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STRAPDOWN INERTIAL NAVIGATION SYSTEM. Part 2 – ERROR MODELS

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Almost all new aircraft are equipped with the Inertial Navigation System (INS). The accuracy in calculations of velocity, position and attitude of aircraft depends on measurement instruments, initial alignment errors, and navigation computer errors and can be improved by analysing and solving of differential equation of error. Solution of the equations allows one to correct output signals from the navigation system. In this work, three error models for the Strapdown Inertial Navigation System (SDINS) are derived – one into computed co-ordinate system in matrix notation and two models in a local-level co-ordinate system, in matrix and in quaternion notations, respectively. Some simplifications to be introduced into error equations are proposed.

Key words: navigation, strapdown, error equations

Notation

$\boldsymbol{\varepsilon}_B$ – vector of gyroscope drift rates expressed in the B-frame
$[\boldsymbol{\varepsilon}_B]$ – skew-symmetric matrix formed by the components of vector $\boldsymbol{\varepsilon}_B$
$[\boldsymbol{a}_B]$ – skew-symmetric matrix formed by the components of vector $\boldsymbol{a}_B$
$\widetilde{\boldsymbol{r}}$ – erroneous vector

see Ortyl and Gosiewski (1998) for the remaining part of notation.

Co-ordinate systems

C-frame – computed frame of reference with origin in computed aircraft position
P-frame – "analytic-platform" frame of reference.

1. Introduction

Testing and verification of SDINS is a very important phase in the investigation and synthesis of SDINS algorithms. SDINS error dynamics analysis is the main one. Analysing INS errors brings us to the following obvious questions:

- How do the errors change in time?
- Can we measure all the error components?
- If we are able to measure only a part of them, can we estimate the rest?
- Is it possible to control the errors?
- How do the errors change during various phases of the INS operation?

To answer these questions, at the first step we derive the error propagation equations. In this paper the error propagation equations for the SDINS system are derived in the form of a set of first-order linear non-stationary differential equations (state-space notation). The standard form of these equations is

$$\dot{\boldsymbol{x}}(t) = \mathsf{A}(t)\boldsymbol{x}(t) + \boldsymbol{w}(t) \tag{1.1}$$

where

$\boldsymbol{x}(t)$ – error state vector
$\mathsf{A}(t)$ – error state matrix
$\boldsymbol{w}(t)$ – excitation vector.

2. Frames of reference

Inertial frame I, earth-fixed frame E, and true local-level frame T have been described in Polish Standards [13] and are summarised in Ortyl (1996). Others frames of reference, needed to derive appropriate error equations, are presented below.

2.1. Body-fixed frame of reference B-frame

The frame is defined by sensor axes and rotates with them. We assume that measurement axes of sensors (three accelerometers and three gyroscopes) are perpendicular to each other. Furthermore, we assume that the B-frame coincides with the aircraft-fixed frame Z or we know the transformation matrix between these frames.

2.2. "Analytic-platform" frame of reference P-frame

A navigational computer computes a transformation function (transformation matrix, quaternions, Huddle (1983)), which under ideal conditions, transforms vectors from the B-frame to the T-frame. It is due to computation errors that this function does not transform vectors to the T-frame but to the P-frame. This P-frame is called the "analytic-platform" frame of reference.

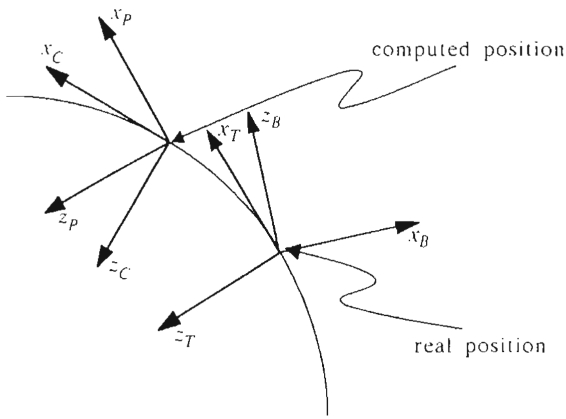

Fig. 1. Localisation and orientation of the frames of reference T, P, B and C

Relation between the P-frame and the T-frame is described by the equation (cf Benson, 1975; Huddle, 1983)

$$\boldsymbol{r}_P = (\mathsf{I} - [\boldsymbol{\Phi}]) \cdot \boldsymbol{r}_T \tag{2.1}$$

where $[\boldsymbol{\Phi}]$ is the skew-symmetric matrix

$$[\boldsymbol{\Phi}] = \begin{bmatrix} 0 & -\phi_z & \phi_y \\ \phi_z & 0 & -\phi_x \\ -\phi_y & \phi_x & o \end{bmatrix}$$

the variables ϕ_x, ϕ_y represent the tilt errors of analytic-platform frame of reference, and ϕ_z is its azimuth error.

For a small angle ϕ_x, ϕ_y, ϕ_z the transformations matrix between T and B frames is described by the relation

$$\mathsf{D}_T^B = (\mathsf{I} + [\boldsymbol{\Phi}]) \cdot \mathsf{D}_P^B \tag{2.2}$$

From elements of the transposition matrix $\mathsf{B} = (\mathsf{D}_T^B)^\top$ we can calculate the aircraft attitude angles: pitch, roll, and yaw (see Ortyl and Gosiewski, 1998; [13]).

2.3. Computed frame of reference C-frame

The computed frame of reference C is generated by a navigational computer from the measurement data. Computation errors cause that instead of T-frame computer generates C-frame. The origin of C-frame is attached to a computed point on the earth surface. All axes are coincident with the true local-level frame at this point. Numerical calculation errors, measurement errors and other errors cause that C-frame differs in angular orientation from the true local-level frame T by three small independent rotations due to errors in computed geodetic position (Huddle, 1983)

$$\delta\varphi = \varphi_C - \varphi_T = -\delta\theta_E \tag{2.3}$$

is the error in the computed geodetic latitude; it is positive in counterclockwise revolution about the east axis, and

$$\delta\lambda = \lambda_C - \lambda_T \tag{2.4}$$

is the error in the computed longitude; it is positive in counterclockwise revolution about the Earth's polar axis. This error can be projected onto the local north and vertical axes for a given latitude φ_T as follows

$$\delta\theta_N = \delta\lambda\cos\varphi_T \qquad\qquad \delta\theta_V = \delta\lambda\sin\varphi_T \tag{2.5}$$

For a given wander azimuth α three errors of angular rotation can be expressed in the true local-level frame (in this work we will assume that the azimuth equals null)

$$\begin{bmatrix} \delta\theta_x \\ \delta\theta_y \\ \delta\theta_z \end{bmatrix}_T = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \delta\theta_N \\ \delta\theta_E \\ \delta\theta_V \end{bmatrix} \stackrel{\text{for } \alpha=0}{=} \begin{bmatrix} \delta\theta_N \\ \delta\theta_E \\ -\delta\theta_V \end{bmatrix} \tag{2.6}$$

These three rotations describe completely the difference in the orientation of computed frame C in relation to the true local-level frame T

$$\boldsymbol{r}_C = (\mathsf{I} - [\delta\theta])\boldsymbol{r}_T \tag{2.7}$$

where $[\delta\theta]$ is the skew-symmetric matrix defined by the following formula

$$[\delta\theta] = \begin{bmatrix} 0 & -\delta\theta_z & \delta\theta_y \\ \delta\theta_z & 0 & -\delta\theta_x \\ -\delta\theta_y & \delta\theta_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & \delta\lambda\sin\varphi_T & -\delta\varphi \\ -\delta\lambda\sin\varphi_T & 0 & -\delta\lambda\cos\varphi_T \\ \delta\varphi & \delta\lambda\cos\varphi_T & 0 \end{bmatrix} \tag{2.8}$$

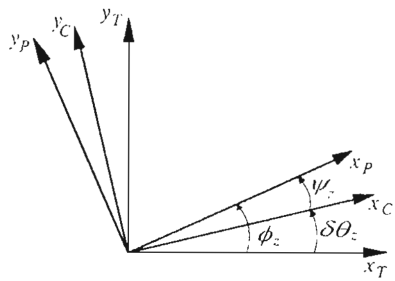

Fig. 2. Angular relations between the frames P, T, C (rotation only about z_T axis)

and are used to construct the transformation matrix D_E^C as a function of the computed latitude and longitude $[\varphi, \lambda]_C$. The matrix $\mathsf{D}_E^C(\varphi_C, \lambda_C)$ describes orientation of the C-frame relative to the E-frame

$$[\mathsf{D}_E^C]_C = \begin{bmatrix} -\sin\varphi_C \sin\lambda_C & \cos\lambda_C & -\cos\varphi_C \sin\lambda_C \\ -\sin\varphi_C \cos\lambda_C & -\sin\lambda_C & -\cos\varphi_C \cos\lambda_C \\ -\cos\varphi_C & 0 & \sin\varphi_C \end{bmatrix} \tag{2.9}$$

Angular relations between the frames: true T, computed C, and analytic-platform P are presented in Fig.2.

