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BASIC MATHEMATICAL RELATIONS OF FLUID DYNAMICS FOR MODIFIED PANEL METHODS

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Classical panel methods need some modifications in order to account for important features of real flow fields. The list of these features includes a wide spectrum of effects (appearing in many fluid dynamics problems), and their mathematical description is non-standard in some cases. Fundamental relationships and boundary conditions can have different, although equivalent forms. The paper presents foundation of potential flows and is prepared as a common foundation for any next papers devoted to the review of selected modified panel methods and their application to complex flow field calculations.

Key words: aerodynamics, panel methods, mathematical foundations

1. Introduction

Among all methods of the computational fluid dynamics (CFD) perhaps most known are the *panel methods*. It is truth that the Euler and even Navier-Stokes models can be today solved with the aid of supercomputers. Nevertheless, applying the Navier-Stokes model to flight mechanics problems, e.g. stability and control of elastic airplanes, is not only impractical today, but may not satisfy other necessary criteria such as high speed and low costs of computations which are most important. Therefore, the panel methods based on potential flow models are still most popular tool in numerical aircraft-oriented aerodynamics, especially in preliminary design of complex configurations.

However, the potential flow models need some modifications in order to account for important features of non-potential flow. The list of needed modifications was pointed out by Goraj and Pietrucha (1995b). The present paper gives mathematical relations for modified panel methods in the area of three.

rather homogeneous problems: (1) compressibility effects; (2) unsteady effects; (3) phenomena at high angles of attack.

Subjects related to these problems are numerous, and in many cases are also complex. Mathematical description sometimes need a non-standard formulation, very often applied in original and review papers. We want to emphasize that the paper is not a review of the panel methods that have been published since a long time on several occasions. Our main purpose is to present a rather complete set of the popular frequently used descriptions.

2. Main features of classical panel methods

2.1. Classical and modified panel methods

When one writes about the panel methods, one has to bear in mind that the main feature of classical panel methods consist in their capability to predict reliably aerodynamics in linear approach only, because they are numerical schemes for solving the *Laplace equation*

$$\varphi_{xx} + \varphi_{yy} + \varphi_{zz} \equiv \nabla^2 \varphi = 0 \tag{2.1}$$

where $\{x, y, z\}$ denotes a movable (body-fixed) frame of reference and φ is the perturbation velocity potential.

In order to complete the problem we need to formulate the proper boundary conditions on the body surface, at the trailing edge, and at infinity.

The first boundary condition requires zero normal velocity across the body solid boundaries

$$\nabla\varphi \cdot \boldsymbol{n} = 0 \tag{2.2}$$

where $\boldsymbol{n}$ is a unit vector normal to the body surface.

Along the wing trailing edges the velocity has to be limited in order to fix the rear stagnation line and therefore

$$\nabla\varphi < \infty \tag{2.3}$$

The third boundary condition requires that the flow disturbance due to the body motion through the fluid should diminish far from the aircraft

$$\nabla\varphi \rightarrow 0 \tag{2.4}$$

Therefore, from the *Navier-Stokes model* point of view, Eq (2.1) demands for the following assumptions to be accepted: no viscosity terms; no compressibility effects; no unsteady effects. However, in practice numerous existing panel methods incorporate processes which allow some account to be taken of compressibility and unsteady effects, phenomena at high angles of attack, and even finite Reynolds numbers. Such panel methods we are calling the *modified* panel methods.

Thus, in our opinion the following assumptions constitute the *classical* panel methods: (1) linearity of the governing equations and boundary conditions; (2) flatness of the vortex surface; (3) shedding-up of the wake from the trailing edge only; (4) incompressible and steady flow.

In the panel method approach, differential Eq (2.1) is converted to an integral one over the configuration surface by means of the third identity of Green (Kellog, 1967) which here is named simply *Green's Theorem.*

2.2. Boundary integral formulation

There are two main forms of the boundary-value problems, namely the *Dirichlet form* and the *Neumann form.* Both forms can be formulated using *internal* or *external* approaches. In aircraft aerodynamics the external Neumann conditions are usually considered, because in most cases we do not know the potential distribution, whereas we know the potential derivatives, normal to the surface and equal to the velocity components. Therefore, we limit our considerations to the external Neumann conditions.

The boundary-value problem for the Laplace equation consists in finding φ, satisfying this equation in some region, and such that

$$\left.\frac{\partial\varphi}{\partial n}\right|_S = g(x, y, z) \tag{2.5}$$

where g is a function given *a priori.*

In order to formulate the Neumann problem in terms of boundary integral equations, we use Green's theorem in accord with, the potential at a point P exterior to S is

$$\varphi(P) = \frac{1}{4\pi}\iint\limits_S \Big[\varphi\frac{\partial}{\partial n}\Big(\frac{1}{r}\Big) - \frac{1}{r}\frac{\partial\varphi}{\partial n}\Big]\, dS \tag{2.6}$$

If the values of φ or $\partial\varphi/\partial n$ on the boundary surface S are known, Eq (2.6) may be used to obtain the values of φ at any point in the flow field.

