{H}$ and H_i are the eigenvalues of $\mathbf{H}$. The functions $\mathbf{B}(\mathbf{H})$ and $\overline{\mathbf{B}}(\mathbf{H})$ are isotropic tensor functions of the second- and fourth- order, respectively. General representations of the above tensor functions depending on symmetric second-order tensors were derived by Jemioło and Telega (1997a). Let us consider now an approximation of the functions $\mathbf{B}(\mathbf{H})$ and $\overline{\mathbf{B}}(\mathbf{H})$. Since both the scalar r and fabric tensor $\mathbf{H}$ are not "small" (in the sense of the small deformation tensor $\mathbf{e}$), therefore the functions $a(r)$, $b(r)$, $\mathbf{B}(\mathbf{H})$ and $\overline{\mathbf{B}}(\mathbf{H})$ are not, in general, linear in r and $\mathbf{H}$. We propose the following approximation of the functions $a(r)$ and $b(r)$

$$a(r) = \sum_{i=1}^{N} a_i r^i \qquad b(r) = \sum_{i=1}^{N} b_i r^i \qquad a_i, b_i = \text{const} \tag{3.13}$$

The tensor functions $\mathbf{B}(\mathbf{H})$ and $\overline{\mathbf{B}}(\mathbf{H})$ are approximated as follows

$$\mathbf{B}(\mathbf{H}) = \sum_{j=1}^{M} c_j (H_1^j \mathbf{H}_1 + H_2^j \mathbf{H}_2 + H_3^j \mathbf{H}_3) \tag{3.14}$$

$$\begin{aligned}\overline{\mathbf{B}}(\mathbf{H}) = \sum_{k=1}^{K} \Big[& \alpha_k (H_1^{2k} \mathbf{H}_1 \otimes \mathbf{H}_1 + H_2^{2k} \mathbf{H}_2 \otimes \mathbf{H}_2 + H_3^{2k} \mathbf{H}_3 \otimes \mathbf{H}_3) + \\ & + \frac{1}{2}\beta_k H_1^k H_2^k (\mathbf{H}_1 \otimes \mathbf{H}_2 + \mathbf{H}_2 \otimes \mathbf{H}_1) + \frac{1}{2}\beta_k H_1^k H_3^k (\mathbf{H}_1 \otimes \mathbf{H}_3 + \mathbf{H}_3 \otimes \mathbf{H}_1) + \\ & + \frac{1}{2}\beta_k H_2^k H_3^k (\mathbf{H}_2 \otimes \mathbf{H}_3 + \mathbf{H}_3 \otimes \mathbf{H}_2) + \frac{1}{4}\gamma_k H_1^k H_2^k (\mathbf{H}_1 \diamond \mathbf{H}_2 + \mathbf{H}_2 \diamond \mathbf{H}_1) + \\ & + \frac{1}{4}\gamma_k H_1^k H_3^k (\mathbf{H}_1 \diamond \mathbf{H}_3 + \mathbf{H}_3 \diamond \mathbf{H}_1) + \frac{1}{4}\gamma_k H_2^k H_3^k (\mathbf{H}_2 \diamond \mathbf{H}_3 + \mathbf{H}_3 \diamond \mathbf{H}_2) \Big]\end{aligned} \tag{3.15}$$

where c_j $(j = 1, ..., M)$, α_k, β_k and γ_k $(k = 1, ..., K)$ are constants and

$$(\mathbf{A} \diamond \mathbf{B})_{ijkl} = \frac{1}{2}(A_{ik} B_{jl} + A_{il} B_{jk})$$

Similarly, the function $\mathbf{a}(r, \mathbf{H})$ appearing in Eq (3.8) is assumed in the form

$$\begin{aligned}\mathbf{a}(r, \mathbf{H}) = \sum_{l=1}^{L} \Big[& \widetilde{\alpha}_l(r) (H_1^{2l} \mathbf{H}_1 \otimes \mathbf{H}_1 + ...) + \\ & + \frac{1}{2}\widetilde{\beta}_l(r) H_1^l H_2^l (\mathbf{H}_1 \otimes \mathbf{H}_2 + \mathbf{H}_2 \otimes \mathbf{H}_1) + ...) + \\ & + \frac{1}{4}\widetilde{\gamma}_l(r) H_1^l H_2^l (\mathbf{H}_1 \diamond \mathbf{H}_2 + \mathbf{H}_2 \diamond \mathbf{H}_1 + ...) \Big]\end{aligned} \tag{3.16}$$