3. General SDINS error equations

SDINS error equations can be derived in:

- Computed frame C (Pitman, 1962)
- True frame T (Benson, 1975).

Each approach contains two groups of equation; i.e., position and velocity error, attitude error.

Taking either approach one can use transformation matrix, quaternion, or other notation.

Besides these errors, we ought to calculate the gravity acceleration errors (basing on a spherical or elliptical Earth model and on a gravity vector model) and sensor (gyroscopes and accelerometers) errors (it is assumed that we know total for each sensor – sum of bias, scale factor, non-orthogonality, etc.).

All the SDINS navigation equations which are needed to the error model derivation are presented by Ortyl and Gosiewski (1998).

3.1. Error model in the computed frame in terms of matrix calculus

3.1.1. *Equation of attitude error*

The differential equation which describes a relation between the transformation matrix and measured angular velocity vector reads (Ortyl, 1996; Ortyl and Gosiewski, 1998)

$$\dot{\mathsf{D}}_T^B = \mathsf{D}_T^B \mathbf{\Omega}_B^{B\to I} + (\mathbf{\Omega}_T^{T\to I})^\top \mathsf{D}_T^B = \mathsf{D}_T^B \mathbf{\Omega}_B^{B\to I} - \mathbf{\Omega}_T^{T\to I} \mathsf{D}_T^B \tag{3.1}$$

where the following relation has been taken into account $(\mathbf{\Omega}_T^{T\to I})^\top = -\mathbf{\Omega}_T^{T\to I}$.

To solve Eq (3.1) computer uses the incorrect angular velocities. Instead of $\boldsymbol{\omega}_B^{B\to I}$ the term $\widetilde{\boldsymbol{\omega}}_B^{B\to I}$ is employed that is the measured angular velocity vector obtained from the gyroscope outputs

$$\widetilde{\boldsymbol{\omega}}_B^{B\to I} = \boldsymbol{\omega}_B^{B\to I} + \boldsymbol{\varepsilon}_B \tag{3.2}$$

The following skew-symmetric matrix corresponds to the measured angular velocity vector

$$\widetilde{\mathbf{\Omega}}_B^{B\to I} = \mathbf{\Omega}_B^{B\to I} + [\boldsymbol{\varepsilon}_B] \tag{3.3}$$

In practise, the second matrix of angular velocity in Eq (3.1) is known with error. The computed erroneous position is used to derive this angular velocity vector, as well as the the computed matrix of angular velocity vector $\mathbf{\Omega}_C^{C\to I}$ rather than the true vector $\mathbf{\Omega}_T^{T\to I}$. Using that erroneous matrix results in the fact that instead the transformation matrix between the B and T-frames, the one between the B and P-frame is calculated. Let us denote this erroneous matrix by D_P^B, then

$$\dot{\mathsf{D}}_P^B = \mathsf{D}_P^B \widetilde{\mathbf{\Omega}}_B^{B\to I} - \mathbf{\Omega}_C^{C\to I} \mathsf{D}_P^B \tag{3.4}$$

The matrix D_P^B defines the "analytic-platform" frame. The error matrix is defined as follows

$$\varDelta \mathsf{D} = \mathsf{D}_P^B - \mathsf{D}_C^B \tag{3.5}$$

The matrix D_C^B in Eq (3.5) can be transformed to the factorized form

$$\mathsf{D}_C^B = \mathsf{D}_C^P \mathsf{D}_P^B \tag{3.6}$$

Assuming that the P and C-frames are rotated about each other through small angles it is possible to express D_C^P as follows

$$\mathsf{D}_C^P = \mathsf{I} + [\boldsymbol{\varPsi}_C] \tag{3.7}$$

where $[\boldsymbol{\Psi}_C]$ is the skew-symmetric matrix constructed of the elements of vector $\boldsymbol{\Psi}_C$, which expresses small angles of the attitude difference between these frames

$$[\boldsymbol{\Psi}_C] = \begin{bmatrix} 0 & -\psi_z & \psi_y \\ \psi_z & 0 & -\psi_x \\ -\psi_y & \psi_x & 0 \end{bmatrix} \tag{3.8}$$

After complicated transformations (cf Weinred and Bar-Itzhack, 1978) we can write

$$\dot{\boldsymbol{\Psi}}_C = -\boldsymbol{\varepsilon}_C - \boldsymbol{\Omega}_C^{C\to I}\boldsymbol{\Psi}_C \tag{3.9}$$

or denoting the time derivative of $\boldsymbol{\Psi}_C$ by $\boldsymbol{\Psi}^{(C)}$ when it is assumed that the C-frame is a non-rotating coordinate system

$$\boldsymbol{\Psi}^{(C)} = -\boldsymbol{\varepsilon} - \boldsymbol{\omega}^{C\to I} \times \boldsymbol{\Psi} \tag{3.10}$$

For the I-frame from Equation (3.10) we have

$$-\boldsymbol{\Psi}^{(I)} = \boldsymbol{\varepsilon} \tag{3.11}$$

what denotes that the change in time of $\boldsymbol{\Psi}$, observed from the I-frame, is simply the generalised drift rate of the system gyros projected on the I-frame axes.

Similarly to Eq (3.10) we have for T-frame

$$\boldsymbol{\Psi}^{(T)} + \boldsymbol{\omega}^{T\to I} \times \boldsymbol{\Psi} = -\boldsymbol{\varepsilon} \tag{3.12}$$

The above equation describes the attitude error between the computed and "analytic-platform" frames.

3.1.2. Equation of velocity error

As a matter of fact the equations for the actual velocity are solved in the C-frame in which the velocity components are also expressed so we should use the velocity equations (see Ortyl and Gosiewski, 1998) in which the subscripts T are changed by the subscripts C

$$\dot{\boldsymbol{v}}_C + (\boldsymbol{\Omega}_C^{E\to I} + \boldsymbol{\Omega}_C^{C\to I})\boldsymbol{v}_C - \boldsymbol{g}_C = \boldsymbol{a}_C \tag{3.13}$$

where

$\boldsymbol{v}_C$ – computed velocity with respect to the earth expressed in the C-frame

$\boldsymbol{\Omega}_C^{E\to I}, \boldsymbol{\Omega}_C^{C\to I}$ – skew-symmetric matrices.

Excitation perturbations cause output perturbations so the true velocity in Eq (3.13) is as follows

$$\tilde{\dot{\boldsymbol{v}}}_C = \dot{\boldsymbol{v}}_C + \delta\dot{\boldsymbol{v}}_C \tag{3.14}$$

where
$\tilde{\boldsymbol{v}}_C$ – velocity vector with three components indicated by the navigation system
$\delta \boldsymbol{V}_C$ – velocity error in this approach.

Note that the direction and magnitude of the gravity vector relative to the C-frame are functions of the computed latitude, longitude, elevation as well as gravity random errors. This problem will be discussed in the next section. Here, we assume that the actual gravity is described as follows

$$\tilde{\boldsymbol{g}}_C = \boldsymbol{g}_C + \delta\boldsymbol{g}_C \tag{3.15}$$

where $\tilde{\boldsymbol{g}}_C$ is the computed (erroneous) value of the gravity.

The matrices of angular rates $\boldsymbol{\Omega}_C^{E\to I}$, $\boldsymbol{\Omega}_C^{C\to I}$ are known precisely and therefore they are not perturbed in the C-frame. The same notion is true for the transformation matrices to and from the C-frame.