However, only $\partial\varphi/\partial n$ is prescribed on S (see Eq (2.5)). Thus, we need an equation for evaluating φ on S. Such an equation is obtained by noting that, as the point P approaches a point Q on the surface S, the value of $\varphi(P)$ approaches the value of φ at this point Q on S. This yields

$$2\pi\varphi(Q) = \iint\limits_S \Big[\varphi\frac{\partial}{\partial n}\Big(\frac{1}{r}\Big) - \frac{1}{r}\frac{\partial\varphi}{\partial n}\Big]\; dS \tag{2.7}$$

This integral equation and its extensions are crucial for the panel methods methodology: once φ is known on S, we may evaluate (anyway, on the outside of S) the potential φ from Eq (2.6), the velocity $\boldsymbol{V}$ (see Eq (3.6)), and the pressure p (using the Bernoulli equation, see Section 4.6).

Solution to the problem is based on the distribution of singularities on the boundary surface S. The most general approach consists in using sources and doublets as singularities, which are interrelated by the formulae

$$\sigma = \Big(\frac{\partial\varphi}{\partial n}\Big)_U - \Big(\frac{\partial\varphi}{\partial n}\Big)_L \qquad\qquad \mu = -(\varphi_U - \varphi_L) \tag{2.8}$$

where U and L denote upper and lower sides of the surface, respectively, with respect to the outward versor $\boldsymbol{n}$.

The most known formulation of Hess and Smith (see Section 4.2.2 in Goraj and Pietrucha, 1995a) consists in choosing $\mu = 0$, that yields

$$\varphi(P) = -\frac{1}{4\pi}\iint\limits_S \frac{1}{r}\sigma\; dS \tag{2.9}$$

Therefore, the corresponding integral equation in σ is

$$2\pi\sigma(Q) - \iint\limits_S \Big[\sigma\frac{\partial}{\partial n}\Big(\frac{1}{r}\Big)\Big]\; dS = 4\pi g \tag{2.10}$$

From the mathematical point of view this is the Fredholm equation of the second kind. Once σ is known, Eq (2.9) gives the potential φ.

An implementation of a procedure for obtaining a numerical solution of Eq (2.10) and the like is called a panel method (for details see Goraj and Pietrucha, 1995a; Kubryński, 1993; Morino, 1993).

3. Governing equations

3.1. Euler model and its simplification

The starting point for the various potential formulations is the Euler model

– it is generally accepted as the basic model governing most fluid dynamics phenomena of interest in flight mechanics. The fundamental assumptions for a physical model of potential flows are that the fluid is inviscid and non-conducting. External forces and heat sources are not taken into account. The governing equations in the conservation form (written either in unmovable or movable inertial frame of reference) for such a model are as follows (cf Anderson, 1990; Ward, 1955):

– continuity equation

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \boldsymbol{V} = 0 \tag{3.1}$$

— momentum equation

$$\rho \frac{D\boldsymbol{V}}{Dt} = -\operatorname{grad} p \tag{3.2}$$

— energy equation

$$\rho \frac{D}{Dt}\Big(e + \frac{1}{2}V^2\Big) + \operatorname{div}(p\boldsymbol{V}) = 0 \tag{3.3}$$

where

ρ – density
$\boldsymbol{V}$ – velocity vector
p – pressure
e – internal energy per unit mass, and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{V} \cdot \nabla \tag{3.4}$$

The local flow velocity vector $\boldsymbol{V}$ should correspond to the frame of reference (i.e. is observed either from unmovable or movable frame of reference).

The set of Eqs (3.1) ÷ (3.3) is called the *Euler model* and it will represent inviscid rotational flow within the whole speed range. This model consists of five scalar equations in six scalar unknowns: ρ, p, e, and three scalar components of velocity $\boldsymbol{V}$.

The most impressive simplification of this model is obtained on the assumption that the fluid is isentropic, i.e. isentropy equation

$$\frac{p}{p_\infty} = \Big(\frac{\rho}{\rho_\infty}\Big)^\gamma \tag{3.5}$$

where γ is the ratio of specific heats, and the subscript ∞ denotes the free stream conditions.

From the momentum equation in Crocco's form (Hirsch, 1988) it can be concluded that there exists the relation between vorticity and entropy. This relation shows that entropy variations generate vorticity and, inversely, vorticity creates entropy gradients. So, the assumption that fluid is isentropic leads to the conclusion that a single scalar function, $\phi(x, y, z, t)$ exists and can be defined by

$$\nabla\phi \equiv \mathrm{grad}\phi = \boldsymbol{V} \tag{3.6}$$

which is equivalent to the irrationality condition

$$\mathrm{rot}\boldsymbol{V} = \boldsymbol{0} \tag{3.7}$$

Function $\phi(x, y, z, t)$ is named the *full* (or *total*) *velocity potential.*

3.2. Unsteady Bernoulli equation

Combining Eq (3.2) with the definition (3.6) one obtains the least restricted form of Bernoulli equation, called sometimes the *Bernoulli-Kelvin equation* written either in a movable frame of reference

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(\mathrm{grad}\phi)^2 + \int\limits_{p_\infty}^{p} \frac{1}{\rho}\,dp = \Big[\frac{\partial\phi}{\partial t} + \frac{1}{2}(\mathrm{grad}\phi)^2\Big]_\infty \tag{3.8}$$

or in an unmovable frame of reference

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}(\mathrm{grad}\varphi)^2 + \int\limits_{p_\infty}^{p} \frac{1}{\rho}\,dp = 0 \tag{3.9}$$

where φ is the velocity potential in an unmovable frame of reference.

