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COMPUTATIONAL STUDY OF THE WAKE CONTROL PROBLEM[1]

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In the paper we are concerned with the problem of flow control. We consider two-dimensional (2D) turbulent wake flows past rotating obstacles where the control objective is to minimize the drag force. Results of numerical simulations are presented which indicate that substantial drag reduction can be obtained using an open-loop algorithm. This finding is compared with available experimental data. In the second part of the paper we derive a rigorous feedback method for optimal flow control.

Key words: flow control, drag reduction, vortex method

1. Introduction

The problem of flow control is one of the greatest challenges of Fluid Dynamics. Its importance cannot be overestimated, both as regards the perspective of Theoretical Physics and Applied Engineering. The problem becomes even more complicated when turbulent flow regimes are taken into account. This is in fact what happens in most situations of engineering interest. The flow of viscous incompressible fluid is described by the system of the Navier-Stokes equations. At the moment fairly little is known about the qualitative and quantitative properties of the solutions of this system (cf Doering and Gibbon, 1995), particularly as regards long time evolution. Rigorous prediction of the fluid flow is not possible at the moment which means that one cannot foresee the influence that any arbitrary disturbance may have on the flow. Consequently, all attempts at flow control have to be based on *ad hoc* assumptions. Similar problems are encountered in laboratory experiments, as turbulent flows

[1]The paper won one of the two first prizes awarded at the contest for the best work in the field of fluid mechanics organized by the local branch of PTMTS in Częstochowa in 1998

involve time- and length-scales still beyond reach of modern laboratory equipment. Another issue is repeatability which can hardly be attained in laboratory conditions.

At the same time techniques of flow control attract ever expanding interest of the Engineering Community all over the world. The fields of potential applications range from chemical and process industry to off-shore and aerospace technologies. In the context of the latter, one of the central issues is the separation control which is closely related to drag-to-lift ratio of a lifting surface.

Control techniques vary both according to the control objective and the strategy applied. As regards the former, one may wish to maximize/minimize any of the components of the hydrodynamic force, turbulence level (*the flow relaminarization problem*), heat exchange rate, mixing, etc. According to whether external energy is added to the flow or not, one may distinguish active and passive flow control techniques. Active strategies usually involve continuous displacement of the flow boundaries (e.g. moving boundaries, systems of flow actuators, blowing and suction, etc.) (cf Bushnell and McGinley, 1989; Gad-el-Hak and Bushnell, 1991) and/or interaction with a body force (e.g. magnetic force acting on a ferro-fluid). Passive techniques rely on stationary modifications of the original geometry (e.g. riblets, flaps, vortex generators, etc.). Flow control based on introducing some flow additives (polymers) may be regarded as a separate technique and will not be discussed here. Active methods can be divided into open- and closed-loop techniques, depending on how the control rule is generated. In the case of open-loop algorithms the control rule is established *a priori* and makes no reference to the flow evolution. Conversely, closed-loop strategies determine the control using instantaneous flow field information, as well as flow history. They are also called *feedback* algorithms.

In the present paper we are concerned with the problem of drag minimization in the incompressible wake flow generated by a circular cylinder with the Reynolds number corresponding to turbulent flow. The objectives of the work are twofold: first, we show evidence coming from numerical simulations that a properly designed open-loop algorithm is indeed capable of performing effective flow control resulting in significant drag reduction, and then we develop a rigorous feedback (i.e. closed-loop) algorithm based on the Optimal Control Theory. In both cases control is performed by rotary motion of the obstacle.

Throughat the paper the following denotations of vector and tensor operations will be used: $(\boldsymbol{a}\cdot\boldsymbol{b})$ will represent the scalar product of the two vectors (a_ib_i), $(\boldsymbol{ab})$ will stand for the dyadic product, i.e. the tensor (a_ib_j),

$(\boldsymbol{a} \times \boldsymbol{b})$ will denote the cross-product of the two vectors $(\varepsilon_{ijk} a_j b_k)$, whereas $(\mathsf{A} : \mathsf{B})$ will be the scalar product, i.e. contraction, of the two tensors $(A_{ij} B_{ji})$. The symbols $(\boldsymbol{a}\mathsf{B})$ and $(\mathsf{B}\boldsymbol{a})$ will represent the multiplications $(B_{ij} a_i)$ and $(B_{ij} a_j)$, respectively.

2. Hydrodynamic forces in wake flows

In this section we will focus on the phenomena which accompany the origin of the hydrodynamic force. Wake flows, both in laminar and in turbulent regimes, are characterized by the formation of a staggered array of counterrotating vortices which are shed behind the obstacle. This phenomenon is called the von Karman vortex street and is fairly persistent with respect to the variation of the Reynolds number. Below we will focus on the relation between the changes of the vorticity distribution in the wake and changes of the drag force. We will arrive at certain conclusions which will be useful in designing the open-loop algorithm.