In Eq (3.13) the excitation function $\boldsymbol{a}_C$ is the one available as a value of $\tilde{\boldsymbol{a}}_P = \mathsf{D}_P^B(\boldsymbol{a}_B + \delta\boldsymbol{a}_B)$ in which measured signals $\boldsymbol{a}_B$ are perturbed by the measurement errors $\delta\boldsymbol{a}_B$ (sum of bias, scale factors, etc.). The perturbed equation (3.13) is

$$\tilde{\dot{\boldsymbol{v}}}_C + (\boldsymbol{\Omega}_C^{E\to I} + \boldsymbol{\Omega}_C^{C\to I})\tilde{\boldsymbol{v}}_C - \tilde{\boldsymbol{g}}_C = \mathsf{D}_P^B(\boldsymbol{a}_B + \delta\boldsymbol{a}_B) \tag{3.16}$$

If the P-frame is rotated with respect to the C-frame through vector $\boldsymbol{\Psi}$, then the specific force vector and its error expressed in the P-frame are described by the following relations

$$\begin{aligned} \mathsf{D}_P^B\boldsymbol{a}_B &= \mathsf{D}_P^C\boldsymbol{a}_C = \boldsymbol{a}_C - [\boldsymbol{\Psi}]\boldsymbol{a}_C \\ \mathsf{D}_P^B\delta\boldsymbol{a}_B &= \mathsf{D}_P^C\delta\boldsymbol{a}_C = \delta\boldsymbol{a}_C - [\boldsymbol{\Psi}]\delta\boldsymbol{a}_C \approx \delta\boldsymbol{a}_C \end{aligned} \tag{3.17}$$

Substituting Eqs (3.14), (3.15), and (3.17) into (3.16), upon subtraction of Eq (3.13) from the obtained result and omission of higher order terms, the following equation of velocity error is obtained

$$\delta\dot{\boldsymbol{v}}_C + (\boldsymbol{\Omega}_C^{C\to I} + \boldsymbol{\Omega}_C^{E\to I})\delta\boldsymbol{v}_C - \delta\boldsymbol{g}_C = -[\boldsymbol{\Psi}]\boldsymbol{a}_C + \delta\boldsymbol{a}_C \tag{3.18}$$

3.1.3. Equation of position error

The exact position (in Cartesian coordinates) is given (for zero initial conditions) by

$$\boldsymbol{r}_E = \int_0^t \mathsf{D}_E^C \boldsymbol{v}_C \, d\tau \tag{3.19}$$

A navigation system $\boldsymbol{v}_C$ replaces uses $\widetilde{\boldsymbol{v}}_C$ with to generate $\widetilde{\boldsymbol{r}}_E$, while by definition of the C-frame the matrix D_E^C is known without error, thus

$$\widetilde{\boldsymbol{r}}_E = \int_0^t \mathsf{D}_E^C \widetilde{\boldsymbol{v}}_C \, d\tau \tag{3.20}$$

where $\widetilde{\boldsymbol{r}}_E = \boldsymbol{r}_E + \delta\boldsymbol{r}_E$.

Subtracting Eq (3.19) from the above equation we obtain the equation of position error

$$\delta\boldsymbol{r}_E = \int_0^t \mathsf{D}_E^C \delta\boldsymbol{v}_C \, d\tau \tag{3.21}$$

or

$$\delta\boldsymbol{r}_T = \mathsf{D}_T^E \delta\boldsymbol{r}_E = \mathsf{D}_T^E \int_0^t \mathsf{D}_E^C \delta\boldsymbol{v}_C \, d\tau \tag{3.22}$$

Differentiation of Eq (3.22) yields

$$\delta\dot{\boldsymbol{r}}_T = -\boldsymbol{\Omega}_T^{T\to E} \delta\boldsymbol{r}_T + \delta\boldsymbol{v}_C \tag{3.23}$$

3.2. Error model in the T-frame in terms of matrix calculus

In this approach, the frame in which the navigation equations are to be solved is the T-frame and the transformation matrix is used to determine attitude.

3.2.1. Equation of attitude error

The navigation system assumes that the "analytic-platform" axes are coincident with the T-frame. However, the computed angular velocity $\widetilde{\boldsymbol{\omega}}_T^{T\to I}$, required for attitude computations, is available only with error $\boldsymbol{\omega}_T^{T\to I} + \delta\boldsymbol{\omega}_T^{T\to I}$. The measured angular velocity does not equal the true angular velocity but is perturbed by the gyro error – the so-called drift rate $\boldsymbol{\varepsilon}_P = \mathsf{D}_P^B \boldsymbol{\varepsilon}_B$.

Taking above into account the angular velocity of P-frame rotating with relative to the I-frame is described in the following form

$$\boldsymbol{\omega}_P^{P\to I} = \boldsymbol{\omega}_T^{T\to I} + \delta\boldsymbol{\omega}_T^{T\to I} + \mathsf{D}_P^B\boldsymbol{\varepsilon}_B \tag{3.24}$$

Substituting Eq (2.1) into Eq (3.24) and neglecting higher order terms the following relation is obtained

$$\boldsymbol{\omega}_P^{P\to T} = [\boldsymbol{\Phi}]\boldsymbol{\omega}_P^{P\to I} + \delta\boldsymbol{\omega}_P^{T\to I} + \mathsf{D}_P^B\boldsymbol{\varepsilon}_B \tag{3.25}$$

The angular velocity $\boldsymbol{\omega}_P^{P\to T}$ is the angular velocity of the P-frame relative to the T-frame and for small angles it can be expressed as $\dot{\boldsymbol{\Phi}}$, thus

$$\boldsymbol{\omega}_P^{P\to T} = \dot{\boldsymbol{\Phi}} = -\boldsymbol{\Omega}_P^{P\to I}\boldsymbol{\Phi} + \delta\boldsymbol{\omega}_P^{T\to I} + \mathsf{D}_P^B\boldsymbol{\varepsilon}_B \tag{3.26}$$

3.2.2. Equation of velocity error

The exact velocity equation is as follows

$$\dot{\boldsymbol{v}}_T + (\boldsymbol{\Omega}_T^{E\to I} + \boldsymbol{\Omega}_T^{T\to I})\boldsymbol{v}_T - \boldsymbol{g}_T = \boldsymbol{a}_T \tag{3.27}$$

Due to errors in the navigation system the following variables appear in the above equation

$$\begin{aligned} &\widetilde{\boldsymbol{v}} = \boldsymbol{v}_T + \delta\boldsymbol{v}_T && \widetilde{\boldsymbol{\Omega}}_T^{T\to I} = \boldsymbol{\Omega}_T^{T\to I} + \delta\boldsymbol{\Omega}_T^{T\to I} \\ &\widetilde{\boldsymbol{g}}_T = \boldsymbol{g}_T + \delta\boldsymbol{g}_T && \widetilde{\boldsymbol{\Omega}}_T^{E\to I} = \boldsymbol{\Omega}_T^{E\to I} + \delta\boldsymbol{\Omega}_T^{E\to I} \end{aligned} \tag{3.28}$$

Instead of $\boldsymbol{a}_T$ the true excitation is calculated from the relation $\mathsf{D}_P^B(\boldsymbol{a}_B + \delta\boldsymbol{a}_B)$, where the noisy acceleration $\boldsymbol{a}_B + \delta\boldsymbol{a}_B$ is measured using accelerometers. Thus, the navigation system solves the following velocity equation

$$\dot{\widetilde{\boldsymbol{v}}} + (\widetilde{\boldsymbol{\Omega}}_T^{T\to I} + \widetilde{\boldsymbol{\Omega}}_T^{E\to I})\widetilde{\boldsymbol{v}} - \widetilde{\boldsymbol{g}}_T = \mathsf{D}_P^B(\boldsymbol{a}_B + \delta\boldsymbol{a}_B) \tag{3.29}$$

Substituting (3.28) and (2.2) into (3.29) and subtracting Eq (3.27) we obtain the following equation of velocity error

$$\delta\dot{\boldsymbol{v}}_T = -(\boldsymbol{\Omega}_T^{E\to I} + \boldsymbol{\Omega}_T^{T\to I})\delta\boldsymbol{v}_T - (\delta\boldsymbol{\Omega}_T^{E\to I} + \delta\boldsymbol{\Omega}_T^{T\to I})\boldsymbol{v}_T + \delta\boldsymbol{g}_T - [\boldsymbol{\Phi}]\boldsymbol{a}_T + \delta\boldsymbol{a}_T \tag{3.30}$$

3.2.3. Equation of position error

The exact position (in Cartesian co-ordinates) is given (for zero initial conditions) by

$$\boldsymbol{r}_E = \int\limits_0^t \mathsf{D}_E^T\boldsymbol{v}_T\, d\tau \tag{3.31}$$

however, the navigation system generates

$$\widetilde{\boldsymbol{r}}_E = \int\limits_0^t \widetilde{\mathsf{D}}_E^T \widetilde{\boldsymbol{v}}_T \, d\tau \tag{3.32}$$

where $\widetilde{\mathsf{D}}_E^T = \mathsf{D}_E^T(\mathsf{I} + [\delta\theta])$, $\widetilde{\boldsymbol{r}}_E = \boldsymbol{r}_E + \delta\boldsymbol{r}_E$.

