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ANTIPLANE PROBLEMS FOR ANISOTROPIC LAYERED MEDIA WITH THIN ELASTIC INCLUSIONS UNDER CONCENTRATED FORCES AND SCREW DISLOCATIONS[1]

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The antiplane problem of elasticity theory for a layered anisotropic medium containing the plane ribbon inhomogeneities is solved using the jump function method. The external load is determined by the boundary conditions, concentrated forces and screw dislocations inside layers. The inclusions are modelled by jumps of the stress and displacement vectors on the middle surfaces. Using the Fourier integral transform we obtain the relation between the stress tensor and displacement vector components and the external load and unknown functions of jumps. Taking into account the conditions interaction of between thin inclusion and anisotropic environment, the problem is reduced to a system of singular integral equations in the functions of jumps. In a general case the last is solved by means the collocation method. Some example is considered to illustrate the method.

Key words: elastic inclusion, antiplane state of strain, concentrated force, dislocation

1. Introduction

The structure of real materials is far from being ideal and the crystal bodies have frequently occuring flaws, i.e. the inclusions. Thin inhomogeneities can play a role of the composite reinforcement, e.g., adhesive interlayers during filling in the gaps to increase the machine or structure life (cf Klymenko and

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Liubchak, 1987). A thin interlayer can also describe accurately the imperfection of a real contact of bodies, where initiates the plastic strain and fracture due to the stress concentration caused by thin flaws.

The work of Frenchko and Tkach (1978) was the first in investigation into the antiplane problem of elasticity theory for a medium with inclusions; namely for the problem of inclusion in terms of the simplified Winkler model along the circular arc. There was no stress jump. Somewhat earlier (cf Berezhnytski et al., 1977) the generalized stress intensity factors (GSIF) were introduced and one-term asymptotic relations between stresses and displacements near a point elastic inclusion were constructed. The possibility of a synchronous jump of stresses and displacements was demonstrated by Sulym (1981). The solution was found on the basis of the jump function method (JFM), the conditions interaction between a thin inclusion and isotropic matrix were constructed. At the same time Opanasovych and Drahan (1981) obtained the similar results and later Opanasovych and Drahan (1984) basing on the method of linear potential development, as well. General results for a thin elastic inclusion on the interface of two anisotropic materials were obtained basing on the JFM (cf Sulym, 1987).

This paper applies the JFM and Fourier integral transform method to solution the antiplane problem for a packege of anisotropic layers, containing the plane ribbon inhomogeneities. The concentrated forces and screw dislocations inside the layers form the external load.

2. Problem statement

Consider the antiplane problem for the anisotropic layers S_j $(j = -M, \ldots, L)$ with the height H_j and elastic constants a^j_{km} $(k, m = 4, 5)$. The outer layers can be of a finite or infinitely large height (Fig.1). Let the axis Ox of the main coordinate system xOy be directed along the line between S_0 and S_{-1}.

The thin elastic inhomogeneities are situated inside the layers S_j along the segments L'_j that are inclined at the angles α_j to the material interfaces $y = d_j$ $(j = -M + 1, \ldots, L)$. There may be some other inclusions in the layer or none of them. The inclusions may also be located in the interphase. In the center O_j of the segment L'_j we locate the origin of two local coordinate systems $x_jO_jy_j$ and $s_jO_jn_j$ $(O_js_j||L'_j,\ O_jx_j||Ox)$, that are related by the

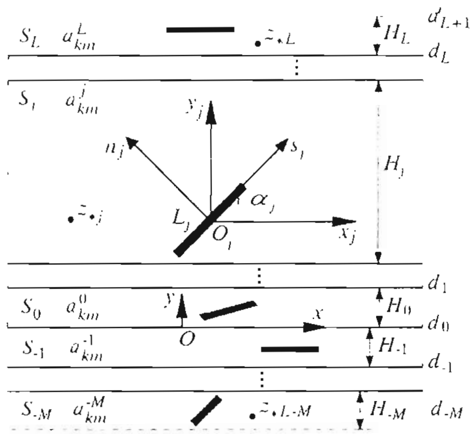

Fig. 1.

equations

$$z \equiv x + \mathrm{i}y = z_j + z_{0j} \qquad z_j \equiv x_j + \mathrm{i}y_j \qquad s_j + \mathrm{i}n_j = z_j \mathrm{e}^{-\mathrm{i}\alpha_j} \tag{2.1}$$

The coordinates of the points O_j in the main coordinate system xOy are denoted as z_{0j}.

The ideal mechanical contact conditions are satisfied along the lines $y = d_j$ between the layers S_j and S_{j-1}

$$\begin{array}{lll} \sigma^j_{yz} = \sigma^{j-1}_{yz} & \dfrac{\partial w^j}{\partial x} = \dfrac{\partial w^{j-1}}{\partial x} & \\ y = d_j & -\infty < x < +\infty & j = -M+1, \dots, L \end{array} \tag{2.2}$$

On the boundary of the package the stresses or displacements are defined

$$\begin{array}{lll} \sigma^L_{yz}(z) = f^+(x) & \dfrac{\partial w^L(z)}{\partial x} = g^+(x) & y = d_{L+1} \\ \sigma^{-M}_{yz}(z) = f^-(x) & \dfrac{\partial w^{-M}(z)}{\partial x} = g^-(x) & y = d_{-M} \end{array} \tag{2.3}$$

As $|x| \to \infty$ the stresses, which are equal to $\sigma^{\infty j}_{xz} = \tau_j$ $(j = -M, \dots, L)$, are given for each layer. The external load is described also by the concetrated forces Q^j and screw dislocations with the Burgers vectors b^j inside the layers S_j at the points $z_{*j} = x_{*j} + \mathrm{i}y_{*j}$.

3. Jump function method

The concept of the method is based on the following two main postulates:

- Conjugation principle of different dimension continua
- Interaction conditions between thin inclusion and environment relate the stresses and displacements at the opposite matrix surfaces by the two functional relations (for every inclusion)

$$\Psi_i^j\left(\sigma_{n_j z}^{j\pm}, \frac{\partial w^{j\pm}}{\partial s_j}\right) = 0 \qquad s_j \in L_j' \qquad i = 1,2 \tag{3.1}$$

which depend on the type of inclusion material (liquid, elastic, elastico-plastic, etc.), its mechanical properties and thickness.