Throughout the paper we will consider two-dimensional (2D) flows in the domain Ω extending to infinity. The obstacle is a circular cylinder with the boundary Γ_0. The origin of the coordinate system is located in the center of the obstacle. The hydrodynamic force is equal to

$$\boldsymbol{F} = -\oint_{\partial\Omega} (-p\boldsymbol{n} + \mu\mathsf{D}\boldsymbol{n})\, d\sigma = \oint_{\partial\Omega} (p\boldsymbol{n} + \mu\boldsymbol{n} \times \boldsymbol{\omega})\, d\sigma \qquad (2.1)$$

where

- p – pressure
- $\boldsymbol{n}$ – unit normal vector directed into the body
- μ – fluid viscosity
- D – rate-of-deformation tensor, i.e. the symmetric part of the velocity gradient tensor

$$\mathsf{D} = \frac{1}{2}\left[\nabla\boldsymbol{V} + (\nabla\boldsymbol{V})^{\mathrm{T}}\right] \qquad (2.2)$$

and $\boldsymbol{\omega}$ denotes vorticity (in 2D it becomes a scalar $\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$). Formula (2.1) remains valid even if the contour performs an arbitrary motion.

Equivalently, the hydrodynamic force can be expressed as the rate of change of fluid momentum (Batchelor, 1970)

$$\boldsymbol{F} = -\frac{d}{dt}\int_{\Omega} \boldsymbol{V}\, d\Omega - \oint_{\Gamma_\infty} p\boldsymbol{n}\, d\sigma \qquad (2.3)$$

where Γ_∞ denotes the outer circumference placed at infinity and the fluid density ρ was put equal to unity. The above expression can be transformed into a more tractable form using the *impulse identity* (Saffman, 1993)

$$\int_\Omega \boldsymbol{a}\, d\Omega = \int_\Omega \boldsymbol{r} \times (\nabla \times \boldsymbol{a})\, d\Omega - \oint_{\Gamma_0 \cup \Gamma_\infty} \boldsymbol{r} \times (\boldsymbol{n} \times \boldsymbol{a})\, d\sigma \tag{2.4}$$

where $\boldsymbol{a}$ is a given vector field. It must be observed that the above form of the *impulse identity* is valid in the 2D case only (in the 3D case there is a factor of $1/2$ in front of the area integral on the RHS in (2.4)). Using Eqs (2.3) and (2.4) one obtains

$$\boldsymbol{F} = -\frac{d}{dt}\int_\Omega \boldsymbol{r} \times \boldsymbol{\omega}\, d\Omega \tag{2.5}$$

where the pressure contribution in Eq (2.3) and the boundary integral over Γ_∞ in Eq (2.4) can be shown to cancel each other. The integral over Γ_0 in Eq (2.4) vanishes due to the symmetry properties of the obstacle. The drag force F_D (i.e. the horizontal component of the hydrodynamic force) is thus the time derivative of the integral vorticity moment with respect to the X-axis (i.e. the axis of the flow)

$$F_D = -\frac{d}{dt}\int_\Omega \omega y\, d\Omega \tag{2.6}$$

This has a clear physical interpretation in terms of vorticity structure in the wake. Vortex shedding is a quasi periodic phenomenon. Every half-cycle vorticity from the separated boundary layer accumulates in the recirculation zone of the obstacle and is then shed in the form of a new vortex. In every cycle two counterrotating vortices are born and are subsequently advected downstream with the off-axis separation Δy related to the sign of the vortex (Fig.1). Consequently, the expression for the vorticity moment in Eq (2.6) can be roughly approximated as

$$F_D \simeq -\frac{d}{dt}\sum_i \Gamma_i \Delta y_i = -\frac{d}{dt}\Big[(+\Gamma_1)(-\Delta y_1) + (-\Gamma_2)(+\Delta y_2) + \dots\Big] \tag{2.7}$$

where the Γ_i's represent the circulations of the the consecutive vortices that are shed.

