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# STRESS INTENSITY FACTORS IN PLANE BODIES WEAKENED BY CRACK-TYPE DEFECTS[1]

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A technique of approximation of stress intensity factors in a plane body weakened by crack-type defects with the use of direct numerical methods is presented. Efficiency of the invented technique is shown in the case of an unlimited layer with a boundary crack solved by the boundary element method.

*Key words:* stress intensity factor, crack, elastic body

## 1. Formulation of the problem

Strength investigation of a mechanical design weakened by a crack-type defect frequently reduces to solving the corresponding plane elasticity problem and analysing the stress-strain state near the defect tip.

The formulation of the failure criteria within the framework of fracture mechanics is based on the concept of Stress Intensity Factors (SIFs) (Savruk, 1988). They are defined as some factors near the square root singularity in the formulae for stresses at the crack tip expressed in the local polar co-ordinate system $(\rho, \theta)$.

Using of analytical methods allows for precise SIF calculation, however, they are applicable mainly to unlimited bodies. Application of direct numerical and experimental methods is very difficult since:

- they do not allow the stress field to be calculated with a sufficient accuracy near a point defect (e.g. crack) tip involving, therefore, bigger SIF evaluation errors;

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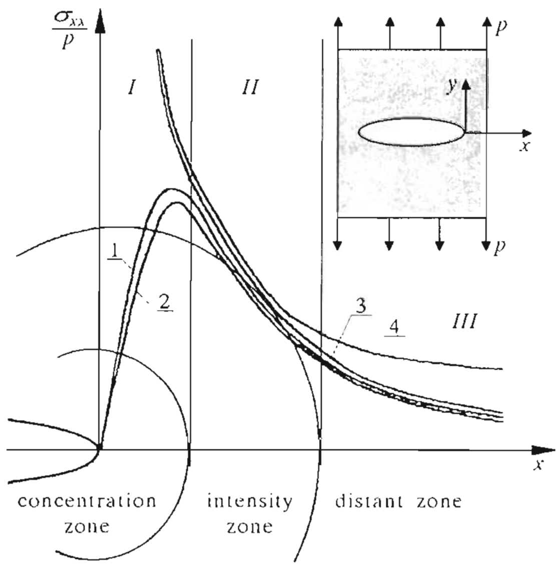



Fig. 1. Distribution of stresses along the axes thin elliptic hole in the neighbourhood of its tip

- though the SIF is defined from asymptotic equations, in the case of crack-type inclusions of finite curvature the error of asymptotic representation increases;
- therefore, it seems to be reasonable to take the following approach: represent the point defect by a hole or inclusion with a certain finite curvature of the tip;
- find the zone, where the asymptotic representations are closest to the exact solution;
- check out the direct numerical methods to see to what extent they allow the stress field in this zone to be determined. It will enable us to find the optimum (in the viewpoint of exactness) non-linear SIF solution by means of a numerical method.

## 2. Solution to the problem

We discuss this problem in detail by considering a thin elliptic hole in a plane loaded at infinity by a uniform stress field perpendicular to its greater semiaxis $a$ (Cherepanov, 1983). In the neighbourhood of the defect tip it is possible to distinguish the three following zones (Fig.1); zone of stress concentration ($I$), zone of stress intensity ($II$) and distant zone ($III$). In most cases it is extremely difficult to obtain the complete solution (the exact analytical one), which would describe the real stress field in all those zones (curve 1). The complete approximate solution which is, to some extent, close to the complete one can be arrived at by using numerical methods (curve 2), e.g., the boundary element method. Sometimes it is possible to model a real defect by a slit or a thin strip with the properties of inclusion and solve precisely this approximate problem. For example, such a solution can be obtained by using the jump function method (JFM), which is a singular one and describes adequately the stress-strain state only in the second and third zones (curve 3).

By virtue of the microscope principle, in the part of zone $I$ close to the defect, the asymptotic methods allow for obtaining solutions near the edge of heterogeneity and very close to those obtained by using the asymptotic solution.

The asymptotic representation of the stress tensor components in terms of the stress intensity factors $K_1$, $K_2$ (SIF) in the neighbourhood of the defect tip is very important in mechanics of brittle fracture. In the polar co-ordinates $(\rho, \theta)$ they can be written as follows (Savruk, 1988)

$$\begin{aligned}\begin{bmatrix} \sigma_{yy} \\ \sigma_{xx} \\ \sigma_{xy} \end{bmatrix} &= \frac{K_1}{4\sqrt{2\pi r}} \begin{bmatrix} 5\cos\theta_1 - \cos\theta_5 \\ 3\cos\theta_1 + \cos\theta_5 \\ -\sin\theta_1 + \sin\theta_5 \end{bmatrix} + \\ &+ \frac{K_2}{4\sqrt{2\pi r}} \begin{bmatrix} -\sin\theta_1 + \sin\theta_5 \\ -7\sin\theta_1 - \sin\theta_5 \\ 3\cos\theta_1 + \cos\theta_5 \end{bmatrix} + O(r^{1/2})\end{aligned} \tag{2.1}$$

where $\theta_p = p\theta/2$.