According to the conjugation principle the effect of thin inclusion on the stress-strain state of the body is reduced to construction of a stress jump function and derivative of displacement jump function on L_j'

$$\sigma_{n_j z}^{j-} - \sigma_{n_j z}^{j+} = f_5^j(s_j) \qquad \frac{\partial}{\partial s_j}[w^{j-} - w^{j+}] = f_6^j(s_j) \qquad s_j \in L_j' \tag{3.2}$$

Moreover, $f_5^j(s_j) = f_6^j(s_j) = 0$ if $s_j \notin L_j'$. Generally speaking, the jump functions are the unknown ones.

For a longitudinal shear in the direction of Oz axis for the anisotropic medium the relations of Hooke's law and equilibrium equations are of the form (cf Lechnytski, 1977)

$$\frac{\partial w}{\partial y} = a_{44}\sigma_{yz} + a_{45}\sigma_{xz} \qquad \frac{\partial w}{\partial x} = a_{45}\sigma_{yz} + a_{55}\sigma_{xz} \tag{3.3}$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} = 0 \tag{3.4}$$

The difference between the standard notation z for the complex variable and the coordinate is evident. If we introduce the stress function F, according to the definition

$$\sigma_{xz} = \frac{\partial F}{\partial y} \qquad \sigma_{yz} = -\frac{\partial F}{\partial x} \tag{3.5}$$

then Eq (3.4) is satisfied identically and Eq (3.3) gives the differential equation

$$a_{55}\frac{\partial^2 F}{\partial y^2} - 2a_{45}\frac{\partial^2 F}{\partial x \partial y} + a_{44}\frac{\partial^2 F}{\partial x^2} = 0 \tag{3.6}$$

The stress function F^j in the layer S_j can be the presented in the form of superposition of the homogeneous solution F^{0j}, induced by the external loading without inclusions, and the disturbed solution $\widehat{F}^j$. In turn, $\widehat{F}^j$ is the sum of the principal disturbed solutions $\widehat{F}^{0j}$ for an infinite plane with the same mechanical properties and inclusions as for S_j (if there are no inclusions in S_j, then $\widehat{F}^{0j} = 0$) and the disturbed corrective solution $\widehat{F}^{1j}$ which should involve the effect of neighbouring layers and should not initiate the stress and displacement jumps

$$F^j(z) = F^{0j}(z) + \widehat{F}^j(z) = F^{0j}(z) + \widehat{F}^{0j}(z) + \widehat{F}^{1j}(z) \qquad z \in S_j \tag{3.7}$$

The relation between the stresses at infinity is obtained from the condition for rotation absence at infinity of the interfaces

$$\begin{aligned} &\frac{\partial}{\partial x}[w^j - w^{j-1}]\Big|_{|x|\to\infty} = \frac{\partial}{\partial x}[w^{0j} - w^{0,j-1}]\Big|_{|x|\to\infty} = \\ &= \Big(a_{45}^j\sigma_{yz}^{0j} + a_{55}^j\sigma_{xz}^{0j} - a_{45}^{j-1}\sigma_{yz}^{0,j-1} - a_{55}^{j-1}\sigma_{xz}^{0,j-1}\Big)\Big|_{|x|\to\infty} = \\ &= a_{45}^j\tau + a_{55}^j\tau_j - a_{45}^{j-1}\tau - a_{55}^{j-1}\tau_{j-1} = 0 \end{aligned} \tag{3.8}$$

The homogeneous solution corresponds to the external loading inside each layer and satisfies the boundary conditions (2.3) and the ideal mechanical contact conditions (2.2). This solution does not induce the stress and displacement jumps. Therefore, the disturbed solution should satisfy the zero boundary conditions (one of the two equations on each boundary)

$$\begin{aligned} &\widehat{\sigma}_{yz}^L \equiv \widehat{\sigma}_{yz}^{0L} + \widehat{\sigma}_{yz}^{1L} = 0 \\ &\frac{\partial \widehat{w}^L}{\partial x} \equiv \frac{\partial \widehat{w}^{0L}}{\partial x} + \frac{\partial \widehat{w}^{1L}}{\partial x} = 0 \qquad y = d_{L+1} \end{aligned} \tag{3.9}$$

$$\begin{aligned} &\widehat{\sigma}_{yz}^{-M} \equiv \widehat{\sigma}_{yz}^{0,-M} + \widehat{\sigma}_{yz}^{1,-M} = 0 \\ &\frac{\partial \widehat{w}^{-M}}{\partial x} \equiv \frac{\partial \widehat{w}^{0,-M}}{\partial x} + \frac{\partial \widehat{w}^{1,-M}}{\partial x} = 0 \qquad y = d_{-M} \end{aligned} \tag{3.10}$$

the ideal mechanical contact conditions on the material interfaces

$$\begin{aligned} &\widehat{\sigma}_{yz}^j \equiv \widehat{\sigma}_{yz}^{0j} + \widehat{\sigma}_{yz}^{1j} = \widehat{\sigma}_{yz}^{j-1} \equiv \widehat{\sigma}_{yz}^{0,j-1} + \widehat{\sigma}_{yz}^{1,j-1} \\ &\frac{\partial \widehat{w}^j}{\partial x} \equiv \frac{\partial \widehat{w}^{0j}}{\partial x} + \frac{\partial \widehat{w}^{1j}}{\partial x} = \frac{\partial \widehat{w}^{j-1}}{\partial x} \equiv \frac{\partial \widehat{w}^{0,j-1}}{\partial x} + \frac{\partial \widehat{w}^{1,j-1}}{\partial x} \\ &y = d_j \qquad -\infty < x < +\infty \qquad j = -M+1,\dots,L \end{aligned} \tag{3.11}$$

and should induce the stress jump and jump of the displacement derivative on L'_j

$$\widehat{\sigma}^{j-}_{n_j z} - \widehat{\sigma}^{j+}_{n_j z} = f^j_5(s_j) \qquad \frac{\partial}{\partial s_j}[\widehat{w}^{j-} - \widehat{w}^{j+}] = f^j_6(s_j) \qquad s_j \in L'_j \tag{3.12}$$

