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INFLUENCE OF VARIABLE ORTHOTROPY UPON THE STABILITY OF THIN-WALLED RECTANGULAR PLATES

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The author analyse the stability loss in thin-walled orthotropic plates, their principal directions of orthotropy being parallel to the wall edges, for plates with the orthotropy ratio, $\beta = E_x/E_y$ varying widthwise. The analysis was carried out within the elastic range. Numerical tests were made on thin plates with the loaded edges simply supported; unloaded edges were tested for different kinds of support (clamped, simply supported, free-edge). Plates were subject to loads causing uniform and linearly varying displacement of edges. The problem was solved using Koiter's first-order asymptotic theory of conservative systems stability (Koiter, 1963).
Results of numerical calculations were presented as graphs describing the relationship between a critical value (force or moment) and the parameter defining orthotropy across the plate width.

Key words: stability, thin-walled structures, orthotropy

1. Introduction

Elastic buckling of isotropic and orthotropic plates and girders has been dealt with in a number of publications (Lekhnitskii, 1947; Chandra and Raju, 1973; Kołakowski, 1993, 1994; Królak, 1995). The test results presented in these papers aim at showing the designers new possibilities of creating light, safe and reliable structures. However, in the literature on stability problems there is a shortage of analysis of how a widthwise variation of the plate (wall) orthotropy may influence critical load values and buckling modes.

At present more and more carrying elements are made of composite materials with different, often very high degrees of orthotropy.

Aiming to use composite materials with required strength properties as carrying elements in thin-walled structures, the designers seek information about the behaviour of those parts under different kinds of load. Particularly dangerous for thin-walled structures are loads which cause a stability loss (buckling) of their carrying elements.

Composite components give a substantial opportunity for variation of composite global properties in selected directions or areas; hence it is possible to obtain plates revealing variable strength properties. Such materials are, e.g., fibrous composites with adequately displaced (concentrated or diluted) fibres. Composites are usually modelled as orthotropic materials. Taking into account the variation of orthotropy ratio, $\beta(y) = E_x(y)/E_y$, which can be a function of y variable (Fig.1), enables the assumed calculation model to approximate real materials.

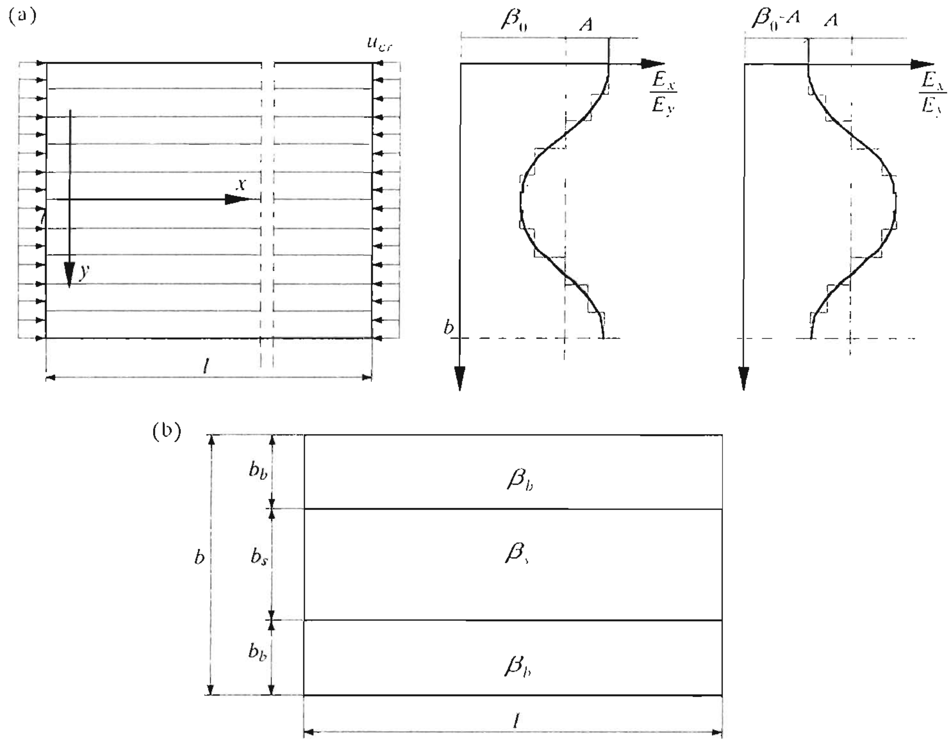

Fig. 1. Band model of variable orthotropy plates; (a) sinusoidal orthotropy, (b) orthotropic stiffening of edges ($\beta_0 = 3.2292$ – constant value of orthotropy ratio; A – amplitude; b – plate width; l – plate length; $\beta = E_x/E_y$ – orthotropy ratio equal to the ratio of the Young moduli along and across the direction of compression, respectively)

The present work discusses the problem of stability loss in the elastic range of homogeneous orthotropic plates with a widthwise variable β ratio.

2. Formulation of the problem

Assumption has been made that the principal orthotropy axes of plates are parallel to their edges. Plates of loaded edges rested on simple supports and longitudinal edges were simply supported, clamped or free, and were loaded so as to cause uniform or linearly distributed displacement of edges (Fig.2). In order to characterize the loading mode a special load factor $\kappa = u_2/u_1$ was introduced, where u_1 and u_2 (Fig.2) are displacement values in the range of -1 to 1. All plates tested had geometrical and material axes of symmetry.