The above shows that in the natural mode of vortex shedding the integral vorticity moment about the X-axis always decreases resulting in the observed drag force. This simple argument gives an idea of what could be done in order to reduce the drag:

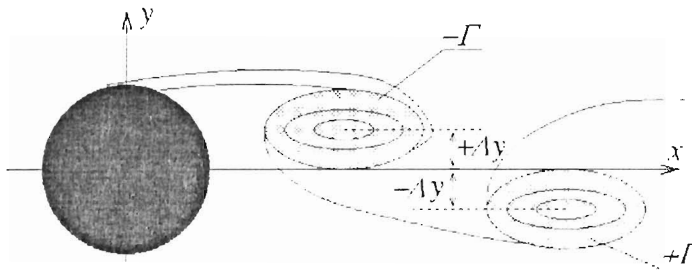

Fig. 1. Sketch illustrating the distinctive features of the vorticity distribution in the von Karman vortex street behind an obstacle. The symbols $\pm\Gamma$ denote the *circulations of the vortices*

- Decrease the off-axis separation Δy_i at which the eddies are advected downstream
- Reduce the strength of the vortices, thus effectively killing the von Karman vortex street.

Tokumaru and Dimotakis (1991) reported on a successful laboratory attempt to implement this strategy in a turbulent wake flow at $\mathrm{Re} = 15000$. The flow control was accomplished by performing rotary oscillations of the circular cylinder. Appropriate choice of the control parameters (i.e. the amplitude and frequency of the rotary oscillations) resulted in dramatic drag reduction, by as much as 80% in some cases. Drag reduction was accompanied by effectively suppressing the vortex shedding. We have reproduced these results using numerical simulation, thus validating the control strategy and at the same time verifying the robustness of the numerical code.

3. Numerical simulation – Random Vortex Blob Method

Below we outline the numerical algorithm that is used in the flow simulations. We use the *Random Vortex Blob Method* which relies on the formal similarity between the 2D vorticity equation and the Planck-Fokker equation describing the evolution of the stochastic Wiener process. Vorticity field is first discretized on a Lagrangian mesh consisting of a family of vorticity carriers (regularized point vortices). Every vorticity particle then evolves according to the stochastic Ito equation. Its displacement consists of the deterministic part (advection) and the random walk which models viscous diffusion. New vorticity is created on solid boundaries in such a way that the *no-slip* condition for the velocity is rigorously enforced. All the numerical parameters (the diameter of the regularized vortex particle, time step, etc.) are chosen to assure the numerical resolution corresponding to the assumed Reynolds number. In the evaluation of the vortex induction the fast summation algorithm is used which significantly reduces the computational cost. Details of the numerical method in the present implementation can be found in Styczek and Wald (1995).

The proposed method appears particularly useful when the flow domain extends to infinity (i.e. the case of open flow systems), as it is capable of properly accounting for the velocity boundary conditions at infinity. Furthermore, in view of the intrinsic relation between the hydrodynamic force and the vorticity distribution, the vorticity form of the momentum equations seems well suited for the study of forces in wake flows.

In fact, recovering the force from the vorticity and velocity fields is not a trivial task. Direct application of the defining formula (2.1) is not possible, because pressure may only be obtained as a solution of a separate problem (Gresho, 1991). Eq (2.5) is physically transparent, but has a number of computational disadvantages. In order to circumvent the difficulties we compute forces using a variational approach. For its description, as well as a more elaborate discussion of the aforementioned problems, the reader is referred to Protas et al. [9].

4. Open-loop algorithm – results of the numerical simulations

In order to verify the effectiveness of the open-loop control strategy we used our numerical simulations to reproduce the laboratory results of Tokumaru and Dimotakis (1991). The angular velocity of the circular cylinder was given by

$$\dot{\varphi}(t) = A_0 \sin(\omega_0 t) \tag{4.1}$$

where the amplitude A_0 and the frequency ω_0 are the two adjustable parameters. They can be normalized in the following way

$$\widetilde{A} = \frac{A_0 R}{U_\infty} \qquad \mathrm{St} = \frac{\omega_0 2R}{U_\infty} \tag{4.2}$$

where R is the radius of the cylinder and U_∞ denotes the free stream velocity at infinity. The parameter St is called the *Strouhal number*.

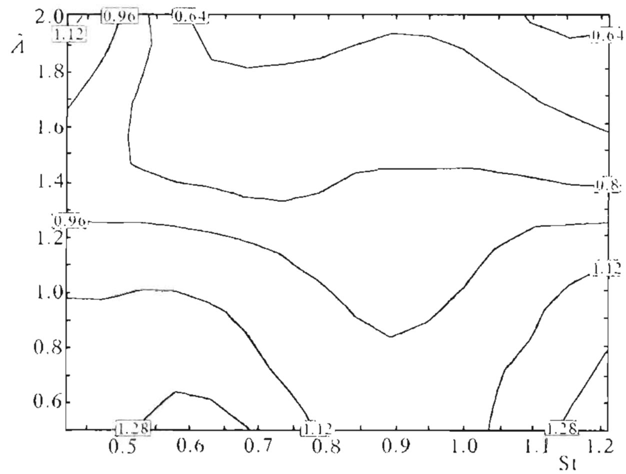

Fig. 2. Contour plot presenting the dependence of the drag coefficient c_D on the control parameters $\widetilde{A}$ and St. The iso-lines of c_D corresponding to the values of 0.64, 0.8, 0.96, 1.12 and 1.28 are indicated