The difference between Eqs (3.32) and (3.31) yields

$$\delta\boldsymbol{r}_E = \int\limits_0^t \mathsf{D}_E^T\Big(\delta\boldsymbol{v}_T + [\delta\theta]\boldsymbol{v}_T + \underbrace{[\delta\theta]\delta\boldsymbol{v}_T}_{\approx 0}\Big) \, d\tau \tag{3.33}$$

or after transformation to the T-frame

$$\delta\boldsymbol{r}_T = \mathsf{D}_T^E \int\limits_0^t \mathsf{D}_E^T\Big(\delta\boldsymbol{v}_T + [\delta\theta]\boldsymbol{v}_T\Big) \, d\tau \tag{3.34}$$

Differentiation of Eq (3.34) yields

$$\delta\dot{\boldsymbol{r}}_T = -\boldsymbol{\Omega}_T^{T\to E}\delta\boldsymbol{r}_T + \delta\boldsymbol{v}_T + [\delta\theta]\boldsymbol{v}_T \tag{3.35}$$

The relations between the variables in both approaches are as follows

$$\begin{aligned} &\delta\boldsymbol{v}_T = -[\delta\theta]\boldsymbol{v}_C + \delta\boldsymbol{v}_C \qquad && \delta\boldsymbol{g}_T = -[\delta\theta]\boldsymbol{g}_C + \delta\boldsymbol{g}_C \\ &\delta\boldsymbol{\Omega}_T^{E\to I} = -[\delta\theta]\boldsymbol{\Omega}_C^{E\to I} && \end{aligned} \tag{3.36}$$

Eq $(3.36)_1$ expresses the relation between the velocity errors in T and C-frames (Hutchinson and Nash, 1971). The term $\delta\boldsymbol{v}_T$ is the true velocity error along the true axes (locally-level in relative to Earth). One should note that for low-speed vehicles the term $[\delta\theta]\boldsymbol{v}_C$ is negligible.

3.3. Error model in the T-frame in terms of quaternions calculus

In this approach, the frame in which the navigation equations are to be solved is the T-frame and the quaternion calculus are used to determine the attitude.

3.3.1. Equation of position error

In this case the position error equation is derived in spherical coordinates. To this end variation of the relation describing these coordinates is calculated (Ortyl and Gosiewski, 1998)

$$\dot{\boldsymbol{r}}^G = \begin{bmatrix} \dot{\varphi} \\ \dot{\lambda} \\ \dot{h} \end{bmatrix} = \mathbf{K}\boldsymbol{v}_T = \begin{bmatrix} \dfrac{1}{R_M+h} & 0 & 0 \\ 0 & \dfrac{\sec\varphi}{R_N+h} & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_N \\ v_E \\ v_V \end{bmatrix}$$

$$\delta\dot{\boldsymbol{v}}^G = \Big[\mathbf{F}_{11}^Q, \mathbf{F}_{12}^Q\Big] \begin{bmatrix} (\delta\boldsymbol{r}^G)^\top \\ \delta\boldsymbol{v}_T^\top \end{bmatrix} \qquad \mathbf{F}_{12}^Q = \mathbf{K} \tag{3.37}$$

$$\mathbf{F}_{11}^Q = \begin{bmatrix} \dfrac{\rho_E R_{M\varphi}}{R_M+h} & 0 & \dfrac{\rho_E}{R_M+h} \\ \rho_N \sec\varphi\Big(\tan\varphi - \dfrac{R_{N\varphi}}{R_N+h}\Big) & 0 & -\dfrac{\rho_N\sec\varphi}{R_N+h} \\ 0 & 0 & 0 \end{bmatrix}$$

where the calculation error of the curvature radius of Earth's main cross-sections is taken into account

$$R_{M\varphi} = \frac{\partial R_M}{\partial\varphi} = 3e^2 a \sin\varphi\cos\varphi \qquad R_{N\varphi} = \frac{\partial R_N}{\partial\varphi} = ae^2\sin\varphi\cos\varphi \tag{3.38}$$

3.3.2. Equation of attitude error

Substitution of indicated variables for the ideal ones into the proper differential equations and next subtraction of the same equations with ideal variables leads to the error equations. Taking into account the errors the attitude equation has the form

$$\dot{\boldsymbol{\Lambda}}_T^B + \delta\dot{\boldsymbol{\Lambda}}_T^B = \frac{1}{2}\overline{\mathbf{M}}(\boldsymbol{\omega}_B^{B\to I} + \boldsymbol{\varepsilon}_B)(\boldsymbol{\Lambda}_T^B + \delta\boldsymbol{\Lambda}_T^B) - \frac{1}{2}\mathbf{M}(\boldsymbol{\omega}_T^{T\to I} + \delta\boldsymbol{\omega}_T^{T\to I})(\boldsymbol{\Lambda}_T^B + \delta\boldsymbol{\Lambda}_T^B) \tag{3.39}$$

where $\delta\boldsymbol{\Lambda}_T^B$ is quaternion error of the transformation from the B-frame to the T-frame. Taking into account definition of the P-frame it can be interpreted as an attitude quaternion of the P-frame in relation to the T-frame: $\delta\boldsymbol{\Lambda}_T^B = \boldsymbol{\Lambda}_T^P$.

Subtracting the ideal form (without errors) of Eq $(3.38)_2$ from the expanded equation (3.40) and neglecting higher order terms we obtain the error equation as follows (Ortyl and Gosiewski, 1998)

$$\delta\dot{\boldsymbol{\Lambda}}_T^B = \frac{1}{2}\overline{\mathbf{M}}(\boldsymbol{\omega}_B^{B\to I})\delta\boldsymbol{\Lambda}_T^B - \frac{1}{2}\mathbf{M}(\boldsymbol{\omega}_T^{T\to I})\delta\boldsymbol{\Lambda}_T^B + \frac{1}{2}\mathbf{Q}(\boldsymbol{\Lambda}_T^B)\boldsymbol{\varepsilon}_B - \frac{1}{2}\mathbf{R}(\boldsymbol{\Lambda}_T^B)\delta\boldsymbol{\omega}_T^{T\to I} \tag{3.40}$$

where $\mathsf{R}(\Lambda)$ is the matrix formed by the components of the quaternion Λ_T^B

$$\mathsf{R}(\Lambda) = \begin{bmatrix} -\lambda_1 & -\lambda_2 & -\lambda_3 \\ \lambda_0 & \lambda_3 & -\lambda_2 \\ -\lambda_3 & \lambda_0 & \lambda_1 \\ \lambda_2 & -\lambda_1 & \lambda_0 \end{bmatrix} \tag{3.41}$$

$\mathsf{Q}(\Lambda)$ is the matrix formed by the components of the quaternion Λ_T^B

$$\mathsf{Q}(\Lambda) = \begin{bmatrix} -\lambda_1 & -\lambda_2 & -\lambda_3 \\ \lambda_0 & -\lambda_3 & \lambda_2 \\ \lambda_3 & \lambda_0 & -\lambda_1 \\ -\lambda_2 & \lambda_1 & \lambda_0 \end{bmatrix} \tag{3.42}$$

$\boldsymbol{\varepsilon}_B = [\varepsilon_x, \varepsilon_y, \varepsilon_z]_B^\top$ is the resultant drift of gyroscopes in the B-frame, and $\delta\boldsymbol{\omega}_T^{T\to I} = [\delta\omega_N, \delta\omega_E, \delta\omega_V]_T^\top$ represents the error vector of the calculated angular velocity of the T-frame relative to the I-frame expressed in the T-frame.

3.3.3. Equation of velocity error

Substitution of indicated variables for the ideal ones into the proper differential equations and next the subtraction of the same equations with ideal variables leads to the error equation. Taking into account the errors the velocity equation has the form

$$\begin{aligned} \delta\dot{\boldsymbol{v}}_T &= -(\boldsymbol{\Omega}_T^{E\to I} + \boldsymbol{\Omega}_T^{T\to I})\delta\boldsymbol{v}_T - (\delta\boldsymbol{\Omega}_T^{E\to I} + \delta\boldsymbol{\Omega}_T^{T\to I})\boldsymbol{v}_T + \delta\boldsymbol{g}_T + \\ &+ \mathsf{D}(\Lambda + \delta\Lambda)(\boldsymbol{a}_B + \delta\boldsymbol{a}_B) - \mathsf{D}(\Lambda)\boldsymbol{a}_B \end{aligned} \tag{3.43}$$

where for simplicity we have written $\Lambda = \Lambda_T^B$.