where

$$\widetilde{\alpha}_l(r) = \sum_{p=1}^{P} \alpha_{lp} r^p \qquad ... \qquad \widetilde{\gamma}_l(r) = \sum_{p=1}^{P} \gamma_{lp} r^p \qquad \alpha_{lp}, ..., \gamma_{lp} = \text{const} \tag{3.17}$$

The evolution equation $(3.11)_2$, linear with respect to $\mathbf{e}$, can be postulated in the alternative form

$$\dot{\mathbf{H}} = \delta_1 \mathbf{I} + \delta_2 \mathbf{H} + \delta_3 \mathbf{H}^2 + \delta_4 \mathbf{e} + \frac{1}{2}\delta_5 (\mathbf{He} + \mathbf{eH}) + \frac{1}{2}\delta_6 (\mathbf{H}^2\mathbf{e} + \mathbf{eH}^2) \tag{3.18}$$

where

$$\delta_i = c_1^i + c_2^i \operatorname{tr} \mathbf{e} + c_3^i \operatorname{tr} \mathbf{He} + c_4^i \operatorname{tr} \mathbf{H}^2 \mathbf{e} \tag{3.19}$$

Here c_q^i $(i = 1,2,3;\ q = 1,...,4)$, δ_4, δ_5 and δ_6 are functions of the invariants of $\mathbf{H}$. We observe that the representation (3.18) does not involve the representation of fourth-order tensor.

Remark 3.1

- If $\mathbf{B}(\mathbf{H}) = \mathbf{0}$ and $\overline{\mathbf{B}}(\mathbf{H}) = \overline{\mathbf{0}}$ then the bone remodelling process does not change the material symmetry group. The bone density can then change, though not the anisotropy directions.
- For $\overline{\mathbf{B}}(\mathbf{H}) = \overline{\mathbf{0}}$ the anisotropy group can change. For instance, an orthotropic material may become transversely isotropic (without rotation of singled out anisotropy axes).
- If in the evolution equation (3.11) all functions do not disappear, then in the process of functional bone adaptation the anisotropy axes can rotate. Consequently, the material symmetry group can change. The simplest evolution law leading to the rotation of the principal axes of bone anisotropy results from Eq (3.18) and has the following form

  $$\dot{\mathbf{H}} = c(\mathbf{He} + \mathbf{eH})$$

  where c is constant.
- The adaptive theory elasticity proposed in Cowin and Hegedus (1976a), Hegedus and Cowin (1976b) and Cowin and Nachlinger (1978) involves only one scalar parameter related to the material density. Numerical applications were given by Cowin et al. (1993), Levenston (1997) and Luo et al. (1995).

Remark 3.2 The existence and uniqueness theorem due to Monnier and Trabucho (1998) can be extended to the initial-boundary value problem of adaptive piezoelectricity. We observe that these authors introduce a nonlocal model involving the elastic moduli of nonlocal type and a parameter η. From the biomechanical point of view such a model involves cells communicating with other cell in a certain neighbourhood. Anyway, the passage to the classical model of adaptive elasticity $(\eta \to 0)$ remains open.

4. Homogenization and bone remodelling

The bone tissue is a hierarchical multiphase material, inhomogeneous and anisotropic. In this section we propose a model of bone remodelling which exploits homogenization. Our approach applies both to compact and cancellous bones.