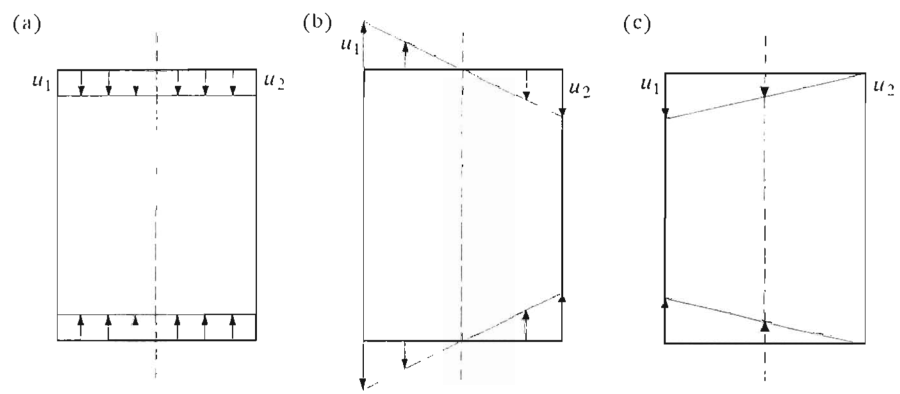

Fig. 2. Loading modes; (a) uniform displacement of edges, $\kappa = 1$, (b) linearly variable displacement of edges, $\kappa \neq 1$

The plate with widthwise variable orthotropy (Fig.1) was considered. A model was assumed to be built of long and narrow orthotropic plates each of which may have a different orthotropy ratio. This enables one to analyse the effect of sinusoidal orthotropy across the plate – plate I. Plate II consists of three bands of different orthotropy ratios; the two outside bands have the same ratio between the Young moduli along and across the direction of compression (this is a model of a plate with constant thickness and stiffened or weakened edge bands).

As a consequence of the above, the critical values are the following: critical force F_{cr} [N] in the case of uniform displacement of edges; critical moment M_{cr} [Nm] in the case of linear displacement corresponding to $\kappa = -1$ (Fig.2b);

critical force F_{cr} [N], and critical moment M_{cr} [Nm], in other cases. It should be noted that, if no axis of symmetry was assumed, the uniform displacement of edges (Fig.2a) could refer to the case of eccentric compression.

The effect of widthwise variable orthotropy upon the stability of thin-walled plates was analysed using the first-order approximation of Koiter's asymptotic theory of stability of conservative systems (Koiter, 1963) and the solution method presented by Królak (1995). The fundamental equations of the method proposed by Królak (1995), Królak and Kołakowski (1995) are presented below.

For the ith part of the plate, exact geometrical relationships are adopted in order to allow for considering both the out-of-plane and in-plane bending of each part of the plate

$$\begin{aligned}
\varepsilon_{xi} &= u_{i,x} + 0.5(u_{i,x}^2 + v_{i,x}^2 + w_{i,x}^2)\\
\varepsilon_{yi} &= u_{i,y} + 0.5(u_{i,y}^2 + v_{i,y}^2 + w_{i,y}^2)\\
\varepsilon_{xyi} &= 0.5(u_{i,y} + v_{i,x} + u_{i,x}u_{i,y} + v_{i,x}v_{i,y} + w_{i,x}w_{i,y})\\
\kappa_{xi} &= -w_{i,xx} \qquad \kappa_{yi} = -w_{i,yy} \qquad \kappa_{xyi} = -w_{i,xy}
\end{aligned} \tag{2.1}$$

Physical relationships for the ith part of the plate are formulated in the following way

$$\varepsilon_{xi} = \frac{N_{xi} - \nu_{xyi}N_{yi}}{E_{xi}h_i} \qquad \varepsilon_{yi} = \frac{N_{yi} - \nu_{yxi}N_{xi}}{E_{yi}h_i} \qquad \varepsilon_{xyi} = \frac{N_{xyi}}{2G_ih_i} \tag{2.2}$$

The relation between the Young moduli and the Poisson ratio in Eqs (2.2) is as follow

$$E_{xi}\nu_{yxi} = E_{yi}\nu_{xyi} \tag{2.3}$$

The differential equilibrium equations resulting from the virtual work principle and corresponding to Exs (2.2) for the ith part of plate can be written as follows

$$\begin{aligned}
&N_{xi,x} + N_{xyi,y} + (N_{xi}u_{i,x})_{,x} + (N_{yi}u_{i,y})_{,y} + (N_{xyi}u_{i,x})_{,y} + (N_{xyi}u_{i,y})_{,x} = 0\\
&N_{yi,y} + N_{xyi,x} + (N_{xi}v_{i,x})_{,x} + (N_{yi}v_{i,y})_{,y} + (N_{xyi}v_{i,x})_{,y} + (N_{xyi}v_{i,y})_{,x} = 0\\
&D_i\nabla\nabla w_i - (N_{xi}w_{i,x})_{,x} + (N_{yi}w_{i,y})_{,y} + (N_{xyi}w_{i,x})_{,y} + (N_{xyi}w_{i,y})_{,x} = 0
\end{aligned} \tag{2.4}$$

The solution of these equations for each plate should satisfy kinematic and static conditions at the junctions of adjacent part of the plate and boundary conditions at the ends $x = 0$ and $x = l$.