Assuming considerably small errors and neglecting higher order terms Eq (3.43) can be rewritten as follows (Friedland, 1978)

$$\begin{aligned} \delta\dot{\boldsymbol{v}}_T &= -(\boldsymbol{\Omega}_T^{E\to I} + \boldsymbol{\Omega}_T^{T\to I})\delta\boldsymbol{v}_T - (\delta\boldsymbol{\Omega}_T^{E\to I} + \delta\boldsymbol{\Omega}_T^{T\to I})\boldsymbol{v}_T + \delta\boldsymbol{g}_T + \\ &+ \Big[(\partial\mathsf{D}/\partial\Lambda)\delta\Lambda\Big]\boldsymbol{a}_B + \mathsf{D}(\Lambda)\delta\boldsymbol{a}_B \end{aligned} \tag{3.44}$$

The component $\Big[(\partial\mathsf{D}/\partial\Lambda)\delta\Lambda\Big]\boldsymbol{a}_B = \mathsf{F}(\Lambda, \boldsymbol{a}_B)\delta\Lambda$, where the matrix $\mathsf{F}^\top$ is defined as follows

$$
\mathsf{F}^\top(\Lambda, \boldsymbol{a}_B) = \tag{3.45}
$$

$$
= 2\begin{bmatrix} \lambda_0 a_x - \lambda_3 a_y + \lambda_2 a_z & \lambda_3 a_x + \lambda_0 a_y - \lambda_1 a_z & -\lambda_2 a_x + \lambda_1 a_y + \lambda_0 a_z \\ \lambda_1 a_x + \lambda_2 a_y + \lambda_3 a_z & \lambda_2 a_x - \lambda_1 a_y - \lambda_0 a_z & \lambda_3 a_x + \lambda_0 a_y - \lambda_1 a_z \\ -\lambda_2 a_x + \lambda_1 a_y + \lambda_0 a_z & \lambda_1 a_x + \lambda_2 a_y + \lambda_3 a_z & -\lambda_0 a_x + \lambda_3 a_y - \lambda_2 a_z \\ -\lambda_3 a_x - \lambda_0 a_y + \lambda_1 a_z & \lambda_0 a_x - \lambda_3 a_y + \lambda_2 a_z & \lambda_1 a_x + \lambda_2 a_y + \lambda_3 a_z \end{bmatrix}
$$

Multiplying $\mathsf{F}^\top$ by $\mathsf{D}(\Lambda)$ we have

$$
\mathsf{F}^\top(\Lambda, \boldsymbol{a}_B)\mathsf{D}(\Lambda) = 2\mathsf{P}(\Lambda)\mathsf{A}(\boldsymbol{a}_B) \tag{3.46}
$$

where

$$
\mathsf{A}(\boldsymbol{a}_B) = \begin{bmatrix} -a_x & -a_y & -a_z \\ \dots & \dots & \dots \\ 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_B^\top \\ \dots \\ [\boldsymbol{a}_B] \end{bmatrix}
$$

$$
\mathsf{P}(\Lambda) = \begin{bmatrix} -\lambda_0 & \vdots & -\lambda_1 & -\lambda_2 & -\lambda_3 \\ -\lambda_0 & \vdots & \lambda_0 & -\lambda_3 & \lambda_2 \\ -\lambda_2 & \vdots & \lambda_3 & \lambda_0 & -\lambda_1 \\ -\lambda_3 & \vdots & -\lambda_2 & \lambda_1 & \lambda_0 \end{bmatrix} = [-\Lambda, \mathbf{Q}(\Lambda)] \tag{3.47}
$$

Multiplying both sides of Eq (3.46) by $\mathsf{D}^\top$ and next transposing the result we obtain

$$
\mathsf{F}(\Lambda, \boldsymbol{a}_B) = \mathsf{F}_{23}^Q = 2\mathsf{D}(\Lambda)\mathsf{A}^\top(\boldsymbol{a}_B)\mathsf{P}^\top(\Lambda) \tag{3.48}
$$

Using the elements of introduced matrices we rewrite Eq (3.45) as follows

$$
\begin{aligned} \delta\dot{\boldsymbol{v}}_T &= -(\boldsymbol{\Omega}_T^{E\to I} + \boldsymbol{\Omega}_T^{T\to I})\delta\boldsymbol{v}_T - (\delta\boldsymbol{\Omega}_T^{E\to I} + \delta\boldsymbol{\Omega}_T^{T\to I})\boldsymbol{v}_T + \delta\boldsymbol{g}_T + \\ &+ 2\mathsf{D}(\Lambda)\mathsf{A}^\top(\boldsymbol{a}_B)\mathsf{P}^\top(\Lambda)\delta\Lambda + \mathsf{D}(\Lambda)\delta\boldsymbol{a}_B \end{aligned} \tag{3.49}
$$

4. Nominal gravity errors

In Eqs (3.18), (3.33) and (3.49) there are the terms of gravity errors $\delta\boldsymbol{g}_C$, $\delta\boldsymbol{g}_T$. In each case those errors are sums of the two components: computation error of nominal gravity value $\Delta \boldsymbol{G}$ (because of limited knowledge of the vehicle present position) and a gravity anomaly $\Delta \boldsymbol{g}$, which is difference between the true value of gravity and the reference value based on some model of the Earth.

4.1. Gravity anomalies

The gravity anomaly vector is the portion of the gravitational force which is not accounted by the formula used to calculate the gravity. In the T-frame, the gravity anomaly vector may be expressed as (cf Nash et al., 1971; Jordan, 1973; Benson, 1975; Mandour and El-Dakiky, 1988)

$$\Delta \boldsymbol{g}_T = [g_T\xi, g_T\eta, -\Delta g]^\top \tag{4.1}$$

where

- ξ, η – north and east components of the vertical deflection; random processes statistically nonisotropic
- Δg – gravity anomaly defined as a difference between the magnitude of the true gravity vector and the magnitude of the reference gravity vector in the direction normal to Earth surface
- g_T – gravity acceleration (Ortyl and Gosiewski, 1998).

4.2. Computation errors of the nominal gravity value

Because of limited knowledge of the vehicle present position (navigation computer errors) the computed value of nominal gravity is known with an error. We assume that for C-frame this error is expressed as follows

$$\Delta \boldsymbol{G}_C = \widetilde{\boldsymbol{g}}_T - \boldsymbol{g}_C \tag{4.2}$$

The nominal gravity vector in the C-frame is expressed by

$$\boldsymbol{g}_C = \mathsf{D}_C^T \boldsymbol{g}_T = \boldsymbol{g}_T - [\delta\theta]\boldsymbol{g}_T \tag{4.3}$$

Latitude and longitude errors can be expressed by the position errors with using the following relations

$$\delta x = (R+h)\delta\varphi \qquad\qquad \delta y = \delta\lambda(R+h)\cos\varphi_T \tag{4.4}$$

Upon substitution of Eq (4.4) into Eq (2.8) and next to Eq (4.3) we have

$$\boldsymbol{g}_C = \begin{bmatrix} \delta x g/(R+h) \\ \delta y g/(R+h) \\ g \end{bmatrix} \tag{4.5}$$

The computed gravity vector in the T-frame can be expressed as follows (Ortyl and Gosiewski, 1998)

$$\bar{\boldsymbol{g}}_T = \begin{bmatrix} 0 \\ 0 \\ g_0 R^2/(R+h_C)^2 \end{bmatrix} \cong \begin{bmatrix} 0 \\ 0 \\ g - 2g\delta h/(R+h_C) \end{bmatrix} \tag{4.6}$$

Substitution of Eqs (4.5) and (4.6) into Eq (4.2) yields

$$\Delta \boldsymbol{G}_C = \begin{bmatrix} g\delta x/(R+h) \\ -g\delta y/(R+h) \\ -2g\delta h/(R+h) \end{bmatrix} \tag{4.7}$$

Using Eq (4.1) and $\delta h = -\delta z$ we have

$$\delta \boldsymbol{g}_C = \underbrace{\begin{bmatrix} -g/(R+h) & 0 & 0 \\ 0 & -g/(R+h) & 0 \\ 0 & 0 & 2g/(R+h) \end{bmatrix}}_{[\delta \boldsymbol{g}]} \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix}_C + \underbrace{\begin{bmatrix} \xi g \\ \eta g \\ -\Delta g \end{bmatrix}}_{\Delta \boldsymbol{g}_C} \tag{4.8}$$

For the T-frame the following relation is true

$$\Delta \boldsymbol{G}_T = \tilde{\boldsymbol{g}}_T - \boldsymbol{g}_T \tag{4.9}$$

Using Eq (4.6) in Eq (4.9) we obtain

$$\Delta \boldsymbol{G}_T = \begin{bmatrix} 0 \\ 0 \\ -2g\delta h/(R+h) \end{bmatrix} \tag{4.10}$$

Using Eqs (4.10) and (4.1) we have

$$\delta \boldsymbol{g}_T = \begin{bmatrix} \xi g \\ \eta g \\ -\Delta g - 2g\delta h/(R+h) \end{bmatrix} \tag{4.11}$$

where R – mean earth's radius.

5. Detailed error equations

5.1. Computed frame

The detailed error equations can be derived using the appropriate expansion of Eqs (3.10), (3.18) and (3.23) combined with proper relations.