Consider the interphase inclusion on the interface of two half-planes S_j and S_{j-1}. For this end we settle a definite value of j and consider the case, when the heights of the corresponding layers S_j, S_{j-1} are unlimited ($H_j, H_{j-1} \to \infty$, $L = j$, $M = 1 - j$). The thin inclusion is characterized by the jumps (3.2), where the index $(\cdot)^+$ concerns S_j and $(\cdot)^-$ concerns S_{j-1}

$$\sigma^-_{y_j z} - \sigma^+_{y_j z} = f^j_5(x_j) \qquad \frac{\partial w^-}{\partial x_j} - \frac{\partial w^+}{\partial x_j} = f^j_6(x_j) \tag{3.13}$$

The solution of Eq (3.6) in the Fourier integral transform space is of the form

$$F^{kF}(\xi, y_j) = A^k_1(\xi)\mathrm{e}^{\lambda^k_1 y_j} + A^k_2(\xi)\mathrm{e}^{\lambda^k_2 y_j} \qquad k = j-1, j \tag{3.14}$$

where

$$\lambda^k_1 = \alpha^k|\xi| - \mathrm{i}\beta^k\xi \qquad \lambda^k_2 = -\alpha^k|\xi| - \mathrm{i}\beta^k\xi$$

$$\alpha^k = \frac{1}{a^k_{55}}\sqrt{a^k_{44}a^k_{55} - (a^k_{45})^2} \qquad \beta^k = \frac{a^k_{45}}{a^k_{55}}$$

and $A^k_q(\xi)$ $(q = 1, 2)$ are arbitrary functions.

Since the stress function (3.14) has to be limited as $y_j \to \pm\infty$, therefore

$$F^F(\xi, y_j) = \begin{cases} A^j_2(\xi)\mathrm{e}^{\lambda^j_2 y_j} & \text{for} \quad y_j > 0 \\ A^{j-1}_1(\xi)\mathrm{e}^{\lambda^{j-1}_1 y_j} & \text{for} \quad y_j < 0 \end{cases} \tag{3.15}$$

In the transform space the following relation is valid

$$\sigma^F_{x_j z} = \frac{\partial F^F(\xi, y_j)}{\partial y_j} \qquad \sigma^F_{y_j z} = \mathrm{i}\xi F^F(\xi, y_j) \tag{3.16}$$

and Eqs (3.13) are reduced to the form

$$\begin{aligned} &\sigma^{F-}_{y_j z} - \sigma^{F+}_{y_j z} = f^{jF}_5(\xi) \\ &a^{j-1}_{45}\sigma^{F-}_{y_j z} + a^{j-1}_{55}\sigma^{F-}_{x_j z} - a^j_{45}\sigma^{F+}_{y_j z} - a^j_{55}\sigma^{F+}_{x_j z} = f^{jF}_6(\xi) \end{aligned} \tag{3.17}$$

Having substituted Eqs (3.14),(3.15) into Eq (3.17), we obtain a system of linear algebraic equations (SLAE) for $A_q^k(\xi)$, the solution of which is

$$A_1^{j-1}(\xi) = \frac{\gamma^j \text{sign}\xi f_6^{jF}(\xi)}{\xi} + \frac{\gamma^j a_{55}^j \alpha^j f_5^{jF}(\xi)}{\mathrm{i}\xi}$$
$$A_2^j(\xi) = \frac{\gamma^j \text{sign}\xi f_6^{jF}(\xi)}{\xi} - \frac{\gamma^j a_{55}^{j-1} \alpha^{j-1} f_5^{jF}(\xi)}{\mathrm{i}\xi} \tag{3.18}$$

where $\gamma^j = (a_{55}^{j-1}\alpha^{j-1} + a_{55}^j\alpha^j)^{-1}$.

On the basis of Eqs (3.15), (3.16), (3.18), the stress state for a piecewise-homogeneous anisotropic plane with the interphasic inclusion is obtained

$$\begin{aligned}\widehat{\sigma}_{y_j z}^{0j} + \mathrm{i}\widehat{\sigma}_{x_j z}^{0j} &= \frac{\gamma^j a_{55}^{j-1}\alpha^{j-1}}{2}\Big[-\overline{g}_m^j t_5^j(z_2^j) + g_p^j t_5^j(z_1^j)\Big] + \\ &+ \frac{\mathrm{i}\gamma^j}{2}\Big[\overline{g}_m^j t_6^j(z_2^j) + g_p^j t_6^j(z_1^j)\Big] \qquad z \in S_j \\ \widehat{\sigma}_{y_j z}^{0j} + \mathrm{i}\widehat{\sigma}_{x_j z}^{0j} &= \frac{\gamma^j a_{55}^{j}\alpha^{j}}{2}\Big[-\overline{g}_m^{j-1} t_5^j(z_2^{j-1}) + g_p^{j-1} t_5^j(z_1^{j-1})\Big] + \\ &+ \frac{\mathrm{i}\gamma^j}{2}\Big[\overline{g}_m^{j-1} t_6^j(z_2^{j-1}) + g_p^{j-1} t_6^j(z_1^{j-1})\Big] \qquad z \in S_{j-1}\end{aligned} \tag{3.19}$$

where

$$t_r^j(z) = \frac{1}{\pi}\int\limits_{L_j'} \frac{f_r^j(t)\,dt}{t-z} \qquad r = 5,6$$

$$g_p^k = \beta^k + \mathrm{i}(\alpha^k + 1) \qquad g_m^k = \beta^k + \mathrm{i}(\alpha^k - 1)$$

$$z_1^k = x_j + \beta^k y_j + \mathrm{i}y_j\alpha^k \qquad z_2^k = \overline{z}_1^k \qquad k = j-1, j$$