The nonlinear problem is solved by the asymptotic Koiter methods. The displacement field $\overline{U}$, and sectional force field $\overline{N}$, are expanded in a power series in the buckling mode amplitudes ξ_n (ξ_n is the amplitude of nth buckling mode divided by the thickness of the first component wall h_1)

$$
\begin{aligned}
\overline{U} &= \lambda \overline{U}_i^{(0)} + \xi_n \overline{U}_i^{(n)} + \dots \\
\overline{N} &= \lambda \overline{N}_i^{(0)} + \xi_n \overline{N}_i^{(n)} + \dots
\end{aligned}
\tag{2.5}
$$

where the prebuckling field are $\overline{U}_i^{(0)}$, $\overline{N}_i^{(0)}$ the buckling mode fields are $\overline{U}_i^{(n)}$, $\overline{N}_i^{(n)}$.

By substituting the expansion (2.5) into the equation of equilibrium (2.4), junction conditions and boundary conditions, the boundary value problems of zero and first order can be obtained. The equations of first order boundary value problem are shown in the literature (cf Królak and Kołakowski (1995), Eqs (2.6) ÷ (2.10) and appendix). The zero approximation describes the prebuckling state while the first approximation, which is the linear problem of stability, enables us to determine the critical load values and the buckling modes. This question can be reduced to a homogeneous system of differential equilibrium equations.

The results were obtained using an appropriate computer programme, in which the procedure of finding determinante zeros was modified, and also convergence of the Godunow ortogonalization method was tested. At present the computer programme allows for analysis of plates and building girders made up of those plates with open and closed cross-sections, respectively. All these structures can consist of 400 narrow bands, each of them having a different (although constant) orthotropy ratio β_i.

A proper choice of orthotropy ratio for a particular band enables one to express the β ratio for the whole wall in the form of a specified function $\beta(y)$.

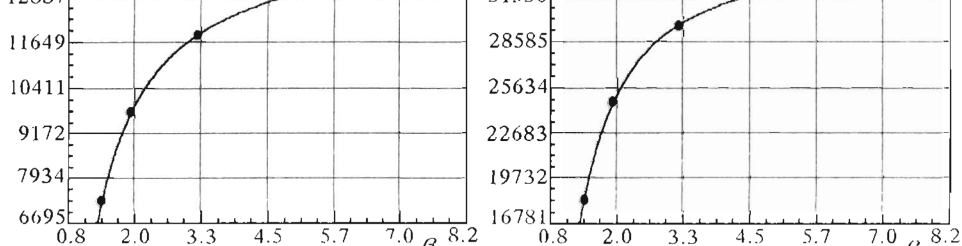

Fig. 3. Relations between the orthotropy ratio and G, E_y; • – from literature, — – obtained from the approximation

Orthotropic materials, unlike the isotropic ones, show no dependence between the Young and shear moduli; however, the literature delivers a number of approximate formulas which define this dependence. In order to obtain a model with a sinusoidal orthotropy ratio β the following values, depending on this ratio: E_x, E_y, G, ν_{xy} must be known. The required dependencies between the orthotropy ratio and E_x, E_y, G, ν_{xy} (Fig.3) were obtained by the approximation of material data published by Chandra and Raju (1973) and have the following form:

- Young modulus along the direction of compression for the ith band

$$E_{xi} = \beta_i E_{yi}$$

- Young modulus across the direction of compression for the ith band

$$E_{yi} = 34807 - \frac{11629}{\beta_i} - \frac{16821}{\beta_i^2}$$

- shear modulus for the ith band

$$G_{yi} = 14605 - \frac{7812}{\beta_i} - \frac{3464}{\beta_i^2}$$

- Poisson ratio for the ith band

$$\nu_{xy,i} = 0.3 \qquad i = 1, ...n$$

3. Numerical results

The results of numerical calculations have been presented in the form of graphs illustrating the dependence of the critical values (force F_{cr}, moment M_{cr}, or both at the same time) upon the parameter that describes the orthotropy variation. The symbols on graphs have the following meaning: letters stand for conditions on the longitudinal edges (p – simply supported, u – clamped, s – free edge), numbers designate geometrical parameters (first are the values of length l to width b ratio where 05 means that $l/b = 0.5$; the second, after a dash, are the values of width b to thick h ratio).

Plate I: a plate with orthotropy ratio β varying sinusoidally across the plate (Fig.1a) – this parameter is the amplitude A of a sinusoid which expresses the orthotropy variation.

Plate II: a plate with stiffened or weakened edges (Fig.1b) – the parameter defining the orthotropy variation is β^* which is a ratio between the of orthotropy ratios in the stiffened and not stiffened bands, respectively.

The following two plates have been considered.

3.1. Plate I (sinusoidal orthotropy variation)

Orthotropy ratios for individual bands in the model were assumed according to the equation

$$\beta_i = \beta_0 + A \cos \frac{2\pi y}{b}$$

where

- β_0 – mean value of the orthotropy ratio across the plate, $\beta_0 = E_x/E_y = 3.2292$
- A – amplitude of the sine curve, $A \in < -2, 2 >$
- y – co-ordinate defining the band position relative to one of the longitudinal edges
- b – plate width