The above formulae can be arrived at by choosing the relevant asymptotic behaviour of singular solution in the neighbourhood of the defect tip or the asymptotic behaviour of the exact solution in the neighbourhood of the defect tip at distances longer than the tip curvature radius $\rho$, but shorter than the defect length $l$ (cf Cherepanov, 1983)

$$\rho \ll r \ll l \tag{2.2}$$

This ring defines the stress intensity zone (in Fig.1 zone $II$) and, at the same time, two other zones. The curve 4 in Fig.1 represents the stresses $\sigma_{xx}$ calculated from Eq (2.1).

The angle of inclination of principal axes of the stress tensor $\beta$ (parameter of isocline, observed in photoelasticity) can be determined from the formula

$$\tan 2\beta = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}} \tag{2.3}$$

In the area where the asymptotic equation (2.1) holds the following relation is true

$$\tan 2\beta = -2\frac{K_1(-\sin\theta_1 + \sin\theta_5) + K_2(3\cos\theta_1 + \cos\theta_5)}{K_1(\cos\theta_1 - \cos\theta_5) + K_2(3\sin\theta_1 + \sin\theta_5)} \tag{2.4}$$

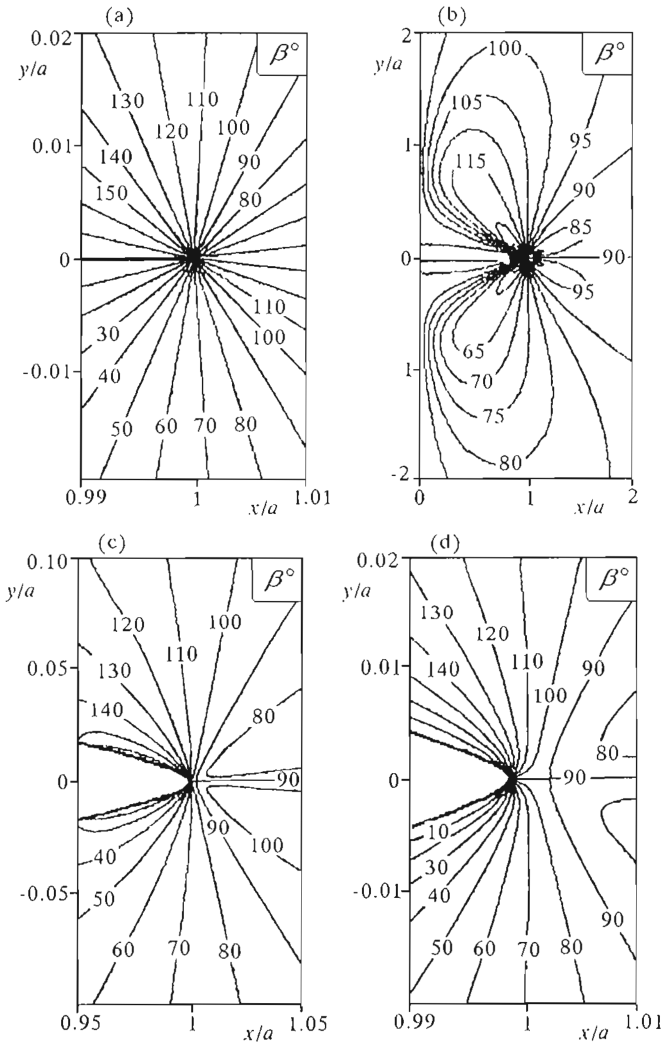
which proves, that along the position vector of the inclusion $\tan\Big(2\beta(r,\theta)\Big)\Big|_{\theta=\text{const}} = \text{const}$ or

$$\frac{\partial \tan\Big(2\beta(r,\theta)\Big)}{\partial r} = 0 \tag{2.5}$$

## 3. Numerical results

Eq (2.5) shows, that the set of isoclines of the asymptotic stress distribution (2.1) in the neighbourhood of the crack tip is represented by a set of rays released from the defect tip (Fig.2). At the same time, the field of isoclines of the complete solution in the tip of elliptic hole differs substantially from the field of isoclines of the asymptotic solutions in zones $I$ and $III$ (Fig.2b and Fig.2d, respectively) and is close to that in the zone of stress intensity (2.2) (Fig.2c), where the field of isoclines of the complete solution satisfies condition (2.5).