If we assume that materials of the half-plane are identical ($a_{km}^j = a_{km}^{j-1}$), then from Eq (3.19) we obtain the stress state of a homogeneous plane

$$\widehat{\sigma}_{y_j z}^{0j} + \mathrm{i}\widehat{\sigma}_{x_j z}^{0j} = 14\Big[-\overline{g}_m^j t_5^j(z_2^j) + g_p^j t_5^j(z_1^j)\Big] + \frac{\mathrm{i}}{4a_{55}^j\alpha^j}\Big[\overline{g}_m^j t_6^j(z_2^j) + g_p^j t_6^j(z_1^j)\Big] \tag{3.20}$$

For the inclusion inside a homogeneous plate, that is turned through an angle α_j relative to the axis O_jx_j, in the coordinate system $s_jO_jn_j$ Eq (3.20) for the stresses will remain, but one needs only to replace x_j and y_j with s_j and n_j, and the constants a_{km}^j, α^j, β^j, g_m^j, g_p^j with $a_{km}^{\prime j}$,

$\alpha'^j = \sqrt{a'^j_{44}a'^j_{55} - (a'^j_{45})^2}/a'^j_{55}$, $\beta'^j = a'^j_{45}/a'^j_{55}$, $g'^j_m = \beta'^j + \mathrm{i}(\alpha'^j - 1)$, $g'^j_p = \beta'^j + \mathrm{i}(\alpha'^j + 1)$, respectively. The constants a'^j_{km} characterize elastic properties of the material in the coordinate system $s_jO_jn_j$ and their relation to a^j_{km} is given by Lechnytski (1977). Using the equation

$$\sigma_{y_jz} + \mathrm{i}\sigma_{x_jz} = \mathrm{e}^{-\mathrm{i}\alpha_j}(\sigma_{n_jz} + \mathrm{i}\sigma_{s_jz}) \tag{3.21}$$

the formulae for stresses in the homogeneous half-plane with the inclusion, which is turned through an angle α_j, is of the form

$$\widehat{\sigma}^{0j}_{y_jz} + \mathrm{i}\widehat{\sigma}^{0j}_{x_jz} = \frac{\mathrm{e}^{-\mathrm{i}\alpha_j}}{4}\left[-\overline{g}'^j_m t^j_5(z^j_2) + g'^j_p t^j_5(z^j_1)\right] + \frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}\alpha_j}}{4a'^j_{55}\alpha'^j}\left[\overline{g}'^j_m t^j_6(z^j_2) + g'^j_p t^j_6(z^j_1)\right] \tag{3.22}$$

Eq (3.22) is the principal disturbed solution for the layer S_j.

The corrective disturbed solution $\widehat{F}^{1j}$ is of the form

$$\begin{aligned}
\widehat{F}^{1j}(x,y) &= \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\left[A^j_1(\xi)\mathrm{e}^{\lambda^j_1 y} + A^j_2(\xi)\mathrm{e}^{\lambda^j_2 y}\right]\mathrm{e}^{-\mathrm{i}\xi x}\,d\xi \\
\widehat{\sigma}^{1j}_{xz}(x,y) &= \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\left[A^j_1(\xi)\lambda^j_1\mathrm{e}^{\lambda^j_1 y} + A^j_2(\xi)\lambda^j_2\mathrm{e}^{\lambda^j_2 y}\right]\mathrm{e}^{-\mathrm{i}\xi x}\,d\xi \\
\widehat{\sigma}^{1j}_{yz}(x,y) &= \frac{\mathrm{i}}{2\pi}\int\limits_{-\infty}^{+\infty}\left[A^j_1(\xi)\mathrm{e}^{\lambda^j_1 y} + A^j_2(\xi)\mathrm{e}^{\lambda^j_2 y}\right]\xi\mathrm{e}^{-\mathrm{i}\xi x}\,d\xi
\end{aligned} \tag{3.23}$$

$A^j_q(\xi)$ are the unknown functions. Moreover, if the height H_j of some layer is infinite, then one of these functions is zero: $A^L_1(\xi) = 0$ for S_L; $A^{-M}_2(\xi) = 0$ for S_{-M}.

Having substituted Eqs (3.22), (3.23) and (2.1) into the conditions (3.9) ÷ (3.11), we obtain the SLAE for $A^j_q(\xi)$

$$\sum_{j=-M}^{L}\Big(\sum_{q=1}^{2} c^q_{kj}(\xi)A^j_q(\xi) - \sum_{r=5}^{6}\int\limits_{L'_p} d^r_{kj}(\xi,t)f^j_r(t)\,dt\Big] = 0 \qquad k = -M,\dots,L) \tag{3.24}$$

the solution of which is

$$A^j_q(\xi) = \sum_{p=-M}^{L}\sum_{r=5}^{6}\int\limits_{L'_p} g^{jr}_{qp}(\xi,t)f^p_r(t)\,dt \qquad q = 1,2 \quad j = -M,\dots,L \tag{3.25}$$

The functions $c^q_{kj}(\xi)$, $d^r_{kj}(\xi,t)$, $g^{jr}_{qp}(\xi,t)$ depend on the elastic constants and the package geometry. The values obtained of $A^j_q(\xi)$ (Eq (3.25)) are substituted into Eq (3.23) and the corrective stresses are determined in the form

$$\widehat{\sigma}^{1j}_{yz}(z) + \mathrm{i}\widehat{\sigma}^{1j}_{xz}(z) = \sum_{p=-M}^{L}\sum_{r=5}^{6}\int_{L'_p} R^{jr}_p(z,t) f^p_r(t)\, dt$$

$$R^{jr}_p(z,t) = \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{+\infty}\Big[g^{jr}_{1p}(\xi,t)(\xi+\lambda^j_1)\mathrm{e}^{\lambda^j_1 y} + g^{jr}_{2p}(\xi,t)(\xi+\lambda^j_2)\mathrm{e}^{\lambda^j_2 y}\Big]\mathrm{e}^{-\mathrm{i}\xi x}\, d\xi \tag{3.26}$$