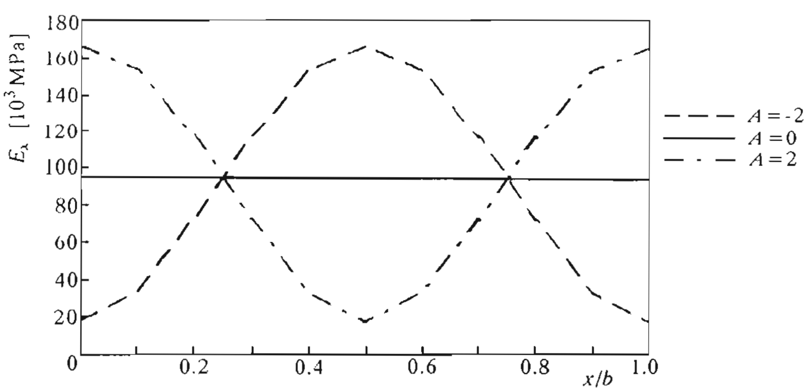

Fig. 4. Variation of the Young modulus for the direction of compression E_x across the plate at different A values

Fig.5 shows the results for flat rectangular plates of sinusoidal orthotropy ratios. The values of A factor were assumed as an amplitude of sine curve between -2 and 2. If $A < 0$ the Young modulus in the direction of compression at the plate edge is lower than inside; if $A > 0$ the plate is more rigid at the edges; if $A = 0$ then the orthotropy ratio β is constant on the whole width of the plate (Fig.4). The results presented in Fig.5 were obtained for a load which causes uniform displacement of the loaded edges $\kappa = u_2/u_1 = 1$. The analysis

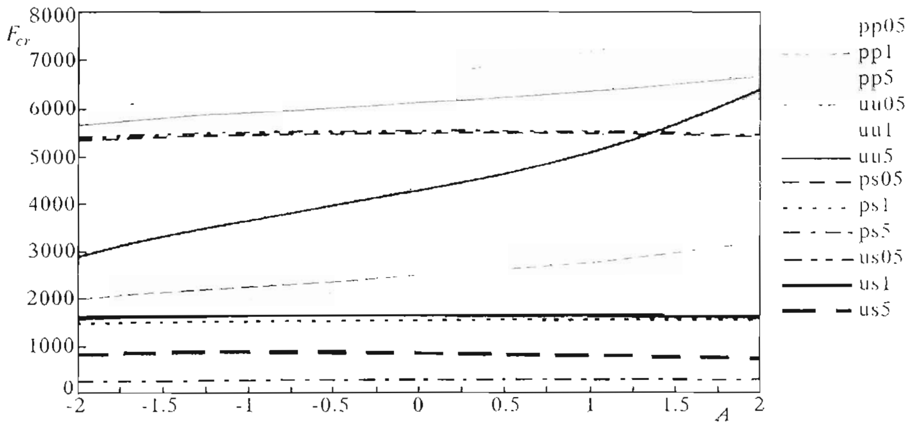

Fig. 5. Critical force as a function of the parameter A defining the orthotropy variation for the plates of $b/h = 60$

was performed for plates of different length to width ratios $(L/b = 0.5; 1; 5)$ and different supports of the unloaded edges (p – simply supported, u – clamped, s – free edge). The curves on the diagram of $F_{cr}(A)$ prove that in the case of plates with one free edge (in Fig.5 designated as "ps" and "us") the introduction of widthwise variable orthotropy was pointless since the value of A parameter in the analysed range had practically no influence upon the value of critical force.

The maximum critical force value in the case of the above plates occurs at $A = 0$, i.e. for a plate with constant orthotropy. Somewhat different are the results obtained for the plates with simply supported or clamped unloaded edges. The weakening of edges, i.e. the lowering of A value from 0 to -2, causes a decrease in the critical force value by about 25% for the square plates and plates with $l/b = 5$. Increasing the A value from 0 to 2 in simply supported plates makes the critical force grow by $\sim 500\,\mathrm{N}$ which gives a 10% increase for short and 25% for the long ones. In comparison with the clamped edge plates the increase in A value causes a significant growth of the critical force, by about $1000\,\mathrm{N}$ (over 50% in case of square plates).

Fig.6 illustrates a dependence of critical force on the A parameter for a plate loading mode which causes uniform displacement of loaded edges $\kappa = 1$. The curves with corresponding to identical edge conditions but different dimensions, i.e. the plate width to thickness ratios $b/h = 60$ and 100 (in the description of Fig.6 the numbers after the dash) and different ratios $l/b = 0.5$; 1; 5 (in the description of Fig.6 the numbers before the dash). Both when $b/h = 60$ (Fig.5) and when $b/h = 100$, plates show a distinct growth of