The developed technique of SIF calculation consists in modelling of a linear crack-type defect by the elliptic hole with semiaxes $a$ and $b$. We solve such an approximate problem by a direct numerical method. Formally, determination of the SIF from Eqs (2.1) is straightforward – one should only calculate or measure using any method the two components of stress tensor at the point of a body of given co-ordinates $r$, $\theta$ and then find $K_1$ and $K_2$ from the obtained system of linear algebraic equations. When considering a numerical solution for stresses near the defect tip close to the asymptotic one, we substitute for the left-hand side of Eq (2.1) the approximate values of components of the stress tensor being found and then using any two relations of Eq (2.1) we find $K_1$ and $K_2$. However, since the monomial asymptotic expression


(a)
β°
y/a
x/a
0.02
0.01
0
-0.01
0.99
1
1.01
130
110
120
100
140
90
150
80
110
30
100
40
50
60
70
80
(b)
β°
y/a
x/a
2
1
0
1
-2
0
1
2
100
105
95
115
90
85
90
95
65
70
75
80
(c)
β°
y/a
x/a
0.10
0.05
0
-0.05
0.95
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1.05
120
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130
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(d)
β°
y/a
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0.02
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0
-0.01
0.99
1
1.01
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120
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Fig. 2.

for $\sigma_{xx}$ appearing in Eq (2.1) differs substantially from the complete one, for defining the SIFs it is reasonable to use the formulae for $\sigma_{yy}$ and $\sigma_{xy}$. It is possible to construct more exact asymptotic representations of second and higher orders, which would take into account constants and subsequent terms of corresponding series expansions. It would allow one to increase the stress intensity zone area.

To obtain the SIF values from the numerical solution which does not take into account the stress behaviour near the defect tip it is important to find the area, where the SIF-asymptotic solution is closest to the numerical solution, i.e. to determine the zone of stress intensity. Therefore, it is necessary to find additional criteria which would allow one to determine an optimum area for the SIF calculations. We propose that Eq (2.5) be the criteria on the assumption that at the points of stress intensity zone at which the angle $\beta$ is calculated from Eq (2.3) on the basis of a direct numerical method Eq (2.5) holds. The direct numerical and SIF-asymptotic solutions are in a very good agreement.

## 4. Example

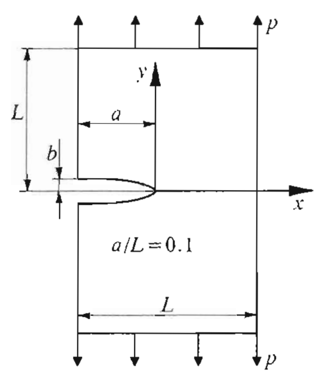


Fig. 3. Scheme of the body

For verification of the presented technique we consider a sample layer with a boundary crack. Calculation have been made by using the indirect boundary element method (IBEM).

The layer of width $L$ is loaded at infinity by uniformly distributed strain $p$. A boundary crack of length $l$ is represented by a half of the ellipse of semiaxes $a = l$ and $b = 0.05l$, we replace the layer by the rectangular plate of length $2L$, see Fig.3. The ratio of the crack length to the layer width is $l/L = 0.1$. The obtained approximate value of the SIF $\widetilde{K}_1$ is compared with that, calculated from the approximate formula

$$K_1^0 = \frac{K_1}{p\sqrt{\pi a}} = 1.12 - 0.231\frac{a}{L} + 10.55\Big(\frac{a}{L}\Big)^2 - 21.72\Big(\frac{a}{L}\Big)^3 + 30.39\Big(\frac{a}{L}\Big)^4 \tag{4.1}$$

$$K_2^0 = \frac{K_2}{p\sqrt{\pi a}} = 0$$

constructed on the basis of the numerical results, obtained by means of the boundary collocation method (Savruk, 1988). Its relative error has not exceeded 0.5% at $0 \leqslant l/L \leqslant 0.6$, this solution, therefore, may be considered as the exact one.

The comparison between the values of $K_1$, $K_2$ and the approximate values of SIFs $\widetilde{K}_1^0 = \widetilde{K}_1/(p\sqrt{\pi a})$ and $\widetilde{K}_2^0 = \widetilde{K}_2/(p\sqrt{\pi a})$ calculated for the IBEM solution in the neighbourhood of the defect tip (Fg.4), proves high accuracy of the obtained SIF values.