Eqs (3.7), (3.22) and (3.26) give the complete solution for the layer S_j. Using Eqs (2.1) and (3.21), it can be rewritten in the local coordinate system $s_jO_jn_j$ and according to the Sochotski-Plemely formulae we can obtain the solution to the boundary value problem on inclusion middle surface

$$\sigma^{j\pm}_{n_jz}(s_j) + \mathrm{i}\sigma^{j\pm}_{s_jz}(s_j) = \sigma^{0j\pm}_{n_jz} + \mathrm{i}\sigma^{0j\pm}_{s_jz} + 14\Big[2\mathrm{i}\alpha'^j t^j_5(s_j) \pm 2(\mathrm{i}\beta'^j - 1)f^j_5(s_j)\Big] +$$

$$-\frac{1}{4a'^j_{55}\alpha'^j}\Big[2(1-\mathrm{i}\beta'^j)t^j_6(s_j) \pm 2\mathrm{i}\alpha'^j f^j_6(s_j)\Big] +$$

$$+\mathrm{e}^{\mathrm{i}\alpha_j}\sum_{p=-M}^{L}\sum_{r=5}^{6}\int_{L'_p} R^{jr}_p(s_j\mathrm{e}^{\mathrm{i}\alpha_j} + z_{0j}, t) f^p_r(t)\, dt$$

$$\frac{\partial w^{j\pm}(s_j)}{\partial s_j} = a'^j_{45}\sigma^{j\pm}_{n_jz}(s_j) + a'^j_{55}\sigma^{j\pm}_{s_jz}(s_j) \tag{3.27}$$

The substitution of Eqs (3.27) into the interaction conditions (3.1) yields a system of singular integral equations (SSIE) for the jump functions $f^j_r(t)$ ($r=5,6$; $j=-M,\dots,L$). By solving it, we determine finally the stress and displacement field at the arbitrary point of the layers package.

Near the inclusion ends in the homogeneous anisotropic material S_1 the stresses will be of a root singularity (cf Muskhelishvily, 1962) and are determined by the formulae (cf Sulym, 1987)

$$\begin{bmatrix} \sigma_{yz} - \sigma^0_{yz} \\ \sigma_{xz} - \sigma^0_{xz} \end{bmatrix} = \frac{K_{31}}{\sqrt{2\pi r}}\begin{bmatrix} 1 \\ -s^1 \end{bmatrix}\frac{1}{\sqrt{\omega_1}} - \frac{K_{32}}{\sqrt{2\pi r}}\begin{bmatrix} 1 \\ -s^1 \end{bmatrix}\frac{1}{\sqrt{\omega_1}} + O(1) \tag{3.28}$$

where

$$\omega_1 = \cos\theta + s^1\sin\theta \qquad s^1 = s^1_1 + \mathrm{i}s^1_2 \qquad s^1_1 = \frac{a^1_{45}}{a^1_{55}}$$

$$s^1_2 = \frac{|r^1|}{a^1_{55}} \qquad r^1 = \sqrt{(a^1_{45})^2 - a^1_{44}a^1_{55}}$$

r and θ are the polar coordinates, K_{31} and K_{32} are the generalized stress intensity factors determined by the formulae $(r = 5, 6)$

$$K_{32} - \mathrm{i}K_{31} = \mp\sqrt{\pi 2}\Big(p_5^{\pm} + \mathrm{i}\frac{p_6^{\pm}}{|r^1|}\Big) \qquad p_r^{\pm} = \lim_{x\to\pm a}\Big(\sqrt{|x \mp a|}f_r(x)\Big) \quad (3.29)$$

Give the jump functions in the form

$$\begin{array}{lll} f_r^j(t) = f_r^j\delta(t) & f_5^j = Q^j & f_6^j = b^j \\ z_{0j} = z_{*j} & r = 5, 6 & j = -M, \ldots, L \end{array} \qquad (3.30)$$

where $\delta(t)$ is the Dirac delta (cf Gelphand and Shilov, 1959). The substitution of Eq (3.30) into the disturbed solution yields the homogeneous solution for the resultant of the concentrated forces Q^j and screw dislocations with the Burgers vectors b^j located at the points z_{*j} of the layers S_j. Therefore Eqs (3.22) and (3.26), taking into account Eq (3.30), give the principal homogeneous and corrective homogeneous solutions, respectively

$$\sigma_{yz}^{00j} + \mathrm{i}\sigma_{xz}^{00j} = \frac{Q^j}{4\pi}\Big(\frac{\overline{g}_m^j}{z_{*2}^j} - \frac{g_p^j}{z_{*1}^j}\Big) - \frac{\mathrm{i}b^j}{4\pi a_{55}^j\alpha^j}\Big(\frac{\overline{g}_m^j}{z_{*2}^j} + \frac{g_p^j}{z_{*1}^j}\Big)$$

$$z_{*1}^j = (x - x_{*j}) + \beta^j(y - y_{*j}) + \mathrm{i}\alpha^j(y - y_{*j}) \qquad z_{*2}^j = \overline{z}_{*1}^j \quad (3.31)$$

$$\sigma_{yz}^{01j} + \mathrm{i}\sigma_{xz}^{01j} = \sum_{p=-M}^{L}\sum_{r=5}^{6} R_p^{jr}(z, 0)f_r^p$$

When the inclusion lies on the interface of two layers S_j and S_{j-1}, then the procedure of solution to the problem is solved. The only singularity is that the principal disturbed solution for such an inclusion is defined by Eq (3.19) for both layers S_j and S_{j-1}. The inclusions located inside the layers mentioned above contribute independently to the complete solution according to the general rule.

4. Example

Consider a piecewise-homogeneous anisotropic plane that consists of two half-planes S_j $(j = -1, 0;\ M = 1;\ L = 0)$. We assume that inside S_0 along the segment L_0' parallel to the interface $y = 0$ of the half-planes there is a thin elastic inclusion; at the point z_{*0} the concentrated force Q^0 and screw dislocatoin with the Burgers vector b^0 act. The coordinates of center of the inclusion L_0' in the main coordinate system xOy are $z_{00} = \mathrm{i}H$. The inclusion has the lenght $2a$ and thickness $2h$.