We performed a parametric study of the phenomenon exploring the ranges of $\widetilde{A}$ and St roughly corresponding to those studied by Tokumaru and Dimotakis (1991). Because of prohibitive computational cost, the Reynolds number we reached in our simulations was $\mathrm{Re} = 5000$, somewhat lower than $\mathrm{Re} = 15000$ used by Tokumaru and Dimotakis (1991). In Fig.2 we present the contour plot of the drag coefficient c_D

$$c_D = \frac{F_D}{\frac{1}{2}\rho U_\infty^2 (2R)}$$

as a function of the two control parameters $c_D = f(\widetilde{A}, \mathrm{St})$ that was obtained in our investigation. Remarkable reduction of the drag coefficient can be observed for properly chosen values of the control parameters $\{\widetilde{A}, \mathrm{St}\}$ which are very close to those indicated by Tokumaru and Dimotakis (1991). In the best case we reached 60% drag reduction, slightly less than 80% reported by Tokumaru and Dimotakis (1991). This discrepancy can be attributed to three factors: slightly lower Reynolds number that characterized the numerical simulations (in the range under consideration, the force still depends on the Reynolds number), underresolution of the boundary layer in the numerical simulations, and 2D description of the problem, whereas the real flows are always 3D.

Inspection of Fig.2 leads to the following conclusions. First, there is some specific value of the Strouhal number for which c_D attains minimum. Pushing