In the C-frame we have assumed that the appropriate angular velocities are known without errors

$$-\mathbf{\Omega}_C^{C\to E} = \begin{bmatrix} 0 & \rho_V & -\rho_E \\ -\rho_V & 0 & \rho_N \\ \rho_E & -\rho_N & 0 \end{bmatrix} \tag{5.1}$$

where

$$\begin{aligned} \rho_N &= \dot{\lambda}\cos\varphi = \frac{v_E}{R_N+h} \qquad\qquad \rho_E = -\dot{\varphi} = -\frac{v_N}{R_M+h} \\ \rho_V &= -\dot{\lambda}\sin\varphi = -\frac{v_E}{R_N+h}\tan\varphi \end{aligned} \tag{5.2}$$

and

$$-\mathbf{\Omega}_C^{C\to I} = \begin{bmatrix} 0 & \Omega_V+\rho_V & -\rho_E \\ -(\Omega_V+\rho_V) & 0 & \Omega_N+\rho_N \\ \rho_E & -(\Omega_N+\rho_N) & 0 \end{bmatrix} \tag{5.3}$$

where

$$\Omega_N = \Omega\cos\varphi \qquad\qquad \Omega_V = -\Omega\sin\varphi \tag{5.4}$$

and $\Omega = 7.292116\cdot 10^{-5}$ is earth angular speed

$$\mathbf{H} = -\mathbf{\Omega}_C^{C\to I} - \mathbf{\Omega}_C^{E\to I} = \begin{bmatrix} 0 & 2\Omega_V+\rho_V & -\rho_E \\ -(2\Omega_V+\rho_V) & 0 & 2\Omega_N+\rho_N \\ \rho_E & -(2\Omega_N+\rho_N) & 0 \end{bmatrix} \tag{5.5}$$

The specific force matrix expressed in the C-frame may be written in the form

$$\begin{aligned} \boldsymbol{a}_C &\approx \boldsymbol{a}_P = \mathbf{D}_P^B\boldsymbol{a}_B \\ [\boldsymbol{a}]_C &\approx [\mathbf{D}_P^B\boldsymbol{a}_B] \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}_C \end{aligned} \tag{5.6}$$

The final form of error equations in this approach is as follows

$$\begin{bmatrix} \dot{\boldsymbol{x}}_{fl} \\ \dot{\boldsymbol{x}}_a \end{bmatrix} = \begin{bmatrix} -\mathbf{\Omega}_C^{C\to E} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ [\delta\boldsymbol{g}] & \mathbf{H} & [\boldsymbol{a}] & \mathbf{0} & \mathbf{D}_P^B \\ \mathbf{0} & \mathbf{0} & -\mathbf{\Omega}_C^{C\to I} & -\mathbf{D}_P^B & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}_{15\times 15} \begin{bmatrix} \boldsymbol{x}_{fl} \\ \boldsymbol{x}_a \end{bmatrix} + \begin{bmatrix} 0_{3\times 1} \\ \Delta g_C \\ 0_{9\times 1} \end{bmatrix} \tag{5.7}$$

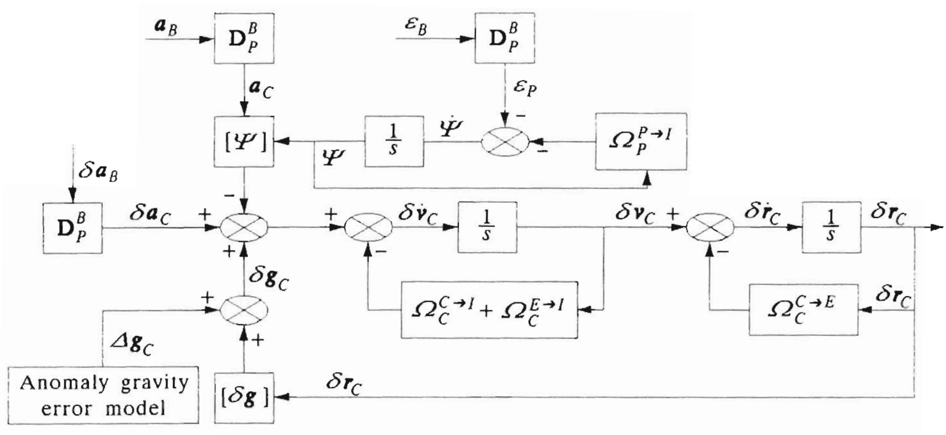

Fig. 3. SDINS error model in the C-frame

where $\boldsymbol{x}_{f1} = [\delta x, \delta y, \delta z, \delta v_N, \delta v_E, \delta v_V, \psi_x, \psi_y, \psi_z]^\top$, $\boldsymbol{x}_a = [\boldsymbol{\varepsilon}_B^\top, \delta\boldsymbol{a}_B^\top]^\top$, $\delta\boldsymbol{a}_B = [\nabla_x, \nabla_y, \nabla_z]^\top$ – resultant accelerometer errors expressed in the B-frame, $\boldsymbol{\varepsilon}_B = [\varepsilon_x, \varepsilon_y, \varepsilon_z]^\top$ – resultant gyro drift rates expressed in the B-frame.

5.2. True frame in terms of matrix calculus

In this case Eqs (3.26), (3.30), and (3.35) are used. Computation errors of angular velocities can be found by means of perturbation of the appropriate angular velocities equations

$$\delta\boldsymbol{\omega}_T^{T\to I} = \mathbf{W}\delta\boldsymbol{\nu} \qquad \delta\boldsymbol{\nu} = [\delta\varphi, \delta h, \delta v_N, \delta v_E]^\top \tag{5.8}$$

$$\delta\varphi = \frac{\delta x}{R_M + h} \qquad \delta h = -\delta z \tag{5.9}$$

$$\mathbf{W} = \begin{bmatrix} \Omega_V - \dfrac{\rho_N R_{N\varphi}}{R_N + h} & -\dfrac{\rho_N}{R_N + h} & 0 & \dfrac{1}{R_N + h} \\ -\dfrac{\rho_E R_{M\varphi}}{R_M + h} & -\dfrac{\rho_E}{R_M + h} & -\dfrac{1}{R_M + h} & 0 \\ W_{31} & \dfrac{\rho_N \tan\varphi}{R_N + h} & 0 & -\dfrac{\tan\varphi}{R_N + h} \end{bmatrix} \tag{5.10}$$

$$\delta\boldsymbol{\omega}_T^{E\to I} = [\Omega_V \delta\varphi, 0, -\Omega_N \delta\varphi]^\top \tag{5.11}$$

where

$$W_{31} = -\Omega_N - \rho_N \sec^2\varphi + \frac{\rho_N R_{N\varphi} \tan\varphi}{R_N + h}$$

Others expansion relations are as follows

$$[\boldsymbol{a}]_T = [\mathsf{D}_T^B \boldsymbol{a}_B] = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \tag{5.12}$$

$$\mathsf{F}_{11} = \begin{bmatrix} \frac{v_V}{v_N}\rho_E & 0 & -\rho_E \\ -\rho_V & \frac{v_N}{v_E}\rho_V - \frac{v_V}{v_E}\rho_N & \rho_N \\ 0 & 0 & 0 \end{bmatrix} \tag{5.13}$$

$$\mathsf{F}_{21}^Q = \begin{bmatrix} \begin{array}{c} -(2\Omega_N + \rho_N \sec^2\varphi)v_E + \\ -\rho_N\rho_V R_{N\varphi} + \frac{\rho_E v_V R_{M\varphi}}{R_M + h} \end{array} & 0 & -\rho_N\rho_V + \frac{\rho_E v_V}{R_M + h} \\ \begin{array}{c} (2\Omega_N + \rho_N \sec^2\varphi)v_N + 2\Omega_V v_V + \\ +(\rho_V v_N - \rho_N v_V)\frac{R_{N\varphi}}{R_N + h} \end{array} & 0 & \frac{\rho_V v_N - \rho_N v_V}{R_N + h} \\ -2\Omega_V v_E + \rho_E^2 R_{M\varphi} + \rho_N^2 R_{N\varphi} & 0 & \rho_E^2 + \rho_N^2 \end{bmatrix} \tag{5.14}$$

$$\mathsf{F}_{21} = \mathsf{F}_{21}^Q \operatorname{diag}\Big[\frac{1}{R_M + h}, 0, -1\Big] \tag{5.15}$$

$$\mathsf{F}_{22} = \begin{bmatrix} -\frac{v_V}{v_N}\rho_E & 2(\Omega_V + \rho_V) & -\rho_E \\ -(2\Omega_V + \rho_V) & \frac{v_V}{v_E}\rho_N - \frac{v_N}{v_E}\rho_V & 2\Omega_N + \rho_N \\ 2\rho_E & -2(\Omega_N + \rho_N) & 0 \end{bmatrix} \tag{5.16}$$

$$\mathsf{F}_{31} = \Big[\frac{1}{R_M + h}\boldsymbol{W}_{i1}, \mathbf{0}_{3\times 1}, -\boldsymbol{W}_{i2}\Big] \tag{5.17}$$

$$\mathsf{F}_{32} = [\boldsymbol{W}_{i3}, \boldsymbol{W}_{i4}, \mathbf{0}_{3\times 1}] \tag{5.18}$$

$$\mathsf{F}_{33} = \begin{bmatrix} 0 & \rho_V + \Omega_V & -\rho_E \\ -(\rho_V + \Omega_V) & 0 & \Omega_N + \rho_N \\ \rho_E & -(\Omega_N + \rho_N) & 0 \end{bmatrix} \tag{5.19}$$