The interaction conditions (3.1) for an elastic anisotropic inclusion are (cf Sulym, 1981, 1987)

$$\boldsymbol{\sigma}^{j+}(x_j) + \boldsymbol{\sigma}^{j-}(x_j) = \Big(2\boldsymbol{N}^j + \frac{1}{h}\int\limits_{-a}^{x_j} \boldsymbol{f}^j(t)\;dt\Big)\mathsf{L} \qquad x_j \in [-a, a] \quad j = 0 \tag{4.1}$$

where $\boldsymbol{f}^j(t) = [f_5^j(t), f_6^j(t)]$ and $\boldsymbol{N}^j = [N_1^j, N_2^j]$ is the vector of a priori constants

$$N_1^j = \sigma_{x_j z}^{0j}(-a)\frac{a_{44}^0}{\max(a_{44}^0, a_{44}^{in})}$$

$$N_2^j = -\frac{\min(a_{44}^0, a_{44}^{in})}{a_{44}^0}\Big[a_{44}^0\sigma_{y_j z}^{0j}(-a) + a_{45}^0\sigma_{x_j z}^{0j}(-a)\Big]$$

$$\boldsymbol{\sigma}^j(z) = \Big[\sigma_{y_j z}^j(z), \frac{\partial w^j(z)}{\partial x_j}\Big]$$

$$\mathsf{L} = \frac{1}{a_{44}^{in}}\begin{bmatrix} -a_{45}^{in} & |r^{in}|^2 \\ -1 & -a_{45}^{in} \end{bmatrix}$$

The principal disturbed solution for the half-plane S_0 (Eq (3.22) at $j = 0$) along the line $y = 0$ is as follows

$$\widehat{\sigma}_{yz}^{0,0}\Big|_{y=0} = \int\limits_{-\infty}^{+\infty} q(\xi, x)\;d\xi \qquad \frac{\partial \widehat{w}^{0,0}}{\partial x}\Big|_{y=0} = \int\limits_{-\infty}^{+\infty} r(\xi, x)\;d\xi$$

$$q(\xi, x) = \frac{1}{4\pi}\mathrm{e}^{-\mathrm{i}\xi x}\mathrm{e}^{-\lambda_1^0 H}\Big(\int\limits_{L_0'} f_5^0(t)\mathrm{e}^{\mathrm{i}\xi t}\;dt + \frac{\mathrm{isign}\xi}{a_{55}^0\alpha^0}\int\limits_{L_0'} f_6^0(t)\mathrm{e}^{\mathrm{i}\xi t}\;dt\Big) \tag{4.2}$$

$$r(\xi, x) = \frac{1}{4\pi}\mathrm{e}^{-\mathrm{i}\xi x}\mathrm{e}^{-\lambda_1^0 H}\Big(-\mathrm{i}a_{55}^0\alpha^0\mathrm{sign}(\xi)\int\limits_{L_0'} f_5^0(t)\mathrm{e}^{\mathrm{i}\xi t}\;dt + \int\limits_{L_0'} f_6^0(t)\mathrm{e}^{\mathrm{i}\xi t}\;dt\Big)$$

The corrective disturbed solutions in the half-planes S_{-1} and S_0 are obtained from Eq (3.23) at $j = -1, 0$, respectively, considering that $A_1^0(\xi) = A_2^{-1}(\xi) = 0$

$$\widehat{\sigma}_{yz}^{1,0}(x,y) = \frac{\mathrm{i}}{2\pi}\int\limits_{-\infty}^{+\infty} \xi A_2^0(\xi)\mathrm{e}^{\lambda_2^0 y}\mathrm{e}^{-\mathrm{i}\xi x}\,d\xi$$

$$\widehat{\sigma}_{xz}^{1,0}(x,y) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} A_2^0(\xi)\lambda_2^0\mathrm{e}^{\lambda_2^0 y}\mathrm{e}^{-\mathrm{i}\xi x}\,d\xi$$

$$\widehat{\sigma}_{yz}^{1,-1}(x,y) = \frac{\mathrm{i}}{2\pi}\int\limits_{-\infty}^{+\infty} \xi A_1^{-1}(\xi)\mathrm{e}^{\lambda_1^{-1} y}\mathrm{e}^{-\mathrm{i}\xi x}\,d\xi \tag{4.3}$$

$$\widehat{\sigma}_{xz}^{1,-1}(x,y) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} A_1^{-1}(\xi)\lambda_1^{-1}\mathrm{e}^{\lambda_1^{-1} y}\mathrm{e}^{-\mathrm{i}\xi x}\,d\xi$$

The displacement derivatives $\partial\widehat{w}^{1,-1}/\partial x$ and $\partial\widehat{w}^{1,0}/\partial x$ are calculated by Hooke's law (3.3).

If we substitute Eqs (4.2), (4.3) into Eq (3.11) at $L=0$, $M=1$, we obtain the SLAE

$$A_1^{-1}(\xi) - A_2^0(\xi) = \frac{2\pi q(\xi,x)\mathrm{e}^{\mathrm{i}\xi x}}{\mathrm{i}\xi}$$

$$a_{55}^{-1}\alpha^{-1}A_1^{-1}(\xi) - a_{55}^0\alpha^0 A_2^0(\xi) = \frac{2\pi r(\xi,x)\mathrm{e}^{\mathrm{i}\xi x}}{|\xi|} \tag{4.4}$$

the solution of which is

$$A_1^{-1}(\xi) = \frac{2\pi\gamma^0 r(\xi,x)\mathrm{e}^{\mathrm{i}\xi x}}{|\xi|} + \frac{a_{55}^0\alpha^0 2\pi\gamma^0 q(\xi,x)\mathrm{e}^{\mathrm{i}\xi x}}{\mathrm{i}\xi}$$

$$A_2^0(\xi) = \frac{2\pi\gamma^0 r(\xi,x)\mathrm{e}^{\mathrm{i}\xi x}}{|\xi|} - \frac{a_{55}^{-1}\alpha^{-1} 2\pi\gamma^0 q(\xi,x)\mathrm{e}^{\mathrm{i}\xi x}}{\mathrm{i}\xi} \tag{4.5}$$