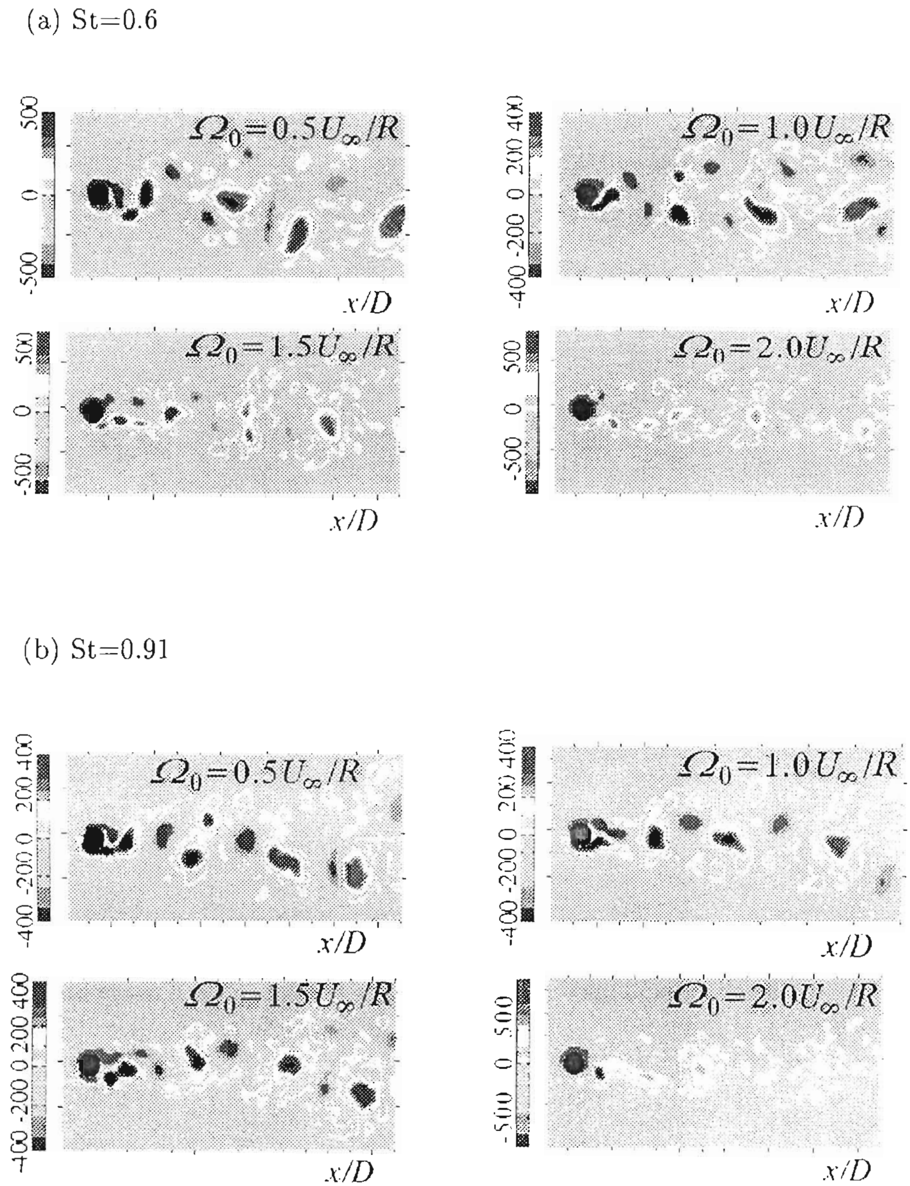

Fig. 3. Instantaneous vorticity distribution in the turbulent wake flow past the rotating circular cylinder at Re = 5000. Values of the normalized control parameters are indicated. Note the gradual suppression of the von Karman vortex street as the rotation frequency increases

the forcing frequency any higher results in the increase of the drag. This value of the Strouhal number is roughly equal to 0.9 and does not indicate any simple relation to its value characterizing the natural mode of vortex shedding which is about 0.2.

In Fig.3 and Fig.4 we show the instantaneous vorticity distributions in the wake which correspond to different choices of the control parameters $\{\tilde{A}, \mathrm{St}\}$. It can be observed that the von Karman vortex street was effectively killed in the cases when the most significant drag reduction was obtained. This substantiates the propositions made in Section 2.

St=1.21

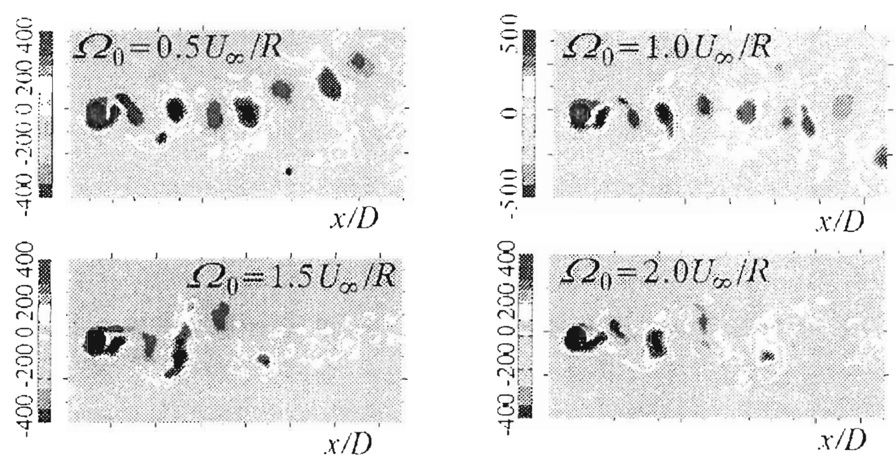

Fig. 4. Instantaneous vorticity distribution in the turbulent wake flow past the rotating circular cylinder at $\mathrm{Re} = 5000$. Values of the normalized control parameters are indicated. Note the gradual suppression of the von Karman vortex street as the rotation frequency increases

5. Towards a rigorous feedback (closed-loop) control algorithm

In the present section we are concerned with the derivation of the gradient algorithm which can be used to determine the optimal control $\dot{\varphi}_{opt}(t)$ in the time interval $[0;T]$, where T is some characteristic time scale of the phenomenon. The control results in the minimum average drag in the interval $[0;T]$. The present method is an extension of the pioneering work (Lions, 1969) and its application to fluid dynamical problems (Abergel and Temam, 1990). Bewley et al. (1997) adopted a similar technique to the problem of drag minimization in the turbulent channel flow.

The starting point is the formulation of the functional that will be minimized. It our case it represents the balance of the work done by the drag force and the work needed to control the flow

$$J(\dot\varphi) = \frac{1}{2}\int_T \left\{ \left[\begin{array}{c} \text{power needed to} \\ \text{control the flow} \end{array} \right] + \left[\begin{array}{c} \text{power related to} \\ \text{the drag force} \end{array} \right] \right\} dt \tag{5.1}$$