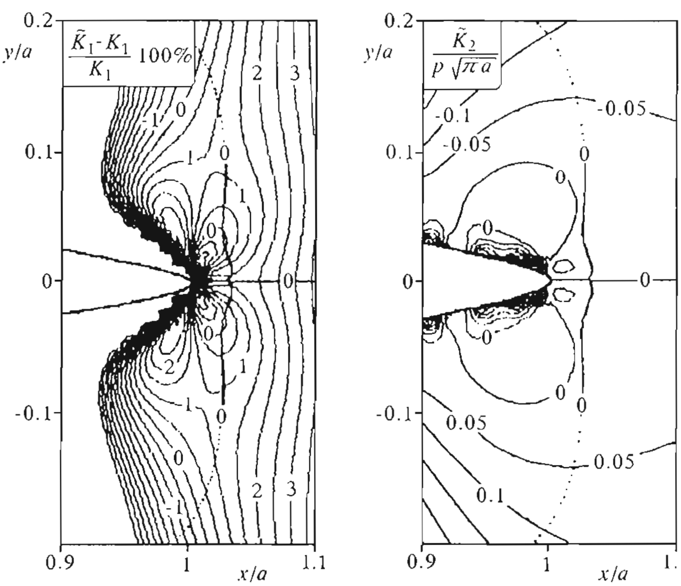


Fig. 4.

To estimate approximate SIF calculation accuracy it is necessary to make several numerical experiments for different ratios of the ellipse semiaxes and to compare the obtained results. In Table 1 the results of the $\widetilde{K}_1^0$ and $\widetilde{K}_2^0$

calculated from the stresses $\sigma_{yy}$ and $\sigma_{xy}$ are given for different ratios of the ellipse semiaxes. Additionally, the heavy line in Fig.4 represents the values of SIF which have been calculated from the stresses at the points, at which the rays $\theta = 30°$, $45°$, $60°$ cross the criterion curve, see Eq (2.5). These SIF values for the distant zone are represented by the dashed line. When changing the ellipse semiaxes ratio the number of boundary elements $N$ varies correspondingly to the change of the hole curvature. The value $K_1^0 = 1.184$ is calculated using Eq (3.1).

**Table 1**

| $\theta$ | | $a/b^2 = 1/\rho$ | | | | |
|---|---|---|---|---|---|---|
| | | 100 | 196 | 324 | 484 | 576 |
| 30° | $r/a$ | 0.080 | 0.053 | 0.038 | 0.028 | 0.025 |
| | $\widetilde{K}_1^0$ | 1.214 | 1.187 | 1.174 | 1.191 | 1.227 |
| | $\widetilde{K}_2^0$ | −0.047 | −0.037 | −0.031 | −0.024 | −0.017 |
| 45° | $r/a$ | 0.084 | 0.055 | 0.040 | 0.030 | 0.025 |
| | $\widetilde{K}_1^0$ | 1.200 | 1.176 | 1.166 | 1.185 | 1.221 |
| | $\widetilde{K}_2^0$ | −0.045 | −0.033 | −0.027 | −0.019 | −0.013 |
| 60° | $r/a$ | 0.097 | 0.068 | 0.048 | 0.035 | 0.028 |
| | $\widetilde{K}_1^0$ | 1.199 | 1.177 | 1.166 | 1.185 | 1.222 |
| | $\widetilde{K}_2^0$ | −0.021 | −0.014 | −0.012 | −0.009 | −0.005 |
| $N$ | | 47 | 51 | 55 | 63 | 67 |

It is interesting that decreasing values of $K_2^0$ are obtained on the rays $\theta = 30°$, $45°$, $60°$, where the tangential stresses $\sigma_{xy}$ are not equal to zero.

At first, due to enlarging of the stress intensity zone, a gradual rise in the ratio $a/b$ allows for improving calculation accuracy of $\widetilde{K}_1^0$ and $\widetilde{K}_2^0$, but further extension impairs the accuracy of numerical method and involves bigger errors of SIF calculation. Due to contraction of the stress concentration zone, when increasing the ellipse semiaxes ratio the stress intensity zone enlarges. The most accurate are the minimum values of SIF, which are obtained for the ratio $a/b = 18$. In our example they are a little bit lower than the values, obtained from Eq (3.1) as a result of replacing the unlimited layer by the finite slice.

Due to the problem symmetry, condition (2.5) allows one to determine with a sufficient accuracy the optimum area for the SIF calculation basing on the IBEM-solution in the stress intensity zone (there is no need for taking the horizontal part of criterion curve into consideration).

## References

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### Przybliżone obliczenia współczynników intensywności naprężeń w płaskich ciałach osłabionych przez szczelino-podobne efekty

W pracy zaproponowana została metoda przybliżonego obliczenia współczynników intensywności naprężeń dla płaskich ciał osłabionych przez szczelino-podobne efekty. Oparta jest ona na zastosowaniu prostej numerycznej metody. Efektywność tego podejścia przy korzystaniu z metody elementów brzegowych potwierdza przykład zagadnienia o wzdłużnym rozciaganiu tarczy z brzegową szczeliną.

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