Then we substitute Eq (3.30) at $j=0$ into Eqs (4.5), (4.3) and obtain the corrective homogeneous solution

$$\sigma_{yz}^{010} + \mathrm{i}\sigma_{xz}^{010} = \frac{Q^0\delta}{4\pi}\Big(\frac{g_p^0}{z_3^0} - \frac{\overline{g}_m^0}{\overline{z}_3^0}\Big) - \frac{\mathrm{i}b^0\delta}{4\pi a_{55}^0\alpha^0}\Big(\frac{g_p^0}{z_3^0} + \frac{\overline{g}_m^0}{\overline{z}_3^0}\Big)$$

$$z_3^j = (x - x_{*j}) + \beta^j(y - y_{*j}) + \mathrm{i}\alpha^j(y + y_{*j}) \tag{4.6}$$

$$\delta = \gamma^0(a_{55}^0\alpha^0 - a_{55}^{-1}\alpha^{-1})$$

for the force Q^0 and dislocation b^0 at the point z_{*0}. The principal homogeneous solution for this load is of the form (3.31) at $j = 0$.

By substituting the complete solution for the upper half-plane S_0 (see Eq (3.7)) in the interaction conditions Eq (4.1), we obtain the SSIE

$$
\begin{aligned}
&\frac{1}{h}\int_{-a}^{x_0} \boldsymbol{f}^0(t)\mathsf{L}\,dt + \boldsymbol{t}^0(x_0)\mathsf{V} + \boldsymbol{t}^0(x_0 - 2\mathrm{i}\alpha^0 H)\mathsf{D} + \boldsymbol{t}^0(x_0 + 2\mathrm{i}\alpha^0 H)\mathsf{B} = \\
&= 2\boldsymbol{\sigma}^{0,0}(x_0,0) - 2\boldsymbol{N}^0\mathsf{L} \qquad x_0 \in [-a,a]
\end{aligned}
\tag{4.7}
$$

where

$$
\boldsymbol{t}^j(z) = \left[t_5^j(z), t_6^j(z)\right] \qquad \mathsf{V} = \begin{bmatrix} 0 & -a_{55}^0\alpha^0 \\ \dfrac{1}{a_{55}^0\alpha^0} & 0 \end{bmatrix}
$$

$$
\mathsf{D} = \begin{bmatrix} -\dfrac{\mathrm{i}\delta}{2} & \dfrac{a_{55}^0\alpha^0\delta}{2} \\ \dfrac{\delta}{2a_{55}^0\alpha^0} & \dfrac{\mathrm{i}\delta}{2} \end{bmatrix} \qquad \mathsf{B} = \begin{bmatrix} \dfrac{\mathrm{i}\delta}{2} & \dfrac{a_{55}^0\alpha^0\delta}{2} \\ \dfrac{\delta}{2a_{55}^0\alpha^0} & -\dfrac{\mathrm{i}\delta}{2} \end{bmatrix}
$$

Requiring that the displacements be unique (when going around the closed contour of inclusion) we obtain the complementary condition

$$
\int_{L_0'} \boldsymbol{f}^0(t)\,dt = 0 \tag{4.8}
$$

The SSIE is solved using the collocation method (cf Bozhydarnyk and Sulym, 1990) with the accuracy of 1%. In calculations we assume that a unidirectional fiber-glass plastic (cf Ashkenazy and Ganov, 1980) is the material of the half-plane S_0 $\{a_{44}^0 = 1/G_{yz}; a_{55}^0 = 1/G_{xz}; a_{45}^0\} = \{0,2; 0,174; 0\} \cdot 10^{-9}\,1/\mathrm{Pa}$; the material of the half-plane S_{-1} is either the normall fiber glass plastic $\{a_{44}^{-1}; a_{55}^{-1}; a_{45}^{-1}\} = \{0,271; 0,273; 0\} \cdot 10^{-9}\,1/\mathrm{Pa}$ (the solid line in Fig.2 ÷ Fig.5) or the absolutely flexible material (the dotted line in Fig.2 and Fig.3), or the absolutely rigid material (the dashed line in Fig.2 and Fig.3). The inclusion is isotropic $a_{55}^{in}/a_{44}^{in} = 1$, $a_{45}^{in} = 0$. The absolute flexibility S_{-1} is equivalent to the problem for the half-space S_0 with a free boundary and the absolute rigidity is equivalent to the one for the half-space with a fixed boundary.

Fig.2 presents the dimensionless GSIF $K_{3j}^0 = K_{3j}\sqrt{a}/(Q^0\sqrt{\pi})$ versus the parameter of the inclusion rigidity $G = \log(a_{44}^{in}/a_{44}^0)$. The inclusion is located on a relative depression $d \equiv H/a = 1$ from the interface $y = 0$. The concentrated force Q^0 acts at the point $(0, 2a)$. Fig.3 presents the analogous

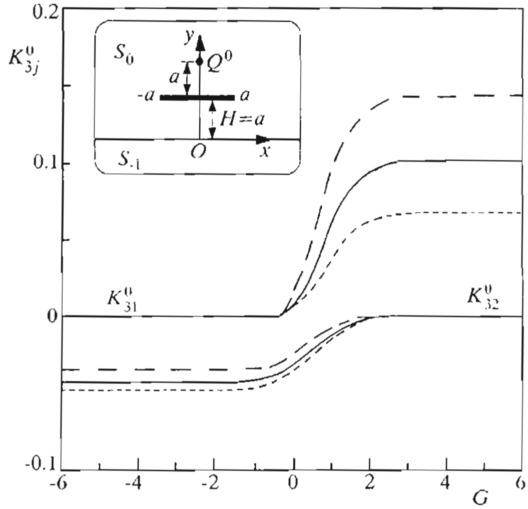
0.2
0.1
0
-0.1
K_{3j}^0
S_0
y
Q^0
a
$-a$
a
$H=a$
S_{-1}
O
x
K_{31}^0
K_{32}^0
-6
-4
-2
0
2
4
6
G

Fig. 2.