The rotational velocity $\dot\varphi(t)$ of the cylinder is the control, therefore the energy needed to control the flow is introduced in the form of work done by a moment of forces applied to the cylinder. Consequently, using the surface force density $\boldsymbol{f}(\dot\varphi)$, the above relation can be expressed as

$$\begin{aligned} J(\dot\varphi) &= \frac{1}{2}\int_T \oint_{\Gamma_0} \left\{ \left[\boldsymbol{f}(\dot\varphi) \times \boldsymbol{r} \right] \cdot (\dot\varphi \boldsymbol{k}) + \boldsymbol{f}(\dot\varphi) \cdot \boldsymbol{U}_\infty \right\} d\sigma dt = \\ &= \frac{1}{2}\int_T \oint_{\Gamma_0} \left\{ \boldsymbol{f}(\dot\varphi) \cdot \left[\boldsymbol{r} \times (\dot\varphi \boldsymbol{k}) + \boldsymbol{U}_\infty \right] \right\} d\sigma dt = \\ &= \frac{1}{2}\int_T \oint_{\Gamma_0} \left\{ \left[-p(\dot\varphi)\boldsymbol{n} + \mu \mathsf{D}(\boldsymbol{u}(\dot\varphi))\boldsymbol{n} \right] \cdot \left[\boldsymbol{r} \times (\dot\varphi \boldsymbol{k}) + \boldsymbol{U}_\infty \right] \right\} d\sigma dt \end{aligned} \tag{5.2}$$

where all the hydrodynamic quantities depend on the control $\dot\varphi$. Next we use the fact proved by Abergel and Temam (1990) that the functional of the type (5.2) is Gâteaux differentiable and that the relation

$$\langle J'(\dot\varphi_{opt}), h \rangle = \frac{DJ(\dot\varphi_{opt})}{D\dot\varphi} \cdot h = 0 \tag{5.3}$$

is a necessary condition characterizing the optimal control $\dot\varphi_{opt}$ and the corresponding optimal state $\{\boldsymbol{u}(\dot\varphi_{opt}); p(\dot\varphi_{opt})\}$. In this expression h denotes the *direction* in which the Gâteaux derivative (i.e. the functional gradient) is taken.

The Gâteaux derivative of the functional (5.2) takes the form

$$
\begin{aligned}
\langle J'(\dot\varphi), h\rangle &= \frac{1}{2}\int_T \oint_{\Gamma_0} \Big\{ \Big\langle \frac{D\boldsymbol{f}(\dot\varphi)}{D\dot\varphi}, h\Big\rangle \cdot \Big[\boldsymbol{r} \times (\dot\varphi \boldsymbol{k}) + \boldsymbol{U}_\infty\Big] + \\
&+ \Big[\boldsymbol{f}(\dot\varphi) \times \boldsymbol{r}\Big] \cdot \Big\langle \frac{D\dot\varphi}{D\dot\varphi}, h\Big\rangle \Big\} \, d\sigma dt = \\
&= \frac{1}{2}\int_T \oint_{\Gamma_0} \Big\{ \Big[-q(h)\boldsymbol{n} + \mu \mathsf{D}(\boldsymbol{w}(h))\boldsymbol{n}\Big] \cdot \Big[\boldsymbol{r} \times (\dot\varphi \boldsymbol{k}) + \boldsymbol{U}_\infty\Big] + \\
&+ \Big[\Big(-p(\dot\varphi)\boldsymbol{n} + \mu \mathsf{D}(\boldsymbol{u}(\dot\varphi))\boldsymbol{n}\Big) \times \boldsymbol{r}\Big] \cdot (h\boldsymbol{k}) \Big\} \, d\sigma dt
\end{aligned}
\tag{5.4}
$$

where the following relations have been used

$$
\begin{aligned}
&\Big\langle \frac{D\boldsymbol{f}(\dot\varphi)}{D\dot\varphi}, h\Big\rangle = -q(h)\boldsymbol{n} + \mu \mathsf{D}(\boldsymbol{w}(h))\boldsymbol{n} \\
&\Big\langle \frac{D\dot\varphi}{D\dot\varphi}, h\Big\rangle = h
\end{aligned}
\tag{5.5}
$$

As was shown by Abergel and Temam (1990), the quantities $\{\boldsymbol{w}(h); q(h)\}$ are related to the Fréchet derivative of the mapping $\dot\varphi \to \{\boldsymbol{u}(\dot\varphi); p(\dot\varphi)\}$. Thus $\{\boldsymbol{w}(h); q(h)\}$ can be obtained as the solution of the linearized form of the Navier-Stokes system

$$
\begin{aligned}
&A\boldsymbol{w} = \begin{bmatrix} \frac{\partial \boldsymbol{w}}{\partial t} + (\boldsymbol{u}_0 \cdot \nabla)\boldsymbol{w} + (\boldsymbol{w} \cdot \nabla)\boldsymbol{u}_0 - \nu \Delta \boldsymbol{w} + \nabla q \\ \nabla \cdot \boldsymbol{w} \end{bmatrix} = 0 \\
&\boldsymbol{w}\Big|_{\Gamma_0} = \boldsymbol{h} \qquad\qquad \boldsymbol{w}(x, y, 0) = 0
\end{aligned}
\tag{5.6}
$$

In the above the field $\boldsymbol{u}_0$ represents the basic state around which the linearization is carried out.