Let Y be a basic cell (representative element) with holes T_p, $p = 1, ..., P$, cf Attouch (1984). First, consider the case where bone tissue occupies the domain $Y \setminus \bigcup\limits_p T_p$. The holes T_p are empty. We assume that the voids T_p can evolve and their evolution depends on Z, where

$$Z = \int\limits_{t_0}^{t} U \, dt \tag{4.1}$$

Here U stands for the velocity of the bone surface remodelling at a surface point $\boldsymbol{y} \in \partial T_p$. Consequently, Z denotes the extent of bone deposition or resorption. The macroscopic elastic potential is given by

$$\begin{aligned} &W_h(\mathbf{e}^h, \xi) = \\ &= \inf\left\{ \frac{1}{2|Y|} \int\limits_{Y \setminus \bigcup\limits_p T_p(Z)} a_{ijkl}[e^y_{ij}(\boldsymbol{v}) + e^h_{ij}][e^y_{kl}(\boldsymbol{v}) + e^h_{kl}] d\boldsymbol{y} \,\middle|\, \boldsymbol{v} \in \Big[H^1_{per}\Big(Y \setminus \bigcup\limits_p T_p(Z)\Big)\Big]^3 \right\} \end{aligned} \tag{4.2}$$

where

$$e^y_{ij} = \frac{1}{2}\Big(\frac{\partial v_i}{\partial y_j} + \frac{\partial v_j}{\partial y_i}\Big) \qquad\qquad \xi = \frac{1}{|Y|}\Big|Y \setminus \bigcup\limits_p T_p(Z)\Big|$$

The function U can be treated as a measurable quantity; otherwise an evolution law has to be specified. The space $H^1_{per}\Big(Y \setminus \bigcup\limits_p T_p(Z)\Big)$ is defined as follows, cf Attouch (1984)

$$\begin{aligned} H^1_{per}\Big(Y \setminus \bigcup\limits_p T_p(Z)\Big) = \Big\{ \boldsymbol{v} \in H^1\Big(Y \setminus \bigcup\limits_p T_p(Z)\Big) \,\Big|\, \\ \Big| \, \boldsymbol{v} \text{ assumes equal values on the opposite faces of } Y \Big\} \end{aligned}$$

The regularity of perforated domains indispensable for performing homogenization was discussed in Acerbi et al. (1992) and Olejnik et al. (1992).

Let us denote by $\overline{\boldsymbol{v}}$ a minimizer solving the minimization problem in the right-hand side of Eq (4.2).

Then $\overline{v}_i = \chi_i^{(kl)} e_{kl}^h$ and the effective elastic moduli are given by

$$a_{ijkl}^h(\xi) = \frac{\partial^2 W_h}{\partial e_{ij}^h \partial e_{kl}^h} = \frac{1}{|Y|} \int\limits_{Y \setminus \bigcup\limits_p T_p(Z)} a_{ijmn}\Big(\delta_{km}\delta_{ln} + \frac{\partial \chi_m^{(kl)}}{\partial y_n}\Big)\, d\boldsymbol{y} \tag{4.3}$$

The coefficients a_{ijkl} appearing in Eq (4.2) were assumed to be constant. In a more general case, important in the study of effective properties of bones, these moduli can be assumed to depend on the macroscopic variable $\boldsymbol{x}$ and on the microscopic variable $\boldsymbol{y} \in Y$. Then $a_{ijkl}^h(\boldsymbol{x}, \xi)$ depends also on $\boldsymbol{x}$.

For the compact bones one distinguishes at least a triple hierarchy, cf Telega et al. (1999). Eq (4.2) can easily be generalized to cover such a more general model of hierarchic perforated material like bone.

Suppose now that the holes T_p, $p = 1, ..., P$, are filled with a "weak" elastic material with elastic moduli ηb_{ijkl}, where $\eta > 0$ is a small parameter. Then the macroscopic elastic potential is given by

$$\begin{aligned} W_h(\mathbf{e}^h, \xi) = \inf \Bigg\{ & \frac{1}{2|Y|} \int\limits_{Y \setminus \bigcup\limits_p T_p(Z)} a_{ijkl}[e_{ij}^y(\boldsymbol{v}) + e_{ij}^h][e_{kl}^y(\boldsymbol{v}) + e_{kl}^h] d\boldsymbol{y} + \\ & + \frac{\eta}{2|Y|} \int\limits_{\bigcup\limits_p T_p(Z)} b_{ijkl}[e_{ij}^y(\boldsymbol{v}) + e_{ij}^h][e_{kl}^y(\boldsymbol{v}) + e_{kl}^h] d\boldsymbol{y} \,\Bigg|\, \boldsymbol{v} \in \Big[H_{per}^1(Y)\Big]^3 \Bigg\} \end{aligned} \tag{4.4}$$

Once the effective potential W_h is known, the macroscopic moduli can be derived similarly as previously, cf Eq (4.3).