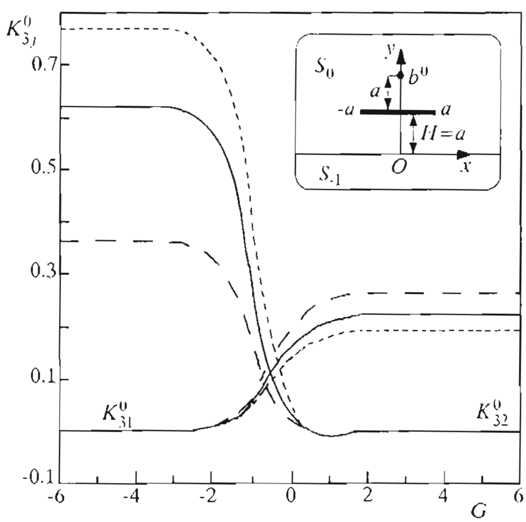
K_{3j}^0
0.7
0.5
0.3
0.1
-0.1
S_0
y
b^0
a
$-a$
a
$H=a$
S_{-1}
O
x
K_{31}^0
K_{32}^0
-6
-4
-2
0
2
4
6
G

Fig. 3.

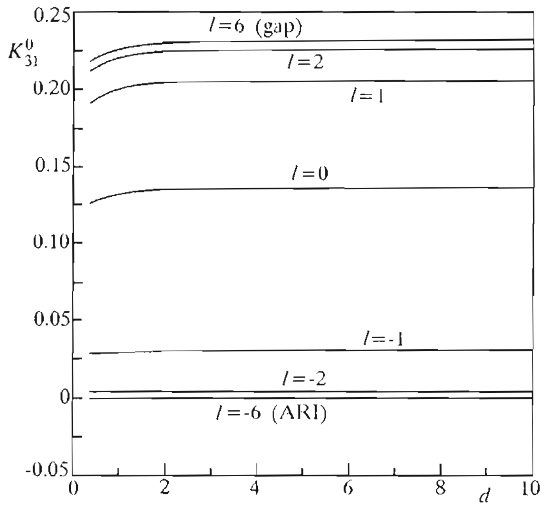
0.25
K_{31}^0
0.20
0.15
0.10
0.05
0
-0.05
0
2
4
6
8
d
10
l=6 (gap)
l=2
l=1
l=0
l=-1
l=-2
l=-6 (ARI)

Fig. 4.

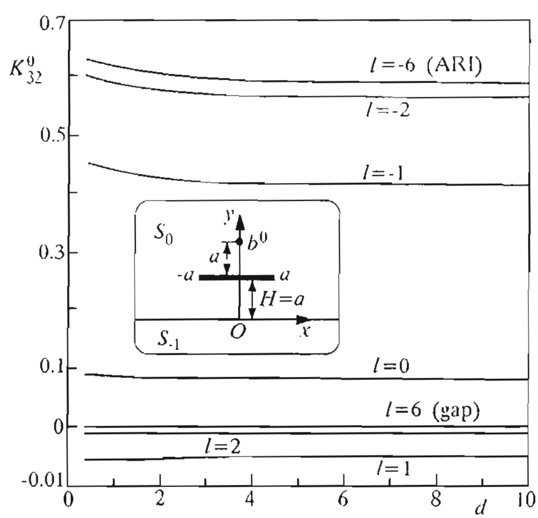
0.7
K_{32}^0
0.5
0.3
0.1
0
-0.01
0
2
4
6
8
d
10
l=-6 (ARI)
l=-2
l=-1
l=0
l=6 (gap)
l=2
l=1
S_0
y
b^0
a
-a
a
H=a
S_{-1}
O
x

Fig. 5.

relation when at the point $(0, 2a)$ there is the dislocation b^0. For such loading $K^0_{3j} = K_{3j}a^0_{55}\sqrt{a}/(0.174b^0\sqrt{\pi})$. When $G > 3$ or $G < -3$, the numerical solutions with the accuracy of 1% yield the boundary values, that are peculiar to the solutions for the gap and ARI, respectively. Whatever the elastic properties of the half-plane S_{-1} are, the corresponding solid line of graphical dependence GSIF will always be located between the dotted and dashed lines that correspond to the free boundary and its rigid fixening.

Fig.4 and Fig.5 present the dependence of $K^0_{3j} = K_{3j}a^0_{55}\sqrt{a}/(0.174b^0\sqrt{\pi})$ $(j = 1, 2$; respectively) the relative inclusion depression d when the dislocation b^0 is located at the point $z_{*0} = (0, H + a)$. Each line in the figures refers to different flexibility of inclusion $a^{in}_{44} = a^{0in}_{44} \cdot 10^l$ $(a^{0in}_{44} = 0.1 \cdot 10^{-9}\,1/\text{Pa})$. As $d \to \infty$ we arrive at the solution for the elastic inclusion an the homogeneous plane that is obtaind with the accuracy of 1% if we assume $d = 10$.

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Antypłaskie zagadnienie dla anizotropowego wielowarstwowego ośrodka z cienkimi sprężystymi inkluzjami poddanego działaniu skupionych sił i dyslokacji śrubowych

Streszczenie

W pracy rozwiązano metodą funkcji skoków antypłaskie zagadnienie teorii sprężystości dla pliku anizotropowych warstw, w których są cienkie laminarne inkluzje. Zadane są naprężenia lub przemieszczenia na granice ośrodka, działanie skupionych sił i dyslokacji śrubowych. Inkluzje są modelowane przez skoki wektorów naprężeń i przemieszczeń na powierzchniach środkowych. Przez zastosowanie wykładniczej transformacji całkowej Fouriera, otrzymujemy zależność współrzędnych tensora naprężeń i pochodnych wektora przemieszczeń od obciążenia zewnętrznego i poszukiwanych funkcji skoków. Z uwzględnieniem warunków oddziaływania cienkiej inkluzji ze środowiskiem anizotropowym zagadnienie jest sprowadzone do układu równań całkowych osobliwych typu Cauchy. W ogólnym przypadku ten układ jest rozwiązywany metodą kolokacji. Metodę ilustrują obliczenia dla konkretnego zagadnienia.

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