The control h does not explicitly appear in the formulation (5.4). In order to factor it out we will use the *adjoint operator* A^* and the *adjoint state* $\{\widetilde{\boldsymbol{w}}; \widetilde{q}\}$ defined by the following relation

$$
(A\boldsymbol{w}, \widetilde{\boldsymbol{w}})_{L^2} = (\boldsymbol{w}, A^*\widetilde{\boldsymbol{w}})_{L^2} + B \tag{5.7}
$$

where $(\cdot,\cdot)_{L^2}$ denotes the scalar product in the Hilbert space L^2 and B stands for the sum of the boundary terms

$$
\begin{aligned}
B \; = & \; -\int_T \oint_{\Gamma_0} \Big[(\boldsymbol{u}_0\boldsymbol{w}) + (\boldsymbol{u}_0\boldsymbol{w})^{\top}\Big] : (\tilde{\boldsymbol{w}}\boldsymbol{n}) \; d\sigma dt \; + \\
& - \; \nu \int_T \oint_{\Gamma_0} \Big\{\tilde{\boldsymbol{w}}\Big[\nabla\boldsymbol{w} + (\nabla\boldsymbol{w})^{\top}\Big] - \boldsymbol{w}\Big[\nabla\tilde{\boldsymbol{w}} + (\nabla\tilde{\boldsymbol{w}})^{\top}\Big]\Big\}\boldsymbol{n} \; d\sigma dt \; + \qquad (5.8) \\
& + \; \int_T \oint_{\Gamma_0} (q\tilde{\boldsymbol{w}} - \tilde{q}\boldsymbol{w}) \cdot \boldsymbol{n} \; d\sigma dt + \Big[\int_{\Omega} \boldsymbol{w} \cdot \tilde{\boldsymbol{w}} \; d\Omega\Big]_{t=0}^{t=T}
\end{aligned}
$$

The adjoint state $\{\tilde{\boldsymbol{w}}; \tilde{q}\}$ is the solution of the problem *adjoint* to the linearized Navier-Stokes system (5.6)

$$
A^*\tilde{\boldsymbol{w}} = \begin{bmatrix} -\frac{\partial \tilde{\boldsymbol{w}}}{\partial t} - \boldsymbol{u}_0[\nabla\tilde{\boldsymbol{w}} + (\nabla\tilde{\boldsymbol{w}})^{\top}] - \nu\Delta\tilde{\boldsymbol{w}} + \nabla\tilde{q} \\ \nabla \cdot \tilde{\boldsymbol{w}} \end{bmatrix} = 0
$$

$$
\tilde{\boldsymbol{w}}|_{\Gamma_0} = -\boldsymbol{g} \qquad \tilde{\boldsymbol{w}}(x, y, T) = 0 \qquad dt < 0 \tag{5.9}
$$

It should be observed that Eqs (5.9) is a *terminal value* problem, i.e. it has to be marched backwards in time. The equation itself is well posed, as using the substitution $\tau = T - t$ we arrive at a problem similar to the standard advection-diffusion equation.

In the case of the rotating circular cylinder the control is limited to the tangential boundary velocity. This implies that

$$
(\boldsymbol{u}_0 \cdot \boldsymbol{n})|_{\Gamma_0} = 0 \qquad (\boldsymbol{w} \cdot \boldsymbol{n})|_{\Gamma_0} = 0 \tag{5.10}
$$

The crucial problem is the determination of the boundary condition for the adjoint state $\tilde{\boldsymbol{w}}$, i.e. the function $\boldsymbol{g}$ in (5.9). It is straightforward to verify that for the particular choice

$$
\boldsymbol{g} = \boldsymbol{r} \times (\dot{\varphi}\boldsymbol{k}) + \boldsymbol{U}_{\infty} \tag{5.11}
$$

the adjoint state introduced above can be used to re-express the Gâteaux derivative (5.4) in such a way that the control h explicitly appears in all the terms and therefore can be factored out

$$
\begin{aligned}
J'(\dot{\varphi})h = & \frac{1}{2}\int_T \oint_{\Gamma_0} \Big\{\mu \mathsf{D}(\tilde{\boldsymbol{w}}) : (\boldsymbol{t}\boldsymbol{n})R \; + \\
& - \Big[\Big(-p(\dot{\varphi})\boldsymbol{n} + \mu\mathsf{D}(\boldsymbol{u}(\dot{\varphi}))\boldsymbol{n}\Big) \times \boldsymbol{r}\Big] \cdot \boldsymbol{k}\Big\} h \; d\sigma dt = \qquad (5.12) \\
= & \frac{1}{2}\int_T \oint_{\Gamma_0} \nabla J(s, t) h \; d\sigma dt
\end{aligned}
$$