Remark 4.1 To better model the real behaviour of wet trabecular bone, the elastic material in pores should be replaced with a viscoelastic material imitating the marrow. Homogenization methods can still be used to describe the macroscopic behaviour of trabecular bone.

5. Bone remodelling as an optimal design problem

The bone tissue may be viewed as a composite material consisting of two phases: the organic phase (1) (mainly collagen fibres) and inorganic phase (2) (mainly hydroxopatite crystals). The elasticity tensor is then written in the form, cf Francford et al. (1995)

$$\mathsf{a}(\boldsymbol{x}) = \chi_1(\boldsymbol{x})\mathsf{a}_1 + \chi_2(\boldsymbol{x})\mathsf{a}_2 \qquad \boldsymbol{x} \in \Omega \tag{5.1}$$

where χ_1 is the characteristic function of the material (1) and $\chi_2 = 1 - \chi_1$. Here Ω is an open bounded set in $I\!R^3$ and its closure $\overline{\Omega}$ is identified with the undeformed configuration of considered bone.

Let us examine first the so called *compliance problem*, cf Allaire and Kohn (1993), Cherkaev and Kohn (1997), Lipton (1994) and the references cited therein. The compliance is the work done in the structural domain Ω against the body forces $\boldsymbol{f} = (f_i)$ $(i = 1,2,3)$ and boundary tractions $\boldsymbol{g} = (g_i)$ by the resulting elastic displacement $\boldsymbol{u} = (u_i)$

$$l(\boldsymbol{u}) = \int\limits_{\Omega} f_i u_i \, d\boldsymbol{x} + \int\limits_{\Gamma} g_i u_i d\Gamma \tag{5.2}$$

where $\Gamma = \partial\Omega$ denotes the boundary of Ω. We assume that $\boldsymbol{f} \in [H^1(\Omega)^3]^*$ and $\boldsymbol{g} \in H^{-1/2}(\Gamma)^3$. Particularly, this assumption includes $\boldsymbol{f} \in L^2(\Omega)^3 = [L^2(\Omega)]^3$ and $\boldsymbol{g} \in L^2(\Omega)^3$.

The displacement field $\boldsymbol{u}$ solves the following variational equation

$$\boldsymbol{u} \in H^1(\Omega)^3 : \quad \int\limits_{\Omega} a_{ijkl}(\boldsymbol{x}) e_{ij}(\boldsymbol{u}) e_{kl}(\boldsymbol{v}) \, d\boldsymbol{x} = l(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in H^1(\Omega)^3 \tag{5.3}$$

Obviously, mixed boundary conditions can be treated similarly. The constitutive equation is classical

$$\mathsf{T} = \mathsf{a}(\boldsymbol{x}) \cdot \mathsf{e}(\boldsymbol{u}) \tag{5.4}$$

To ensure solvability of the problem (2.2) the following condition has to be satisfied

$$l(\boldsymbol{r}) = 0 \tag{5.5}$$

for all rigid-body displacements $\boldsymbol{r}$.

The minimum compliance problem means evaluating

$$(M \subset P) \quad \min_{\mathsf{a}(\boldsymbol{x})} \Big\{ l(\boldsymbol{u}) \Big| \boldsymbol{u} \text{ satisfies Eq (5.3) and } V_2 \geqslant \int\limits_{\Omega} \chi_2 \, d\boldsymbol{x}$$

Here V_2 is the maximum amount of material (2) allowed in the design and $V_2 < |\Omega| = vol(\Omega)$.

