

## DERIVATION OF THE KINETIC EQUATION FOR DISPERSED PARTICLES IN TURBULENT FLOWS

JACEK POZORSKI

*Institute of Fluid-Flow Machinery, Polish Academy of Sciences, Gdańsk*  
*e-mail: jp@galia.imp.pg.gda.pl*

An alternative way to obtain the kinetic equation for dispersed particles in turbulent flows is presented. The method uses some tools developed within the theory of the stochastic differential equations. They are presented first in an outline, including the cumulant expansion of linear stochastic equations as well as the approach to non-linear equations. Then, the general method is applied to the equation of motion of small particles in turbulent flows. The governing kinetic equation for particles is derived; it involves the correlations of the fluctuating fluid velocity along the particle trajectories.

*Key words:* turbulence, particle dispersion, kinetic equation

### 1. Introduction

The governing equations of fluid dynamics, representing principles of mass, momentum and energy conservation, are usually derived in the integral form using the balance of a corresponding physical quantity in some finite volume. Then, the equivalent differential form of the equations is obtained. However, there is an alternative way of proceeding, supposing that one starts the description at a lower, more fundamental level. This means that the dynamics of individual fluid molecules is considered. It is governed by the kinetic equation, i.e. a transport equation of the probability density function (PDF) of position and velocity in the phase space. Various kinetic models of gases and liquids have been proposed in the literature (see, e.g., Résibois and De Leener, 1977). They include the classical Boltzmann kinetic equation (by far the simplest one) and the Enskog equation where the Boltzmann model is improved by treating interacting molecules as hard spheres and not the material points, thus enabling better description of dense fluids. So, starting from a kinetic equation, and defining mean variables like velocity or pressure in terms of the

appropriate moments of the PDF, the "classical" equations of fluid dynamics can be derived using the ensemble-averaging (see e.g. Huang, 1963). We can make an attempt to look at such a procedure from a more general standpoint: if only one manages to propose a sound model that describes the behaviour of a given system at a microscopic level, the corresponding formulae governing the macroscopic description are readily found. For the case of the Boltzmann gas, elastic collision and molecular chaos (*Stosszahlansatz*) hypotheses count among the assumptions of the microscopic model.

At this moment, it may be worthwhile to mention that a similar procedure based on the probability density function approach has recently been proposed for turbulent (one-phase) flow computation (see Pope, 1994 for a review). Here, the governing PDF equation for the fluid instantaneous velocity and possibly also turbulent energy dissipation rate can be shown to be derived from the underlying physical model of fluid particle behaviour (Minier and Pozorski, 1997).

The basic idea of the paper is to perform a somewhat analogous reasoning for the case of two-phase turbulent flows. In the following, we will consider particulate flows with the dispersed phase present in the form of small spherical solid particles (or liquid droplets) suspended in a gaseous or liquid carrier phase. The study of such flows is both of theoretical interest and of considerable practical importance, to mention only particle separators, hydraulic and pneumatic conveyors, combustion chambers, turbine blade cascades in a wet steam flow regime, nuclear engineering, dispersion of pollutants in the atmosphere. The computer codes written for solving two-phase flow problems in industrial and environmental applications usually solve the discretised version of the equations formulated in the Eulerian way: both phases are treated as continuous, interpenetrating fluids. Hence, the conservation equations are written also for the dispersed phase. These equations are, however, more complicated than their fluid equivalents because of the unclosed terms that describe the influence of turbulence on particles. To state our problem precisely: in the paper we limit our attention to dilute suspensions with no interparticle collisions and to the so-called one-way coupling case where the presence of particles does not modify the characteristics of the carrier phase turbulence. Closure proposals for two-phase turbulent flows are a subject of intense research, both in the Eulerian (two-fluid) formalism as well as in the Lagrangian (trajectory) approach.