Finally, the error equations are as follows

$$\begin{bmatrix} \dot{\boldsymbol{x}}_{f2} \\ \dot{\boldsymbol{x}}_a \end{bmatrix} = \begin{bmatrix} \mathsf{F}_{11} & \mathsf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathsf{F}_{21} & \mathsf{F}_{22} & [\mathsf{D}_P^B \boldsymbol{a}_B] & \mathbf{0} & \mathsf{D}_P^B \\ \mathsf{F}_{31} & \mathsf{F}_{32} & \mathsf{F}_{33} & \mathsf{D}_P^B & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}_{15\times 15} \begin{bmatrix} \boldsymbol{x}_{f2} \\ \boldsymbol{x}_a \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{3\times 1} \\ \Delta \boldsymbol{g}_T \\ \mathbf{0}_{9\times 1} \end{bmatrix} \tag{5.20}$$

where $\boldsymbol{x}_{f2} = [\delta x, \delta y, \delta z, \delta v_N, \delta v_E, \delta v_V, \phi_x, \phi_y, \phi_z]^\top$.

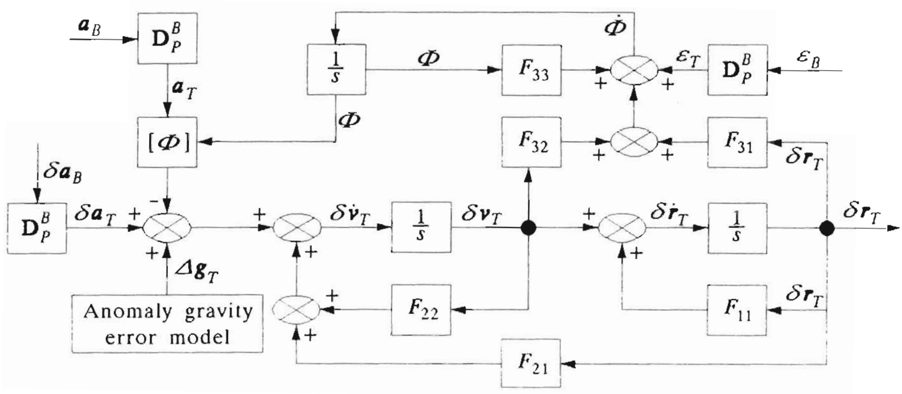

Fig. 4. SDINS error model in T-frame in terms of matrix calculus

5.3. Error model in the T-frame in terms of quaternion calculus

Expanding Eq (3.40) by the use of Eqs (3.41), (3.42), (5.10), equation (3.49) and taking into account Eqs $(3.37)_{2,3,4}$ we obtain

$$
\mathsf{F}_{31}^Q = -\frac{1}{2}[\mathsf{R}(\Lambda)\boldsymbol{W}_{i1}, \mathsf{0}_{4\times 1}, \mathsf{R}(\Lambda)\boldsymbol{W}_{i2}] \qquad \mathsf{F}_{33}^Q = \frac{1}{2}\Big(\overline{\mathsf{M}}(\boldsymbol{\omega}_B^{B\to I}) - \mathsf{M}(\boldsymbol{\omega}_T^{T\to I})\Big) \tag{5.21}
$$

$$
\mathsf{F}_{32}^Q = -\frac{1}{2}[\mathsf{R}(\Lambda)\boldsymbol{W}_{i3}, \mathsf{R}(\Lambda)\boldsymbol{W}_{i4}, \mathsf{0}_{4\times 1}] \qquad \mathsf{F}_{22}^Q = \mathsf{F}_{22}
$$

Finally, the error equations are as follows

$$
\left[\begin{array}{c} \dot{\boldsymbol{x}}_{f3} \\ \dot{\boldsymbol{x}}_a \end{array}\right] = \left[\begin{array}{ccccc} \mathsf{F}_{11}^Q & \mathsf{F}_{12}^Q & \mathsf{0}_{3\times 4} & \mathsf{0}_{3\times 3} & \mathsf{0}_{3\times 3} \\ \mathsf{F}_{21}^Q & \mathsf{F}_{22}^Q & \mathsf{F}_{23}^Q & \mathsf{0}_{3\times 3} & \mathsf{D}(\Lambda) \\ \mathsf{F}_{31}^Q & \mathsf{F}_{32}^Q & \mathsf{F}_{33}^Q & \frac{1}{2}\mathsf{Q}(\Lambda) & \mathsf{0}_{4\times 3} \\ \mathsf{0}_{6\times 3} & \mathsf{0}_{6\times 3} & \mathsf{0}_{6\times 4} & \mathsf{0}_{6\times 3} & \mathsf{0}_{6\times 3} \end{array}\right] \left[\begin{array}{c} \boldsymbol{x}_{f3} \\ \boldsymbol{x}_a \end{array}\right] + \left[\begin{array}{c} \mathsf{0}_{3\times 1} \\ \Delta g_T \\ \mathsf{0}_{10\times 1} \end{array}\right] \tag{5.22}
$$

where $\boldsymbol{x}_{f3} = [\delta\varphi, \delta\lambda, \delta h, \delta v_N, \delta v_E, \delta v_V, \delta\lambda_0, \delta\lambda_1, \delta\lambda_2, \delta\lambda_3]^\top$. The SDINS error model in terms of quaternion calculus is presented in Fig.3.

6. Simplifications of errors equations

Simplifications aim at lowering the requirements imposed on the computer memory and a duty-cycle. Simplifications generally fall into the following two categories (Huddle, 1983).

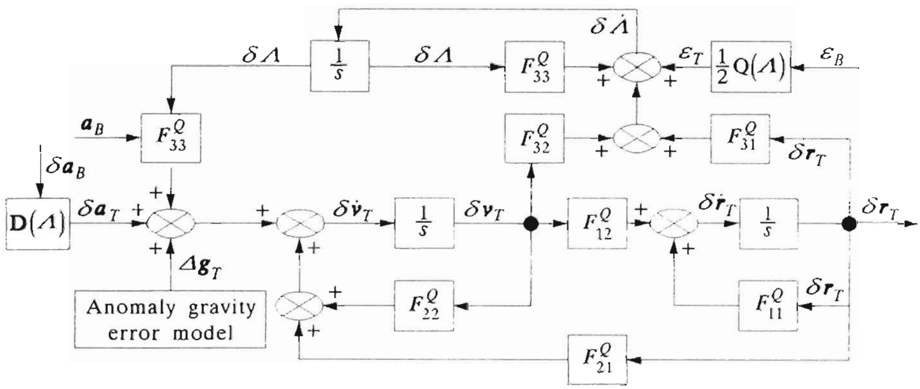

Fig. 5. SDINS error model in the T-frame in terms of quaternion calculus

The first category deals with the question of how many of the instrument-error states need to be included as the states in the Kalman filter. The decision which of these states are incorporated in the Kalman filter design model is made after several design iterations in which the performance with and without various states present in the design model is evaluated and "filter tuning" to accommodate absence of the states in the design model is performed. Such a design process is usually rather lengthy and requires highly sophisticated simulation software.

The second category of simplifications addresses the error dynamics of the navigation system errors. The two types of simplification are of interest:

- Reduction of the modelled error states as previously mentioned
- Reduction of the dynamic coupling between the error states that retained.

The simplifications that have proved especially useful in operational filter design are the following:

- Altitude influence for the Schuler frequency elimination
- Vertical axis model elimination
- Level-axes Coriolis acceleration elimination (in a function of the vertical axis variables)
- Using spherical Earth's model – mean Earth's radius instead of radiuses of Earth's main cross-sections.

6.1. Computed frame model

One of the first simplifications is the assumption that the aircraft is moving near the earth surface $h \ll R$, so instead of relation $g/(R+h)$ we can use g/R. This relation describes the so-called Schuler frequency

$$\nu_s = \sqrt{\frac{g}{R}} \Rightarrow T = 2\pi\sqrt{\frac{g}{R}} \approx 2\pi\sqrt{\frac{6371000\mathrm{m}}{9.81\mathrm{m/s}^2}} = 5064\mathrm{s} \tag{6.1}$$

the SDINS error model in the C-frame is presented in Fig.1.