We observe that the problem $(M \subset P)$ may be viewed as a distributed-parameter optimal control problem, where the control is $\mathbf{a}(\boldsymbol{x})$. By introducing the positive Lagrange multiplier λ associated with the volume constraint

$$V_2 \geqslant \int_{\Omega} \chi_2(\boldsymbol{x})\, d\boldsymbol{x} \tag{5.6}$$

we write the problem $(M \subset P)$ in the form

$$\min_{\mathbf{a}(\boldsymbol{x})} \max_{\boldsymbol{u} \in H^1(\Omega)^3} \Big\{ 2l(\boldsymbol{u}) - \int_{\Omega} \mathbf{e}(\boldsymbol{u}) \cdot \mathbf{a}(\boldsymbol{x}) \cdot \mathbf{e}(\boldsymbol{u})\, d\boldsymbol{x} + \lambda \int_{\Gamma} \chi_2\, d\boldsymbol{x} \Big\} \tag{5.7}$$

It is known that the last problem is ill-posed. Consequently, relaxation is required. Towards this end, the set of layouts is extended to include a composite formed of the original constituents. The set of effective elastic tensors associated with all composites using materials $\mathbf{a}_1$ and $\mathbf{a}_2$ is denoted by G. Let $\theta_2$be the volume fraction of material (2) (or inorganic part) in the bone, $0 \leqslant \theta_2 \leqslant 1$. For fixed volume fraction θ_2, we denote the set of associated effective elastic tensors by G_{θ_2}. An exhaustive characterization of G_{θ_2} is known in special cases only, cf Allaire (1994), Allaire and Kohn (1993), Cherkaev and Kohn (1997), Pedersen and Bendsøe (1999) and the references cited therein.

To extend the design space we introduce the notion of generalized layout. In our case it is given by a local volume fraction θ_2 in $L^\infty(\Omega, [0,1])$ and an associated field $\mathbf{a}$ in $L^\infty(\Omega, T_s)$ taking values in the set $G_{\theta_2(\boldsymbol{x})}$. This set of tensor fields associated with the generalized layouts is denoted by $\widetilde{G}_{\theta_2(\boldsymbol{x})}$. Here T_s^4 denotes the space of fourth-order tensors $\mathbf{C} = (C_{ijkl})$ such that $C_{ijkl} = C_{klij} = C_{jikl}$.

The relaxed form of (5.7) is given by

$$\min_{\theta_2 \in L^\infty(\Omega,[0,1])} \min_{\mathbf{a} \in \widetilde{G}_{\theta_2(\boldsymbol{x})}} \max_{\boldsymbol{u} \in H^1(\Omega)^3} \Big\{ 2l(\boldsymbol{u}) - \int_{\Omega} \mathbf{e}(\boldsymbol{u}) \cdot \mathbf{a}(\boldsymbol{x}) \cdot \mathbf{e}(\boldsymbol{u})\, d\boldsymbol{x} + \lambda \int_{\Gamma} \theta_2(\boldsymbol{x})\, d\boldsymbol{x} \Big\} \tag{5.8}$$

The last problem is solvable. Due to the lack of space the last problem will not be discussed here in detail. The reader is referred to Allaire (1994), Allaire and Kohn (1993), Cherkaev and Kohn (1997), Lipton (1994). The presence of the set $\widetilde{G}_{\theta_2(\boldsymbol{x})}$ in the last problem implies that the optimal structure is realized by a microstructure, particularly a laminar microstructure observed both in cortical and concellous bones.

Remark 5.1 An important point in the purely mechanical generalized minimum compliance problem (5.8) is the functional $l(\boldsymbol{u})$. What loadings $\boldsymbol{f}$ and $\boldsymbol{g}$ should enter this problem? It seems that by changing loads one can simulate both resorption and apposition. The following problem also arises: is it possible to describe by purely mechanical modelling the bone resorption during prolonged inactive stage of lite?

Remark 5.2 The bone adaptation process is evolutionary. The volume fraction θ_2 changes in time, i.e. $\theta_2(\boldsymbol{x},t)$, $t \in [0,T]$, and T is the remodelling time. Its evolution can be described by

$$\frac{d\theta_2}{dt} = F(\boldsymbol{x},t) \qquad \boldsymbol{x} \in \Omega \quad t \in [0,T] \tag{5.9}$$