The practical value of Bernoulli equation is that it allows to relate p to ϕ. Using Eq (3.5) one may compute from Eq (3.8)

$$\begin{aligned} C_p &\equiv \frac{p - p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2} = \frac{p - p_\infty}{\frac{1}{2}\gamma p_\infty M_\infty^2} = \\ &= \frac{2}{\gamma M_\infty^2}\Big\{\Big[1 + \frac{\gamma - 1}{2} M_\infty^2\Big(1 - \frac{V^2 + 2\phi_t}{U_\infty^2}\Big)\Big]^{\frac{\gamma}{\gamma-1}} - 1\Big\} \end{aligned} \tag{3.10}$$

where

$$M_\infty = \frac{U_\infty}{a_\infty} \qquad\qquad a^2 = \frac{dp}{d\rho} = \frac{\gamma p}{\rho} \tag{3.11}$$

and U_∞ denotes the free stream velocity observed from a movable frame of reference.

A very popular form of Bernoulli equation can be obtained if we assume the incompressible flow ($\rho = \text{const}$). Then, from Eq (3.8) we have

$$\frac{p_\infty - p}{\rho} = \frac{1}{2}(\nabla\phi)^2 + \frac{\partial\phi}{\partial t} - \frac{1}{2}U_\infty^2 \tag{3.12}$$

The same can be obtained from Eq (3.9) employing the relations

$$\begin{aligned}
&\left.\frac{\partial\varphi}{\partial t}\right|_{unmovable} = -[\boldsymbol{V}_0 + \boldsymbol{\Omega}\times\boldsymbol{r}]\nabla\varphi + \left.\frac{\partial\varphi}{\partial t}\right|_{movable} \\
&\nabla\varphi = \nabla\phi - U_\infty\boldsymbol{i} \\
&\left.\frac{\partial\varphi}{\partial t}\right|_{movable} = \left.\frac{\partial\phi}{\partial t}\right|_{movable}
\end{aligned} \tag{3.13}$$

what gives

$$\frac{p_\infty - p}{\rho} = -[\boldsymbol{V}_0 + \boldsymbol{\Omega}\times\boldsymbol{r}][\nabla\phi - U_\infty\boldsymbol{i}] + \frac{\partial\phi}{\partial t} + \frac{1}{2}[\nabla\phi - U_\infty\boldsymbol{i}]^2 \tag{3.14}$$

where $\boldsymbol{V}_0$ and $\boldsymbol{\Omega}$ are the velocity and rate of rotation, respectively, (note that $\boldsymbol{V}_0 = -U_\infty\boldsymbol{i}$). It is easy to show that for $\boldsymbol{\Omega} = \boldsymbol{0}$, Eqs (3.14) and (3.12) are equivalent.

The pressure coefficient computed on the base of Eq (3.12) is given by

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2} = 1 - \left(\frac{V}{U_\infty}\right)^2 - \frac{2}{U_\infty^2}\frac{\partial\phi}{\partial t} \tag{3.15}$$

The same result can be obtained from Eq (3.9) with the aid of relations (3.13), namely

$$\frac{p_\infty - p}{\rho} = \frac{1}{2}(\boldsymbol{i}U_\infty - \nabla\phi)^2 + \boldsymbol{i}U_\infty(\nabla\phi - \boldsymbol{i}U_\infty) + \frac{\partial\phi}{\partial t} = \frac{V^2 - U_\infty^2}{2} - \frac{\partial\phi}{\partial t} \tag{3.16}$$

which can be easy converted into Eq (3.15).

3.3. Full potential equation

The equation describing the potential function ϕ is obtained from Eq (3.1) employing the definition (3.6) and has the form

$$\frac{D\rho}{Dt} + \rho\nabla^2\phi = 0 \tag{3.17}$$

After modification the first term of Eq (3.17) by virtue of Eq (3.8), the differential equation satisfied by the velocity potential can be obtained (see Ashley and Landahl, 1965 for details)

$$a^2\nabla^2\phi = \phi_{tt} + \frac{\partial}{\partial t}V^2 + \boldsymbol{V}\cdot \mathrm{grad}\Big(\frac{1}{2}V^2\Big) \tag{3.18}$$

where a is the local speed of sound given for a perfect gas by Eq (3.11).

Eq (3.18) is a single equation in *two* unknowns, ϕ and a. Therefore. a second independent relation between ϕ and a is needed. The simplest method to obtain this relation is to use the definition of the speed of sound (3.11) and the isentropic relation (3.5) in the Bernoulli equation (3.8). Finally, one obtains

$$a^2 = a^2_\infty - \frac{\gamma-1}{2}\Big[2\phi_t + (\nabla\phi)^2 - U^2_\infty\Big] \tag{3.19}$$

From Eqs (3.18) and (3.19), we can obtain the equations corresponding to different ranges of speed.