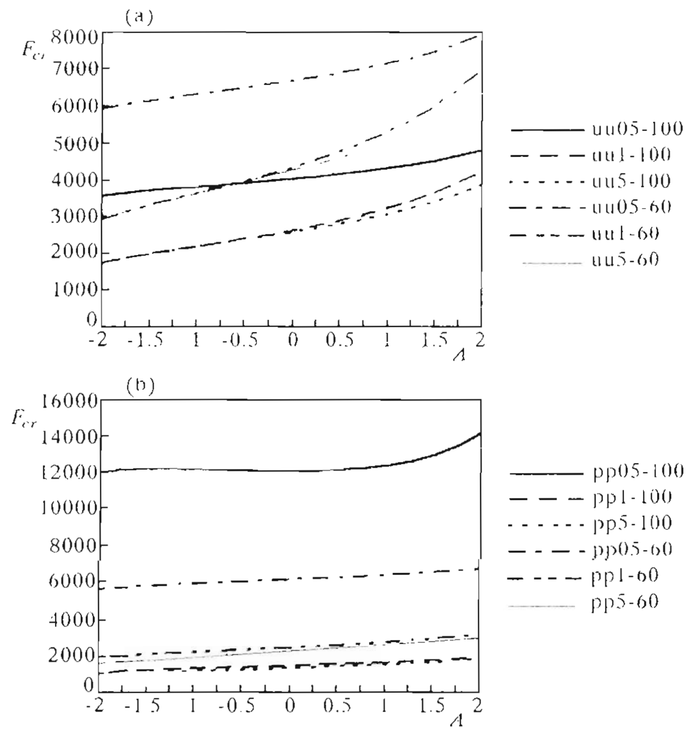

Fig. 6. The influence of parameter A upon the critical force for plates with different l/b and b/h ratios; (a) plates clamped on unloaded edges, (b) plates simply supported on unloaded edges

the critical force values (by about 50% for the clamped plates (Fig.6a) and by about 30% for the simply supported plates (Fig.6b) accompanying an increase in the A parameter from 0 to 2.

The following two graphs (Fig.7 and Fig.8) show the curves illustrating the influence of A parameter upon the critical value M_{cr}, in rectangular plates with sinusoidal orthotropy; the loading mode causes a linear variation in the displacement of edges $\kappa = -1$. In all the analysed cases an increase in A, that is a stiffening of edges, causes an increase in the critical moment value.

Fig.7 refers to the plates with clamped unloaded edges, with different width to thickness ratios $b/h = 60$ and 100 (the after-dash numbers in the description) and with different length to width ratios $l/b = 0.5$, 1 and 5 (the before-dash numbers in the description). In the above cases the critical moment grows by about 25% with A increasing from 0 to 2 and is reduced by about 50% with A changing from 0 to -2. The curves referring to $l/b = 0.5$ and 1 coincide just like the curves corresponding to $l/b = 5$.

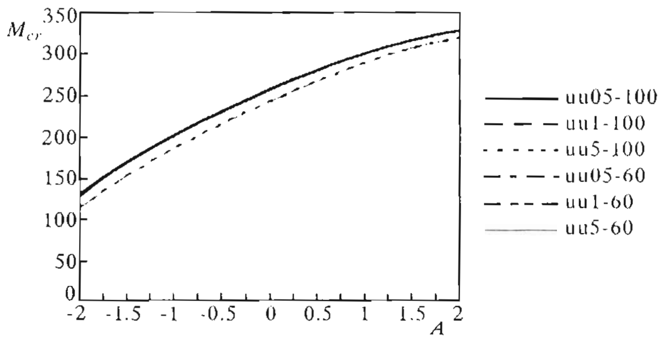

Fig. 7. Critical moment versus the factor A in the case of plates with unloaded edges clamped

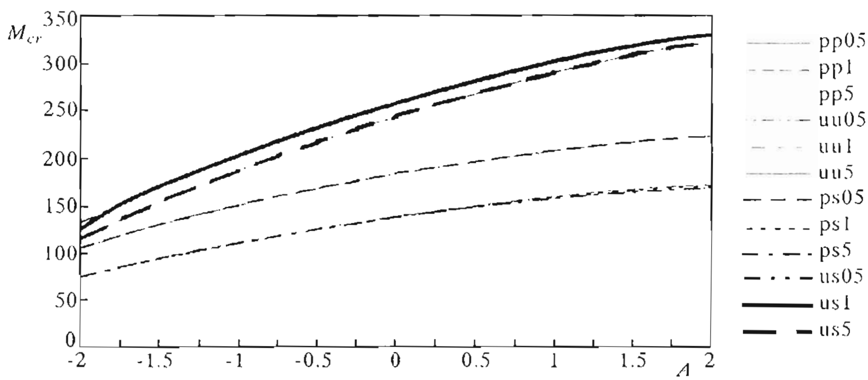

Fig. 8. The influence of factor A upon the critical moment in the case of plates with $b/h = 60$

Fig.8 refers to plates with $b/h = 60$, different length to width ratios $l/h = 0.5$, 1 and 5 (numbers 05, 1 and 5 in the description) and different supporting modes of unloaded edges: simply supported, clamped, free edge (supporting mode is designated by symbols p, u and s, respectively). All the curves are increasing which allows us to say that an increase in A value, i.e. a stiffening of edges, causes increase of the critical value which causes a buckling of a plate under bending. The analysis of curves shows that an increase in A value from 0 to 2 causes an $\sim 25\%$ increase in the critical moment value in all the cases considered while a decrease in A value from 0 to -2 reduces the critical value by about 50%.

The following two figures present the results in the form of graphs which express the dependence of critical values (moment and force) on the amplitude of the sine curve defining the orthotropy variation across the plate width. The results were calculated for rectangular plates subject to loading with $\kappa = 0$ (Fig.2b). The interpretation of curves obtained for such a load is somewhat more difficult than that of the others since for one A value two critical load parameters were obtained, namely the moment M_{cr} and the force F_{cr}.