Using (5.12) in (5.3) we obtain a closed formula characterizing the necessary, though not sufficient, condition for optimality

$$\nabla J(t) = \mu \mathsf{D}(\widetilde{\boldsymbol{w}}) : (\boldsymbol{t}\boldsymbol{n})R - \left[\left(-p(\dot{\varphi})\boldsymbol{n} + \mu \mathsf{D}(\boldsymbol{u}(\dot{\varphi}))\boldsymbol{n}\right) \times \boldsymbol{r}\right] \cdot \boldsymbol{k} = 0 \qquad (5.13)$$

Now an iterative procedure will be presented which can be used to find the optimal control $\dot{\varphi}_{opt}$ and the corresponding optimal state $\left\{\boldsymbol{u}(\dot{\varphi}_{opt}); p(\dot{\varphi}_{opt})\right\}$ in the time interval $\left[0; T\right]$. First, we choose some initial control $\dot{\varphi}^1$ (for instance $\dot{\varphi}^1 \equiv 0$) and then:

1. solve the full nonlinear Navier-Stokes system in order to determine $\{\boldsymbol{u}^i; p^i\}$,
2. solve (5.9) for the adjoint state $\{\widetilde{\boldsymbol{w}}^i; \widetilde{q}^i\}$,
3. use $\{\boldsymbol{u}^i; p^i\}$ and $\{\widetilde{\boldsymbol{w}}^i; \widetilde{q}^i\}$ to determine the instantaneous values of $\nabla J^i(t)$,
4. use $\nabla J^i(t)$ to update the control $\dot{\varphi}^{i+1}(t) = \dot{\varphi}^i(t) - \lambda \nabla J^i(t)$, where λ is some properly tuned descent parameter,
5. iterate 2. through 5. until convergence, i.e. until (5.13) attains.

A remarkable feature of the presented algorithm is that it is global in time. This is however necessary when one attempts to control a nonlinear system in which case the history of the evolution has to be taken into account. In such situations local algorithms are inefficient. Furthermore, as was mentioned by Abergel and Temam (1990), the gradient algorithms of this class converge to a local minimum of the functional which may depend on the choice of the initial control $\dot{\varphi}^1$. Because of the system nonliearity, convergence to the global minimum cannot be established.

6. Final remarks and further perspectives

In the present paper we have discussed the problem of wake control. Related issues were also raised, particularly concerning the methods of computation of forces in open flow systems and the relation between the hydrodynamic forces and the vortex dynamics in wake flows. The latter issue was mentioned in the context of its implications for the open-loop strategies of flow control. Evidence was shown for the efficiency of such approaches as regards the problem of drag minimization.

In the second part of the paper we derived a rigorous feedback algorithm using the tools of the Optimal Control Theory. The performance of this technique should be checked using numerical simulations. For this purpose it will be necessary to generalize the *Random Vortex Blob Method*, so that it will be applicable to the adjoint problem (5.9). The corresponding *vorticity* form of the adjoint problem should also be formulated. No major theoretical difficulties are expected here, though the overall computational cost of the method is immense.

Further developments of the gradient algorithm may consist in its generalization for non-circular geometries of the obstacle. In such case the rotary control will involve not only the tangential, but also the normal velocity component. Consequently, the formula expressing the Gâteaux derivative of the functional will become much more complicated, basically because the boundary conditions (5.10) will be non-homogeneous. Such control however is likely to be more effective, as it involves not only dynamic, but also kinematic forcing. Yet another possible extension is to derive *Robust Control Algorithm* (Bewley et al., 1997) which allows for the presence of some random disturbance in the system. In this case the functional is minimized with respect to the optimized quantity and at the same time maximized with respect to the disturbance.

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Numeryczne badanie sterowanego opływu przeszkody

Streszczenie

W pracy badany jest problem sterowania przepływem. Rozważany jest dwuwymiarowy turbulentny opływ obracającej się przeszkody. Celem sterowania jest minimlizacja siły oporu. Przedstawione wyniki symulacji numerycznych dowodzą, że istotne efekty można uzyskać przy pomocy algorytmów "otwartych". Fakt ten potwierdzają również dostępne dane eksperymentalne. W drugiej części pracy sformułowany jest algorytm sprzężenia zwrotnego oparty o *teorię sterowania optymalnego*.

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