It is of interest that Garrick (see Ashley and Landahl, 1965) pointed out that Eq (3.18) can be reorganized into

$$\nabla^2\phi = \frac{1}{a^2}\Big(\frac{\partial}{\partial t} + \boldsymbol{V}\nabla\Big)\Big(\frac{\partial}{\partial t} + \boldsymbol{V}_c\nabla\Big)\phi \equiv \frac{1}{a^2}\frac{D_c^2\phi}{Dt^2} \tag{3.20}$$

where the subscript c indicates that $\boldsymbol{V}$ is kept as a constant during the second differentiation. Eq (3.20) is just a wave equation (with the propagation speed equal to the local value of a given by Eq (3.19)) when the process is observed relative to a coordinate system moving at the local fluid velocity $\boldsymbol{V}$. Eq (3.20) is known in aeroelasticity as *Garrick's equation.*

It is worth noting that the continuity equation (3.1) with condition (3.6) and on the assumption $\partial\rho/\partial t = 0$, may be written as

$$\nabla(\rho\nabla\phi) = 0 \tag{3.21}$$

which may be expressed as the *Poisson equation*

$$\nabla^2\phi = -\frac{1}{\rho}(\nabla\rho\nabla\phi) \tag{3.22}$$

where the density is given by

$$\frac{\rho}{\rho_\infty} = \Big\{1 + \frac{\gamma-1}{2}M^2_\infty\Big[1 - \Big(\frac{\nabla\phi}{U_\infty}\Big)^2\Big]\Big\}^{\frac{1}{\gamma-1}} \tag{3.23}$$

3.4. Perturbation models

It is a usual procedure to prolongate the simplification assuming additionally that

$$\phi = \varphi_\infty + \varphi \tag{3.24}$$

where φ is the *perturbation potential.*

From Eq (3.18) one obtains the *transonic small-perturbation* model

$$\nabla^2\varphi = (\gamma + 1)M_\infty^2\varphi_x\varphi_{xx} + \frac{1}{a_\infty^2}\Big(U_\infty\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\Big)^2\varphi \tag{3.25}$$

From Eq (3.25) after the neglecting the nonlinear term we obtain the equation

$$\nabla^2\varphi = \frac{1}{a_\infty^2}\Big(U_\infty\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\Big)^2\varphi \tag{3.26}$$

which holds in the subsonic and supersonic flows.

From Eq (3.26) on the assumption $\partial\varphi/\partial t = 0$ one can obtain the *Prandtl-Glauert* equation

$$\beta^2\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0 \tag{3.27}$$

where

$$\beta = \sqrt{1 - M_\infty^2} \tag{3.28}$$

is the *Prandtl-Glauert factor.*

If one assumes that the flow is incompressible $(M_\infty = 0)$ or one performs the Prandtl-Glauert transformation (see Eq (1.3) in Goraj and Pietrucha, 1995a) then from Eq (3.27) can be derived the *Laplace equation*

$$\nabla^2\varphi = 0 \tag{3.29}$$

It should be noted that a similar equation can be obtained at once from Eq (3.17) setting $\rho = \text{const}$. That is why the Laplace equation is sometimes called the continuity equation.

4. Fundamental notions and relationships

4.1. Surface vorticity vector

The potential induced by a surface distribution of doublets with the intensity μ is

$$4\pi\varphi = -\iint\limits_S \mu n \cdot \nabla\frac{1}{r}\,dS \tag{4.1}$$

Differentiating Eq (4.1) with respect to $\boldsymbol{n}$ yields

$$4\pi \boldsymbol{V} = -\int\limits_C \mu \nabla \frac{1}{r} \times dl + \iint\limits_S [n \times \nabla\mu] \times \nabla \frac{1}{r}\, dS \tag{4.2}$$

In this form the velocity can be considered as induced by two vortex distributions (Cantaloube and Rehbach, 1986):

- the first due to a vortex μdl concentrated on the contour C of open surface S, and
- the second due to a *surface distribution of vortices* with the intensity

$$\zeta = \boldsymbol{n} \times \nabla\mu \qquad \text{on} \quad S \tag{4.3}$$

4.2. Modified (complex) velocity potential

For a harmonically oscillating body with the circular frequency ω, it is convenient to introduce the complex velocity potential

$$\Phi = \varphi \exp\Big\{-\mathrm{i}\omega\Big[t + \frac{U_\infty x}{(a_\infty\beta)^2}\Big]\Big\} \tag{4.4}$$

where φ is the perturbation potential (see Eq (3.24)), and β is given by Eq (3.28)).

By virtue of Eq (45) and using the Prandtl-Glauert transformation, Eq (3.26) can be rewritten as the Helmholtz equation

$$\nabla^2\Phi + K^2\Phi = 0 \tag{4.5}$$

where

$$K = \frac{\omega L}{a_\infty \beta^2} \tag{4.6}$$

and L is a characteristic length.

4.3. Acceleration potential

Sometimes a great advantage can be gained using the concept of the *acceleration potential* (instead of the velocity potential), defined as

$$\psi = \frac{D\phi}{Dt} \tag{4.7}$$

Acceleration potential becomes practically useful when disturbances are small, so that $\rho := \rho_\infty$ everywhere. and on the base of Eqs (3.2) and (3.6) we have

$$\psi = \frac{p_\infty - p}{\rho_\infty} \tag{4.8}$$

Eq (4.8) is often treated as a definition of the acceleration potential. Anyway, it differs only by a constant from the local pressure, and such a replacement is used to avoid the necesity for extending surface integration over the wake. The acceleration potential and pressure discontinuities exist only across the lifting surface, whereas the velocity potential discontinuities exist across the lifting surface and wake vortex sheet. The acceleration potential is often used in aerodynamic calculations because it is directly proportional to the pressure perturbation (see Eq (4.8)).