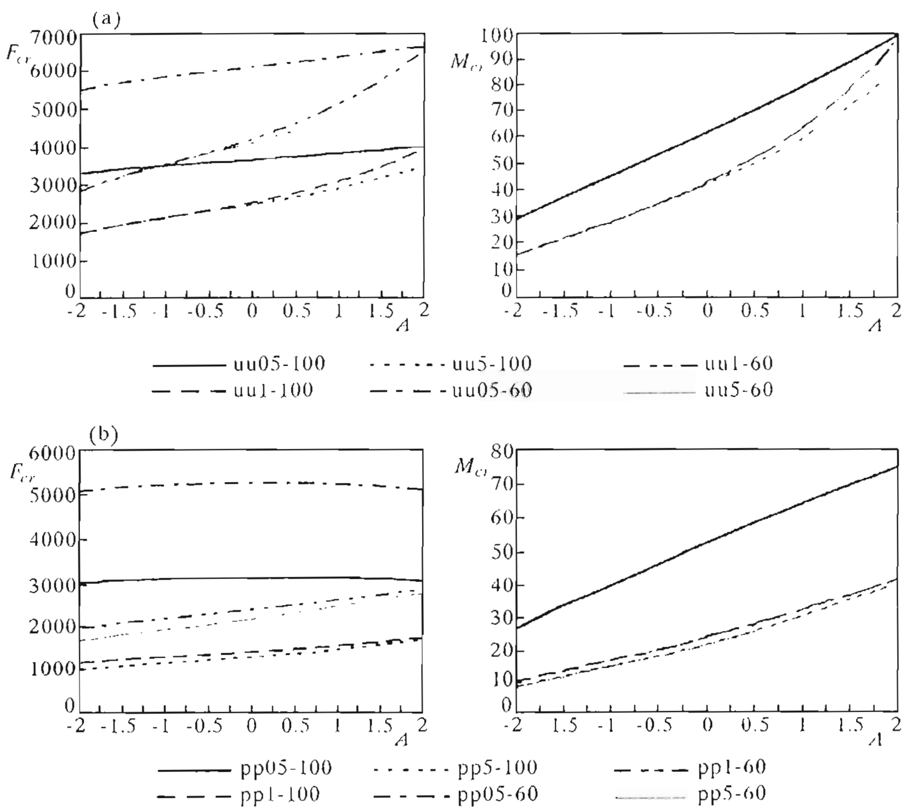

Fig. 9. The influence of orthotropy variation factor A upon critical values; (a) unloaded edges clamped, (b) unloaded edges simply supported

In Fig.9 and Fig.10 the critical moment value M_{cr} grows together with the A value whereas the critical load value not always shows an increase.

The plates with width to thick of plate equal 100 and 60 and length to width of plate $l/b = 0.5$, their unloaded edges being simply supported, show a slight decrease in the critical force value (F_{cr} variation no more than 5%) when A changes from 0 (constant orthotropy) to ± 2. In the two above cases the critical moment M_{cr} grows by about 45% when A goes up from 0 to 2. This observation enables one to conclude that an increase in the amplitude of

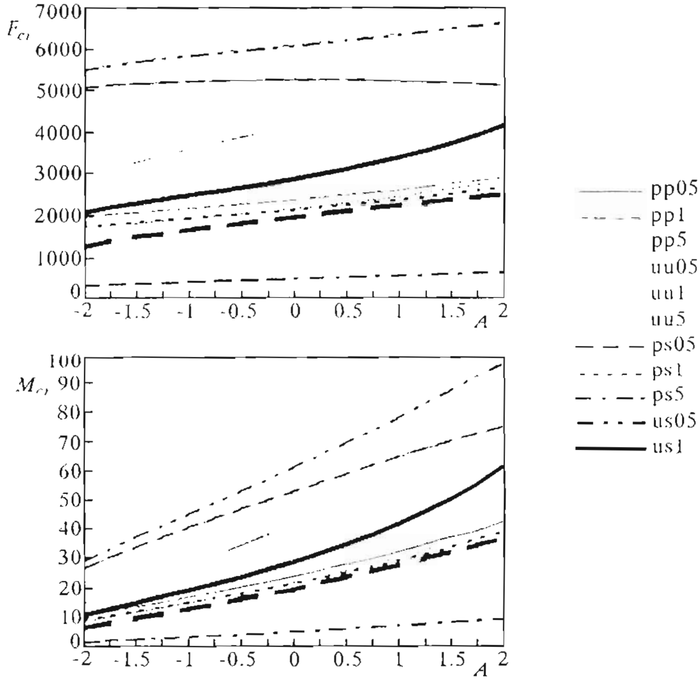

Fig. 10. The influence of factor A upon the critical values in a plate of $b/h = 60$, subject to loading which causes displacement of loaded edges as in the case of eccentric compression

the sinusoid defining the orthotropy variation (orthotropic stiffening of edges) results in an increase in the critical load values causing the stability loss.

3.2. Plate II (stepwise variable orthotropy)

The following geometrical dimensions were assumed: plate width $b = 100$ and 60 mm, length to width ratio $l/b = 0.5$, 1 and 5, thickness $h = 1$ mm.

Below are presented the results of numerical calculations for plates with stepwise variable orthotropy, a model consisting of three bands (Plate II). Analysis was made of plates with stiffened edges (designations: 1b – 20%; 2b – 40%; 3b – 60% of plate width, respectively) and with a stiffened centre (designations: 2s – 20%, 4s – 40%, 6s – 60% of plate width, respectively).

Fig.11 and Fig.12 present curves obtained for plates under a load causing a uniform displacement of edges $(\kappa = 1)$.