The right-hand side of the last equation can include both mechanical and biological stimuli. Mullender and Huiskes (1995) proposed a simple regulatory mechanism, in which the osteocytes act as sensors of a mechanical signal or "mechanoreceptors" and regulators of bone mass by mediating the actor cells – the osteoblasts and osteoclasts. Now the problem is modified as follows

find

$$\left\| \begin{array}{l} \displaystyle \min_{\mathsf{a} \in \widetilde{G}_{\theta_2(\boldsymbol{x},t)}} \max_{\boldsymbol{u} \in \mathcal{L}} \frac{1}{T} \Big\{ 2 \int_0^T l(\boldsymbol{u})\, dt + \\ \displaystyle + \int_0^T \!\! \int_\Omega \mathbf{e}\Big(\boldsymbol{u}(t)\Big) \cdot \mathsf{a}(\boldsymbol{x},t) \cdot \mathbf{e}\Big(\boldsymbol{u}(t)\Big)\, d\boldsymbol{x}dt + \int_0^T \!\! \int_\Gamma \theta_2(\boldsymbol{x})\, d\boldsymbol{x}dt \Big\} \\ \text{subject to (5.9)} \end{array} \right. \tag{5.10}$$

Here $\mathcal{L} = L^\infty\Big((0,T), H^1(\Omega)^3\Big)$ and $\boldsymbol{u}(t) = \Big\{\boldsymbol{u}(\boldsymbol{x},t) \Big| \boldsymbol{x} \in \Omega\Big\}$. We observe that the elastic moduli and the set G_{θ_2} are time-dependent. It means that changes of the local volume fraction θ_2 can change the directional properties of the microstructure of bone.

Remark 5.3 Bone is a microperforated composition. Its actual shape and microstructure can be modelled as a shape optimization problem consisting in seeking minimizers of the sum of the elastic compliance and of the weight of a solid structure under a specified loading. Mathematical framework allowing one such an optimization problem was elaborated by Allaire et al. (1997).

6. Final remarks

Francfort and Marigo (1993) developed a discrete damage evolution model provided that stiffness drops from $\mathbf{a}^0$ to $\mathbf{a}^1$ and

$$\mathbf{a}^0 > \mathbf{a}^1 > \mathbf{0} \tag{6.1}$$

where the inequalities should be understood as inequalities between symmetric fourth-order tensors, i.e.

$$\widetilde{\mathbf{e}} \cdot \mathbf{a}^0 \cdot \widetilde{\mathbf{e}} > \widetilde{\mathbf{e}} \cdot \mathbf{a}^1 \cdot \widetilde{\mathbf{e}} > \mathbf{0} \qquad \forall \widetilde{\mathbf{e}} \in T_s^2 \quad \widetilde{\mathbf{e}} \neq \mathbf{0} \tag{6.2}$$

The damage process examined in Francfort and Marigo (1993) can describe bone resorption (osteoporosis) provided that the final stiffness tensor $\mathbf{a}^1$ is known. This is a weak point of otherwise rigorous approach, which also involves homogenization and relaxation.

Residual strains and stresses had been primarily discoverd in soft tissues, cf Vaishnov and Vossonghi (1983), Chuong and Fung (1986). Next, they were hypothesized to exist also in bone tissues, cf Tanaka and Adachi (1994). Consequently, the following question arises: can bone remodeling be viewed as a shakedown problem? The answer seems to be positive. However, now the elastic moduli are time-dependent and classical proofs of shakedown theorems fail. We recall the that wet bone is an elasto-plastic material cf Cowin (1989).

We have proposed a model of adaptive piezoelectricity. The model of adaptive elasticity can also be generalized in different directions. For instance, one can envisage adaptive thermoelasticity, etc.

Recently, Cowin (1997) severely criticized "Wolff's law" claiming that its rigid form has no sense. For historical comments on bone remdelling the reader is also referred to Martin and Burr (1989) (cf also Currey, 1997).

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Modelowanie anizotropii kości i jej przebudowy

Streszczenie

Przedyskutowano anizotropię tkanki kostnej w zakresie sprężystym i plastycznym. Przeanalizowano równania adaptacyjnej teorii sprężystości z ewoluującą strukturą przy zastosowaniu funkcji tensorowych. Sformułowano równania adaptacyjnej teorii piezoelektryczności. Zaproponowano ogólny model dla przebudowy kości w powiązaniu z homogenizacją. Wysunięto hipotezę, że proces adaptacji kości może być rozpatrywany w ramach teorii przystosowania. Rozpatrzono możliwość badania przebudowy kości jako zadania optymalnego.

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