Although a suitable partial differential equation describing the acceleration potential is not known in a general case, but it satisfies the same equation as the disturbance velocity potential in linearized theory.

4.4. Biot-Savart law

The Biot-Savart law is used in a majority of methods developed for the calculations of the induced flow velocities due to vortices in the high angle of attack flow fields. The basis for the law is the definition of the *vorticity vector* $\boldsymbol{\omega}$

$$\mathrm{rot}\boldsymbol{V} = \boldsymbol{\omega} \tag{4.9}$$

and the definition of incompressible flow (see Eq (3.1))

$$\mathrm{div}\boldsymbol{V} = 0 \tag{4.10}$$

where $\boldsymbol{V}$ is the velocity field.

Eqs (4.9) and (4.10) can be solved to give $\boldsymbol{V}$ as a function of $\boldsymbol{\omega}$

$$\boldsymbol{V}(\boldsymbol{r}_p, t) = \frac{1}{4\pi} \iiint\limits_{\tau} \frac{\boldsymbol{\omega}(\boldsymbol{r}_s) \times (\boldsymbol{r}_p - \boldsymbol{r}_s)}{(\boldsymbol{r}_p - \boldsymbol{r}_s)^2}\, d\tau \tag{4.11}$$

where

$\boldsymbol{r}_p$ – point at which the velocity is being calculated

$\boldsymbol{r}_s$ – location of the vorticity in the volume

τ – volume element.

We want to emphasize that Eq (4.11) is a purely kinematic relationship, so it is true for inviscid and viscous flow fields. The flow field described by Eq (4.11) is incompressible everywhere, and it is irrotational in the region where $\boldsymbol{\omega} = \mathbf{0}$.

When the vorticity is concentrated in a vortex filament of strength Γ and a length of dl, Eq (4.11) reduces to the standard form of the *Biot-Savart law*

$$\boldsymbol{V} = \frac{1}{4\pi} \int_l \frac{1}{r^3} \Gamma \, \boldsymbol{dl} \times \boldsymbol{r} \tag{4.12}$$

where $\boldsymbol{r}$ denotes distance from vortex filament of infinitesimal length $\boldsymbol{dl}$ to a point of interest.

In reality the surface is to be covered not with concentrated line vortices but with a vorticity distributed continuously. So, if instead of discrete filament the vorticity field exist, then (including additionally the compressibility effects) the total velocity vector is sometimes (see Lan, 1989) written as

$$\boldsymbol{V}(\boldsymbol{R}) = \frac{\beta^2}{4\pi} \iint_S \frac{\boldsymbol{\omega}(\boldsymbol{r}_1) \times (\boldsymbol{r} - \boldsymbol{r}_1)}{\boldsymbol{r}_\beta^3} \, dS \tag{4.13}$$

where

$$\boldsymbol{r}_\beta^2 = (x - x_1)^2 + \beta^2 \Big[(y - y_1)^2 + (z - z_1)^2 \Big] \tag{4.14}$$

$\boldsymbol{\omega}(\boldsymbol{r}_1)$ is the vorticity vector at $\boldsymbol{r}_1 = x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}$, and β is the Prandtl-Glauert factor given by Eq (3.28).

The other useful form of the Biot-Savart law can be directly obtain from Eq (4.13) in the following form

$$\boldsymbol{V}(x, y, z) = \frac{\beta^2}{4\pi} \iint_S \frac{1}{r_\beta^3} \det \begin{bmatrix} \boldsymbol{e}_\xi & \boldsymbol{e}_\eta & \boldsymbol{e}_\zeta \\ \omega_\xi & \omega_\eta & \omega_\zeta \\ x - \xi & y - \eta & z - \zeta \end{bmatrix} dS \tag{4.15}$$

where $\{\xi, \eta, \zeta\}$ is a local orthogonal frame of reference, and $dS = d\xi d\eta$ denotes an infinitesimal surface element.

Biot-Savart law in the form of Eq (4.15) was used by Kandil et al. (1984) for parameters $\beta = 1$ and $\zeta = 0$ where r_β was taken as

$$r_\beta = r = \Big[(x - \xi)^2 + (y - \eta)^2 + z^2 \Big]^{\frac{3}{2}} \tag{4.16}$$

4.5. Kelvin and Helmholtz theorems

Assuming an inviscid and incompressible flow we may obtain a very useful mathematical tool known as *Kelvin's circulation theorem*. As a matter of fact, consider the circulation around a *fluid curve* in such a flow and calculate the time rate of the circulation

$$\frac{D\Gamma}{Dt} = \frac{D}{Dt}\int_C V\,dl = \int_C \boldsymbol{a}\,dl \tag{4.17}$$

Substituting the acceleration $\boldsymbol{a}$ obtained from the Euler equation (3.2) into Eq (4.17) yields a constant circulation

$$\frac{D\Gamma}{Dt} = 0 \tag{4.18}$$

Eq (4.18) expresses the theorem of conservation of circulation. This theorem is thus a form of angular momentum conservation and can be used to determine the streamwise strength of vorticity shed into wake: the circulation Γ around a fluid curve enclosing the wing and its wake is conserved.