Fig.11 shows $F_{cr}(\beta^*)$ values for long plates $(l/b = 5)$ with the unloaded edges simply supported (Fig.11a) or clamped (Fig.11b). It can be seen that an

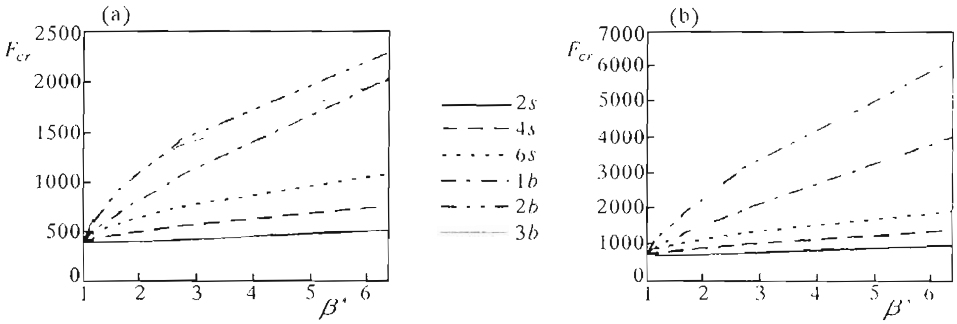

Fig. 11. The influence of parameter β^* upon the critical force value for plates with $l/b = 5$ subject to loading which causes a uniform displacement of edges; (a) unloaded edges simply supported, (b) unloaded edges clamped

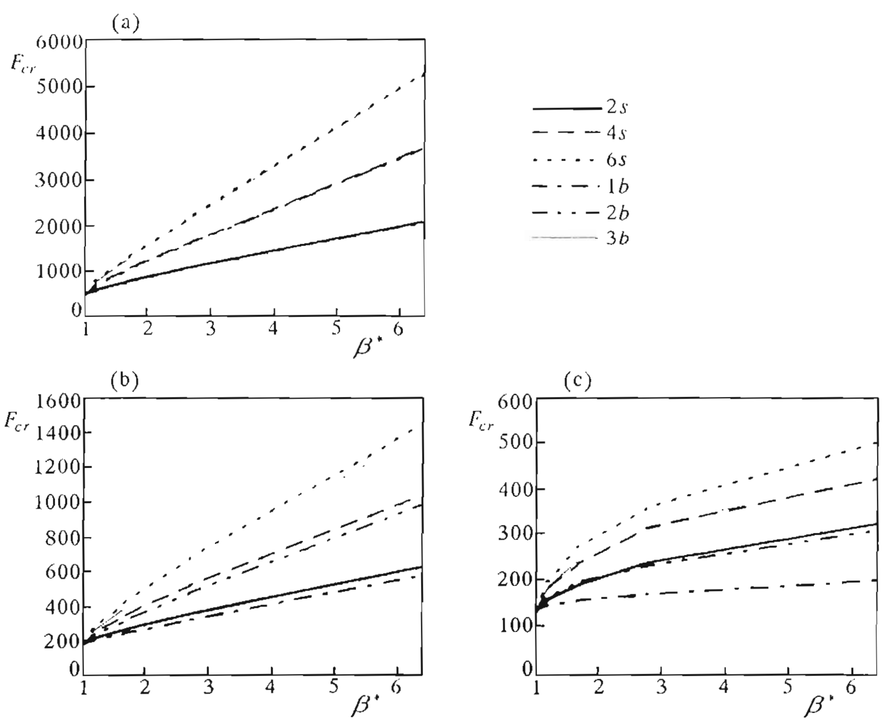

Fig. 12. The influence of β^* upon the critical force value in plates with different l/b ratios subject to a load causing a uniform displacement of edges; one unloaded edge is clamped, the other one free, (a) $l/b = 0.5$, (b) $l/b = 1$, (c) $l/b = 5$

increase in β^* leads to an increase in the critical value. In plates with stiffened edges the growth of β^* from 1 to 7 resulted in an increase in the critical force value by about 4 times; in the case of centrally stiffened plates the respective increase was no more than 50%.

Fig.12 shows $F_{cr}(\beta^*)$ curves for plates of various lengths $(b/h = 100, l/h = 0.5, 1, 5)$ with one unloaded edge simply supported. the other one free. An analysis of these diagrams proves that the variation in critical value depends not only on whether the plate was stiffened on the edge or in the centre (curves in Fig.11) but also on the portion of plate width being stiffened.

In Fig.12 a,b all dependences are linear which is not the case in Fig.12c; this results from a change of the buckling mode when β^* goes up, i.e. it depends on the number of half-waves which are formed along the plate (Table 1). This effect is particularly evident in case of long plates (e.g. $l/b = 5$).

Table 1

The kind of stiffened edge	$\beta^* =$ / l/h	1	1.19	1.65	2.7	6.36
		Buckling mode – number of half-waves m				
3b	1	1	1	1	1	1
	5	3	3	3	2	2
4s	1	1	1	1	1	1
	5	3	3	3	2	2
6s	1	1	1	1	1	1
	5	3	3	3	2	2