6.2. True frame model in terms of matrix calculus

6.2.1. Elimination of vertical axis

From Eq (5.20) and other error-model equations that describe the error propagation for the vertical axis of the T-frame using a spherical earth model we have

$$\begin{aligned}
\delta\dot{z} &= \delta v_V \\
\delta\dot{v}_V &= \underbrace{\frac{2\Omega v_E\sin\varphi}{R+h}\delta x - \frac{v_N^2+v_E^2}{(R+h)^2}\delta z - \frac{2v_N}{R+h}\delta v_N - \Big[\frac{2v_N}{R+h} - 2\Omega\cos\varphi\Big]\delta v_E +}_{\delta C_z} \\
&\quad + \frac{2g}{R+h}\delta z + a_y\phi_x + a_x\phi_y + [\mathbf{D}_P^B\delta a]_z
\end{aligned} \tag{6.2}$$

where δC_z is the vertical component error of the computed Coriolis acceleration.

Example 1. Consider the influence of the normal gravity on the computed altitude error. In this case the aircraft does not change its position $(v_N = v_E = 0, a_x = a_y = 0)$, so Eqs (6.2) are as follows

$$\begin{aligned}
\delta\dot{z} &= \delta v_V \\
\delta\dot{v}_V &= \frac{2g}{R+h}\delta z + \nabla_z
\end{aligned} \tag{6.3}$$

The solution of Eqs (6.3) has the following form (cf Andreev, 1966)

$$\delta z(t) = -\frac{\nabla_z}{2\nu_s^2} + \Big(\delta z^0 + \frac{\nabla_z}{2\nu_s^2}\Big)\cosh(\sqrt{2}\nu_s t) + \frac{\delta v_V^0}{\sqrt{2}\nu_s}\sinh(\sqrt{2}\nu_s t) \tag{6.4}$$

Fig.6 presents the chart of error function for the following initial conditions

$$\nu_s = \sqrt{g/R} \qquad g = 9.81\ \mathrm{m/s^2} \qquad R = 6371000\ \mathrm{m}$$
$$\delta z^0 = 3\ \mathrm{m} \qquad \delta v_V^0 = 1\ \mathrm{m/s} \qquad \nabla_z = 10^{-5}\ \mathrm{m/s^2}$$

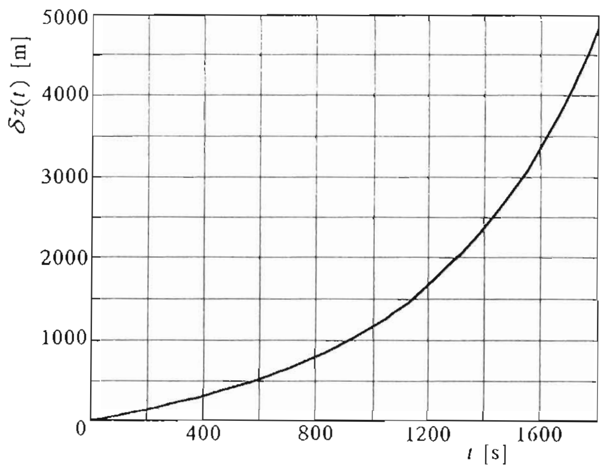

Fig. 6. Computed altitude error caused by accelerometer error and normal gravity

Rapid increase in the altitude error during the system work-time is seen both from Eq (6.4) and in Fig.6. Taking into account the other error components, we can expect much quicker error change. Therefore in the inertial navigation system the external source of altitude (e.g. – barometric altimeter) is used as the reference. Further, since there is a weak cross-coupling between the vertical and level axes, we can omit the vertical axis error states in the model to obtain a fair estimation of the level-axis error states. This is explained in the following sample case.

Example 2. For the following very large values

$$R = 6371000\ \mathrm{m} \qquad \delta z_{bar} = 10\ \mathrm{m} \qquad v_N = 300\ \mathrm{m/s}$$
$$v_V = 30\ \mathrm{m} \qquad \delta x = 10\ \mathrm{NM} = 18520\ \mathrm{m} \qquad \delta y = \delta x$$
$$h = 0 \qquad \delta v_N = \delta v_E = 3\ \mathrm{m/s}$$

the following time derivative of the position error is obtained

$$\delta \dot{x} = -\frac{v_V}{R+h}\delta x + \frac{v_N}{R+h}\delta z + \delta v_N = (-0.0872076 + 4.7088 \cdot 10^{-4} + 3)\ \mathrm{m/s}$$

From the above relation it can be proved that the influence of the position error δz, when compared to that of the velocity error δv_N, is negligible. Also, if the vertical velocity was very large (note, that in most aircraft applications the vertical velocity equals almost zero, except for a small percentage of the flight time) and position error was also very large then the Coriolis components of the acceleration error are rather small. The situation in the second local-level axis is the same. With regard to above we can come to the conclusion that the vertical axis model can be eliminated (in the part computing the position).

6.2.2. Elimination of level axis Coriolis acceleration

The Coriolis acceleration component errors in the computed level axes are as follows

$$
\begin{aligned}
\delta C_x(v_V,\delta v_V,\delta z) &= \frac{v_N v_V - v_E^2\tan\varphi}{(R+h)^2}\delta z + \frac{v_V}{R+h}\delta v_N + \frac{v_N}{R+h}\delta v_V \\
\delta C_y(v_V,\delta v_V,\delta z) &= -\frac{2\Omega v_V\sin\varphi}{R+h}\delta x + \frac{v_E(v_V+v_N\tan\varphi)}{(R+h)^2}\delta z + \\
&= +\frac{v_V}{R+h}\delta v_E + \Big(\frac{v_E}{R+h} + 2\Omega\cos\varphi\Big)\delta v_V
\end{aligned}
\tag{6.5}
$$

The following example will be solved to illustrate the above.

Example 3. For the data

$R = 6371000$ m	$\delta z_{bar} = 10$ m	$v_N = v_E = 300$ m/s
$v_V = 30$ m/s	$\delta x = \delta y = 10$ NM $= 18520$ m	$\varphi = 45°$
$h = 0$	$\delta v_N = \delta v_E = 3$ m/s	$\delta v_V = 1$ m/s

the following result can be obtained

$$\delta C_x \approx 6\ \mu\text{g} \qquad \delta C_y \approx 7.5\ \mu\text{g}$$

These values, in comparison with the gravity-model uncertainty $40 \div 50\ \mu$g, are negligible.

This type of analysis can be extended to cover the Coriolis error components in local-level axes. For example, the component $2\Omega\cos\varphi v_E\delta x/(R+h)$ for the previous data have the magnitude about $9\,\mu$g, which is much smaller than the gravity-model uncertainty.

7. Conclusions

To the best of authors' knowledge in the literature there are neither full, detailed, ready to use, SDINS error models and the analysis of their simplifications available. In the paper there have been developed some SDINS error models for two local-level frames with two notations and examples of simplifications. These error models can be used for the following purposes.

- Error estimation and SDINS output correction.
- Synthesis of the integrated navigation system, which consists of the SDINS and other non-autonomous navigation information source.
- Synthesis of SDINS self-alignment model.
- Evaluation of the influence of the full accelerometer and gyro error models on the SDINS performance. It leads us to the requirements which gyro and accelerometer should satisfy.
- Evaluation of the influence of the gravity, its anomaly, and Earth's shape model on the SDINS performance.

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Inercjalny bezkardanowy system nawigacji. Część 2 – równania błędów

Streszczenie

Większość współczesnych statków powietrznych wyposażona jest w Inercjalne Systemy Nawigacji (ISN). Dokładność wyznaczenia prędkości, pozycji oraz orientacji przestrzennej zależy od błędów elementów pomiarowych (giroskopów, przyspieszeniomierzy), błędów wstępnej orientacji oraz błędów obliczeń realizowanych przez komputer nawigacyjny. Dokładność tę można poprawić przez analizę i rozwiązanie różniczkowych równań błędów. W artykule zostały wyprowadzone trzy modele równań błędów dla Inercjalnego Bezkardanowego Systemu Nawigacji (IBSN). Pierwszy model wyprowadzony został w wyliczonym układzie współrzędnych z wykorzystaniem macierzy cosinusów kierunkowych. Dwa pozostałe wyprowadzono w normalnym układzie współrzędnych z wykorzystaniem odpowiednio: macierzy cosinusów kierunkowych i kwaternionów. Pokazano możliwości uproszczeń równań błędów.

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