Consider now any infinitesimal surface element $\boldsymbol{n}dS$ which moves with the fluid, and denote by $\boldsymbol{\omega}$ the vorticity vector, and by $\boldsymbol{n}$ the unit vector normal to the surface S. Then, by virtue of the Stokes theorem, Eq (4.18) may be put into the form

$$\frac{D}{Dt}(\omega \cdot \boldsymbol{n}dS) = 0 \tag{4.19}$$

which is exactly the Helmholtz theorem. Thus we see that the Kelvin theorem and the Helmholtz theorem are identical.

4.6. Linearized Bernoulli equation

By binomial expansion of Eq (3.10) (for details see Liepmann and Roshko, 1957) with taking into account Eq (3.24), it is possible to obtain the approximate (of second order) pressure formula

$$C_p = -\frac{2}{U_\infty^2}\left[uU_\infty + \frac{\partial\varphi}{\partial t} + (1 - M_\infty^2)u^2 + v^2 + w^2\right] \tag{4.20}$$

in which cubes and higher order powers of perturbation velocities are neglected and, on the base of Eq $(3.13)_3$ there was taken that $\partial\varphi/\partial t = \partial\phi/\partial t$. In the formula (4.20) u, v, w denote the components of local flow velocity seen from the unmovable frame of reference.

If we retain the first order terms only, Eq (4.20) gives the relation, very frequently used in the panel method codes, namely

$$C_p = -\frac{2}{U_\infty^2}\Big[uU_\infty + \frac{\partial\varphi}{\partial t}\Big] \tag{4.21}$$

Eq (4.21) can be easy obtained from Eq (3.15), if we assume that $V = U_\infty + u$. It is the case of two-dimensional flow with the first-order perturbation components to be retained only.

The special case of interest is the flow around a thin plate of infinite aspect ratio. In this case we observe the velocity jump crossing the plate from the upper to lower side. The local tangential components of velocity disturbances u at an arbitrary point along the plate chord, marked as u^U and u^L, respectively, to the upper and lower sides of the plate. We have $u^U = -u^L = u$ and $\varphi^U = -\varphi^L = \varphi$. Putting these relations into Eq (4.21) for the streamlines on upper and lower sides of the plate, and subtracting yields

$$C_p = \frac{\Delta p}{\frac{1}{2}\rho_\infty U_\infty^2} = \frac{p^L - p^U}{\frac{1}{2}\rho_\infty U_\infty^2} = \frac{4}{U_\infty^2}\Big(U_\infty\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\Big)\Delta\varphi \tag{4.22}$$

hence, we have

$$\Delta p = 2\rho_\infty\Big(U_\infty\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\Big)\Delta\varphi \tag{4.23}$$

This formula is often used in unsteady flight dynamics and aeroelasticity to compute the pressure acting upon a vibrating plate (which is used as the wing model).

4.7. Formulae for calculating aerodynamic load

An isolated 2D airfoil in an incompressible inviscid flow feels a force per unit width

$$\boldsymbol{F} = \rho_\infty \boldsymbol{U}_\infty \times \boldsymbol{\Gamma} \tag{4.24}$$

where $\boldsymbol{\Gamma}$ is the net circulation around the airfoil. Eq (4.24) represents the well known Kutta-Joukowski theorem. But there are also another useful formulae for calculating the aerodynamic load.

Denoting by $\boldsymbol{w}$ the *perturbation mass flux vector*

$$\boldsymbol{w} = \rho_\infty[\beta^2\varphi_x, \varphi_y, \varphi_z] \tag{4.25}$$

we may obtain the so-called second order pressure formula

$$p_2 = p_\infty - \left(\rho_\infty \boldsymbol{U}_\infty \boldsymbol{v} + \frac{1}{2}\boldsymbol{v}\boldsymbol{w}\right) \tag{4.26}$$

where $\boldsymbol{v}$ denotes the perturbation velocity. Eq (4.26) agrees with the isentropic formula (3.10) (for $\phi_t \equiv 0$) to the first order in perturbation quantities. Of great importance is the fact that the Eq (4.26) produces consistent force calculations for arbitrary configurations when we define the force in a usual way, i.e.

$$\boldsymbol{F} = \iint_S [V(\boldsymbol{W}\boldsymbol{n}) + p\boldsymbol{n}]\, dS \tag{4.27}$$

where $\boldsymbol{W}$ is the total mass flux vector, defined as follows

$$\boldsymbol{W} = \rho_\infty \boldsymbol{U}_\infty + \boldsymbol{w} \tag{4.28}$$

whilst $\boldsymbol{w}$ is given by Eq (4.25).

Eq (4.27) implies that $\boldsymbol{F}$ is zero when the surface S encloses the fluid only, hence, the momentum is conserved exactly and the force on a given surface may be computed on any enclosing surface.

5. Boundary conditions

5.1. Boundary conditions on the body surface

The body surface S is usually assumed to be impermeable. Hence, the boundary condition (no-penetration condition) at the point S is

$$(V - \boldsymbol{V}_B)\boldsymbol{n} = 0 \qquad \text{or} \qquad V\boldsymbol{n} = \boldsymbol{V}_B\boldsymbol{n} \tag{5.1}$$

where V is the absolute velocity of the fluid particle, and $\boldsymbol{V}_B$ is the absolute velocity of the point belonging to the body.