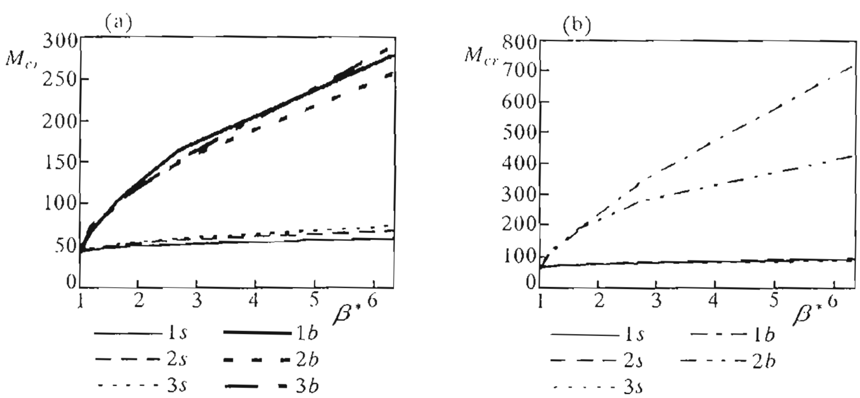

Fig. 13. The influence of β^* upon the critical force value in plates with $l/b = 0.5$ subject to load causing an displacement of edges as under pure bending; (a) unloaded edges simply supported, (b) unloaded edges clamped

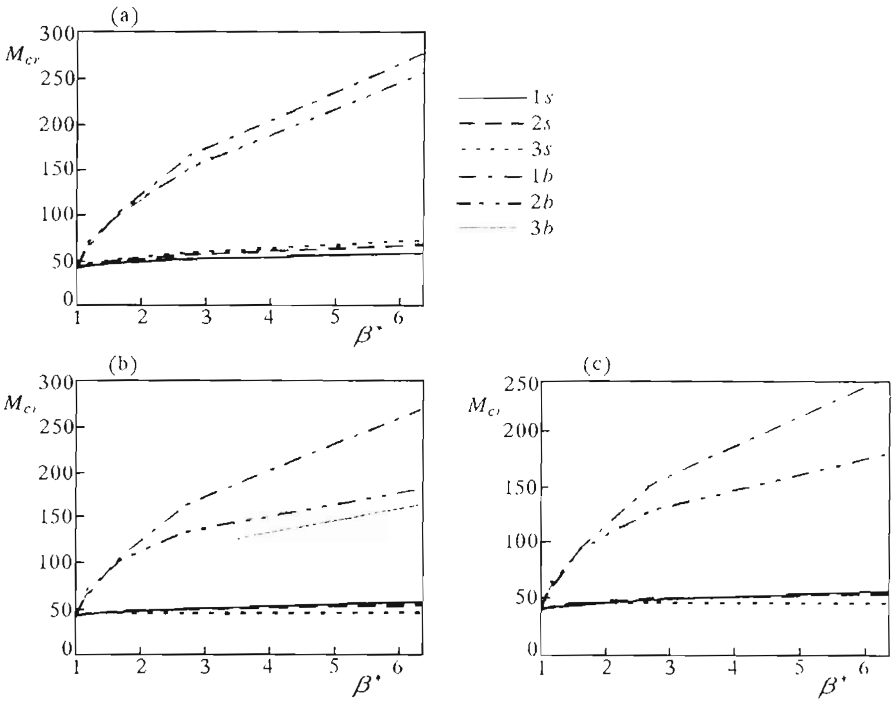

Fig. 14. M_{cr} as a function of β^* in plates where the edge displacement distribution is assumed to be linear $\kappa = -1$

The curves in Fig.13 and Fig.14 illustrate the dependence of critical moment upon the parameter of orthotropy variation β^* for plates where the load distribution can be characterized by $\kappa = -1$ (Fig.2b). Fig.13 presents curves obtained for the plates with $l/b = 0.5$ (short plates) and with different ways of the support of the unloaded edges. Fig.14 shows the curves $M_{cr}(\beta^*)$ for plates with different l/b: 0.5, 1 and 5, and with different boundary conditions on the unloaded edges: simply supported – free edge.

It can be concluded that stiffening makes sense only on the plate edges; stiffening the central part of the plate does not contribute to an increase in the critical load values.

4. Conclusion

The results of numerical calculations presented in this paper prove that, in the structures tested, higher orthotropy ratios at the plate edges result in higher critical load values in plates which are loaded causing a uniform ($\kappa = 1$) or linearly variable ($\kappa = 0$) displacement of the loaded edge. If the plate is subject to a load causing a linear displacement of the loaded edges ($\kappa = -1$) an orthotropic stiffening of the plate centre practically does not bring about any increase in the critical stress values.

The above allows one to conclude that, given the loading mode, edge conditions the geometry of the structure, it is possible to choose for the plate or for each girder wall a function that would describe the orthotropy variation across the plate so that the critical load could have a desired value.

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Wpływ zmiennej ortotropii na stateczność konstrukcji cienkościennych

Streszczenie

W pracy analizowano utratę stateczności cienkościennych płyt ortotropowych o głównych kierunkach ortotropii równoległych do krawędzi ścian. Badano płyty charakteryzujące się zmiennym wzdłuż szerokości współczynnikiem ortotropii $\beta = E_x/E_y$. Analiza prowadzona była w zakresie sprężystym. Badano płyty cienkie podparte na obciążonych brzegach przegubowo, a na nie obciążonych brano pod uwagę różne rodzaje podparcia (utwierdzenie, podparcie przegubowe, swobodny brzeg). Płyty poddano obciążeniom powodującym równomierne i liniowo zmienne zbliżenie brzegów. Zagadnienie rozwiązano przy zastosowaniu pierwszego rzędu asymptotycznej teorii stateczności układów zachowawczych Koitera.

Wyniki obliczeń numerycznych przedstawiono w postaci wykresów opisujących zależność wielkości krytycznej (siły lub momentu) od parametru określającego zmienność ortotropii wzdłuż szerokości płyty.

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