Consider now a body surface, each element of which is moving relative to the fluid (e.g., such a surface is that of a deformable body). This surface may be described by the equation

$$S(\boldsymbol{r}, t) = 0 \tag{5.2}$$

where $S(\boldsymbol{r}, t)$ is a scalar function of the position and time. The normal to the surface at any point is given by

$$\boldsymbol{n} = \frac{\operatorname{grad} S}{|\operatorname{grad} S|} \tag{5.3}$$

Differentiating Eq (5.2) with respect to time, taking into account Eqs (5.1) and (5.3), the boundary condition on the body surface becomes

$$\frac{DS}{Dt} = \frac{\partial S}{\partial t} + \boldsymbol{V} \cdot \operatorname{grad} S = 0 \tag{5.4}$$

A definition of impermeability and pressure appropriate to Eq (3.27) is an open subject. Johnson et al. (1980), made the mathematically natural choice of zero normal mass flux and the second order pressure formula (see Eq (4.26)). Assume that $\boldsymbol{W}$ is equal to $\rho \boldsymbol{V}$ (where $\boldsymbol{W}$ is defined by Eq (4.28)); therefore, the impermeable surface boundary condition can be expressed by

$$\boldsymbol{W} \cdot \boldsymbol{n} = 0 \tag{5.5}$$

Eq (3.27), rewritten as

$$\nabla w = 0 \tag{5.6}$$

expresses the principle of conservation of mass, and then Eq (5.5) guarantees that even if the configuration is such that the assumptions used to derive Eq (3.27) are violated locally, there is still no net production of fluid at the boundary surfaces.

5.2. Kutta-Joukowski condition

Boundary condition for any unviscid theory is simply in formulation, Eq (5.1). However, viscosity changes the flow field considerably, and any unviscid theory, which does not take account of these changes, is completely unrealistic.

At a sharp trailing edge the local flow is controlled by the action of viscous stresses; this determines the circulation around the airfoil and hence the lift. In an inviscid flow the behavior at the trailing edge is no longer controlled, and so a non-unique solution can be found. Therefore, to describe the real viscous flow accurately, an additional condition is necessary. This condition is known as the *Kutta-Joukowski* one which implies that a zero pressure jump at the trailing edge is imposed to give a physically realistic flow. This was traditionally formulated by requiring that the circulation density at the trailing edge (TE) be equal to zero, i.e.

$$\gamma_{TE} = 0 \tag{5.7}$$

The most popular formulation of the Kutta-Joukowski condition follows: the flow leaves the trailing edge of a sharp-tailed airfoil smoothly, i.e., the velocity is finite there. From this postulate one can derive four corollaries (see Moran, 1984) which are often even more useful, e.g., the streamline that leaves a sharp trailing edge is an extension of the bisector of the trailing edge angle.

However, the Kutta-Joukowski condition is limited to sharp-tailed airfoils. which do not exist in the strict sense. The usual procedure is to assume that there emanates from the trailing edge a wake so thin that it cannot support a pressure difference. On sharp edges (leading. side and trailing edge)

$$\Delta C_p = 0 \tag{5.8}$$

For an unsteady flow, a much more careful analysis is required. An extension of the Kutta-Joukowski condition to such a flow was suggested by Giesing and Maskell (cf Morino and Tseng, 1990): for varying bound circulation the stagnation streamline is an extension of one of the two tangents to the airfoil at the sharp trailing edge.

The most severe criticism of the Giesing and Maskell condition centers on the fact that this unsteady condition does not reduce to the classical steady one. Nevertheless, for high reduced frequencies, the experimental evidence (cf Poling and Telionis, 1986) supports the Giesing and Maskell criterion.

5.3. Boundary conditions on the wake

From a physical point of view, the wake is a region where the vorticity generated over the body surfaces is transported. Because the Kelvin-Helmholtz theorem, Eq (4.19), is not applicable in this region, the flow is not necessarily potential for the points of the wake. Hence, in general, we must assume that the wake is a surface of velocity discontinuity. In potential aerodynamics this region is assumed to have zero thickness.

Wakes are defined to be the surfaces of discontinuity on which

$$\Delta(\boldsymbol{V}\boldsymbol{n}) = 0 \tag{5.9}$$

where Δ denotes increament (of normal velocity). Applying the principles of conservation of mass and momentum across a surface of discontinuity (cf Morino and Tseng, 1990), one obtains

$$\Delta p = 0 \tag{5.10}$$

For a flow that is potential on both sides of the wake surface, the condition (5.9) implies

$$\Delta \frac{\partial \phi}{\partial n} = 0 \tag{5.11}$$

Detailed discussion on the boundary conditions for the wake was presented by Morino (1993).

5.4. Boundary conditions far from the body

In a classical approach it is assumed that the flow disturbance, due to the body motion through the fluid, should diminish at infinity (far from the aircraft)

$$\lim_{R\to\infty} \nabla\varphi = 0 \tag{5.12}$$

where φ is the perturbation potential (see Eq (3.24)), and R is the distance from the fixed origin in or near the body.

However, the arguments presented by Ward (1955) show that the linearized theory can not give accurate information about the flow at infinity. In spite of defficiency of linearized solutions, the knowledge of their general behaviour at infinity is useful, particularly for the proof of uniqueness. This behaviour is different for subsonic and supersonic flows.

In a subsonic flow, when there are no vortex sheets, $|R^3\boldsymbol{v}|$ is bounded as $R \to \infty$ ($\boldsymbol{v}$ is the perturbation velocity). When there are vortex sheets, if r is a distance from the nearest point of the body and the vortex sheets, then $|r^2\boldsymbol{v}|$ is bounded as $r \to \infty$, and $|R^2U_\infty\boldsymbol{v}|$ is bounded as $R \to \infty$.

In a supersonic flow, $\boldsymbol{v} = \boldsymbol{0}$ at all points upstream from the characteristic surface which bounds the influence domain of the body.

6. Conclusion

The paper presents foundations of potential flows and is prepared as a common base for any next papers devoted to the review of selected modified panel methods and their application to complex flow field calculations. Fundamental relationships and boundary conditions can have different, although equivalent forms. Sometimes, going through original papers, one can meet a non-standard formulation of basic equation and related conditions, which can make difficult to follow such papers. Authors hope that this paper will be helpful in such a case.

Summing up, this paper may be a useful guide for the beginners in the field of panel methods as well as for the readers who are well familiar with these methods, as at hand advisor.

References

1. ANDERSON J.D., JR., 1990, *Modern Compressible Flow with Historical Perpective*, McGraw-Hill Inter. Eds., 2nd edition

2. ASHLEY H., LANDAHL M., 1965, *Aerodynamics of Wings and Bodies*, Addison-Wesley Publ.Co., Reading, MA

3. CANTALOUBE B., REHBACH C., 1986, Calculation of the Integrals of the Singularities Method, *Rech. Aérosp.*, 1, 15-22

4. GEISSLER W., 1978, Nonlinear Unsteady Potential Flow Calculations for 3-D Oscillating Wings, *AIAA J.*, **16**, 11, 1168-1174

5. GORAJ Z., PIETRUCHA J., 1995a, Classical Panel Methods - a Routine Tool for Aerodynamic Calculation of Complex Configurations: from Concepts to Codes, *J. Theor. Appl. Mech.*, **33**, 4, 843-878

6. GORAJ Z., PIETRUCHA J., 1995b, Physical Modelling of Important Flow Phenomena in Aircraft Aerodynamics, *Proc. Inst. Aviation*, 140, 13-47

7. HIRSCH C., 1988, *Numerical Computation of Internal and External Flows*, **1**, J.Wiley & Sons

8. JOHNSON F.T. ET AL., 1980, 3-D Flow over Wings with Leading-Edge Vortex Separation, *AIAA J.*, **18**, 4, 367-380

9. KANDIL O.A. ET AL., 1984, A Nonlinear Hybrid Vortex Method for Wings at Large Angle of Attack, *AIAA J.*, **22**, 3, 329-336

10. KATZ J., PLOTKIN A., 1991, *Low-Speed Aerodynamics - from Wing Theory to Panel Methods*, McGraw-Hill, Inc., New York

11. KELLOG O.G., 1967, *Foundations of Potential Theory*, Springer-Verlag, Chap. VIII, Sec. 4

12. KUBRYŃSKI K., 1993, Panel Methods in Aerodynamics, (in Polish), *Sci. Papers IMDO of the Wrocław Tech. Univ.*, **59**, 26, 19-30

13. LAN C.E. ET AL., 1989, Calculation of High Angle of Attack Aerodynamics of Fighter Configurations, *7th Appl. Aerod. Confer.*, 205-216

14. LIEPMANN H., ROSHKO A., 1957, *Elements of Gasdynamics*, J.Wiley & Sons

15. MORAN J., 1984, *An Introduction to Theoretical and Computational Aerodynamics*, J.Wiley & Sons

16. MORINO L., TSENG K., 1990, A General Theory of Unsteady Compressible Potential Flows with Applications to Aeroplanes and Rotors, in *"Boundary Methods in Nonlinear Fluid Dynamics"*, Edit. P.K. Banerjee, L. Morino, Elsevier Appl. Sci., London

17. MORINO L., 1993, Boundary Integral Equations in Aerodynamics, *Appl. Mech. Rev.*, **46**, 8, 445-466

18. POLING D.R., TELIONIS D.P., 1986, The Response of Airfoils to Periodic Disturbances - the Unsteady Kutta Condition, *AIAA J.*, **24**, 2, 193-199

19. WARD G.N., 1955, *Linearized Theory of Steady High-Speed Flow*, Cambridge University Press

Podstawowe równania i związki matematyczne mechaniki płynów dla zmodyfikowanych metod panelowych

Streszczenie

Klasyczne metody panelowe wymagają pewnych modyfikacji, aby mogły uwzględniać ważne cechy przepływów rzeczywistych. Zbiór tych cech obejmuje szerokie widmo różnych efektów (pojawiających się w wielu zagadnieniach mechaniki płynów), a ich opis matematyczny jest w pewnych sytuacjach niestandardowy. Związki matematyczne i warunki brzegowe, zarówno na powierzchni bryły, jak na śladzie wirowym i w nieskończoności, mogą mieć bardzo różne, choć równoważne formy. Praca omawia podstawy matematyczne przepływów potencjalnych, zarówno ustalonych jak i nieustalonych. Oprócz sformułowania klasycznego z wykorzystaniem potencjału prędkości omawia również sformułowania niestandardowe, między innymi zastosowanie potencjału przyspieszeń. Jest napisana przy założeniu, że będzie stanowić podstawę opisu matematycznego dla innych prac z dziedziny metod panelowych, zarówno o charakterze przeglądowym, jak i dla prac oryginalnych w zakresie zastosowań do obliczeń przepływów złożonych.

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