A recent idea of Reeks (1991), (1992) has been to establish the corresponding kinetic equation for dispersed particles in a turbulent flow and to derive the conservation equations next. Given the fluid velocity (decomposed into its mean  $\langle U_f \rangle$  and fluctuation  $u_f$ ), the starting point of the reasoning is the

particle equation of motion. In the absence of external forces, the equation is written in the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{v} \quad \frac{d\mathbf{v}}{dt} = -\beta\mathbf{v} + \beta\langle U_f \rangle + \mathbf{f} \quad (1.1)$$

where  $\beta = 1/\tau_p$  is the particle inertia parameter ( $\tau_p$  stands for the relaxation time of particle momentum) and  $\mathbf{f} = \beta\mathbf{u}_f$  is a random part of the force (viscous drag) acting on the particle.

Like in the case of fluid, the kinetic equation for particles is an evolution equation of the probability density function  $P(\mathbf{x}, \mathbf{v}, t)$  in the phase space of particle position  $\mathbf{x}$  and velocity  $\mathbf{v}$ . It was first derived by Reeks (1991) for particles in homogeneous turbulence. The Random Galilean Transformation (RGT) was used there. Then, in the subsequent paper of Reeks (1992), the equation has been obtained for the general case by methods developed within the Lagrangian History Direct Interaction Approximation (LHDIA), first proposed by Kraichnan; see also McComb (1990) for a comprehensive description. Hyland (1995) presented his alternative derivation of the kinetic equation for dispersed particles, based on results from advanced functional calculus. The resulting equation writes

$$\begin{aligned} & \left( \frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i} - \frac{\partial}{\partial v_i} \beta v_i \right) P(\mathbf{x}, \mathbf{v}, t) = \\ & = \frac{\partial}{\partial v_i} \left[ \frac{\partial}{\partial v_j} \mu_{ij}(\mathbf{x}, \mathbf{v}, t) + \frac{\partial}{\partial x_j} \lambda_{ij}(\mathbf{x}, \mathbf{v}, t) + \gamma_i(\mathbf{x}, \mathbf{v}, t) \right] P(\mathbf{x}, \mathbf{v}, t) \end{aligned} \quad (1.2)$$

Here,  $\mu$  and  $\lambda$  are diffusion tensors in the phase space; they depend on  $\beta$  and on the statistics of the random force  $\mathbf{f}$ . The other term,  $\gamma$ , reflects the inhomogeneities of the fluid turbulence.

In the Boltzmann equation for gas, the only mechanism to change the velocity of molecules is via the collision term and the interactions between molecules are supposed to be instantaneous (with no history). On the other hand, in the case of the kinetic equation for particles in turbulent flow, the history terms are present. Roughly speaking, these terms are (as we will see below) the time integrals over the correlations of fluid velocity along the particle trajectory (fluid "seen" by particles).

In this paper, we present a different way to obtain the kinetic equation (1.2). The method uses some tools developed within the theory of the stochastic differential equations. It is believed that this alternative derivation will be, both mathematically and physically, more easily understandable, as it avoids the explicit introduction of Kraichnan's (1977) LHDIA. The presented method is based on the cumulant expansion of the governing stochastic

differential equation (SDE) which is the particle equation of motion with a random term.

The mathematical formalism relating to stochastic differential equations presented below is taken basically from Van Kampen (1992).

## 2. General method

### 2.1. Characteristic functions and cumulants

As we want the presentation of the mathematical method to be more comprehensive, we first recall a handful of definitions that will be used in the following. Moments  $m$  and central moments  $\mu$  of a random variable  $X$ , respectively, are denoted by

$$m_j = \langle X^j \rangle \quad \mu_j = \langle (X - \langle X \rangle)^j \rangle = \langle \langle X^j \rangle \rangle \quad j = 1, 2, \dots$$

As often used, angle brackets indicate the ensemble mean whereas double angle brackets are a shorthand notation for moments about the mean value  $\langle X \rangle$ . The characteristic function  $\varphi$  of a random variable  $X$  is defined as

$$\varphi(\omega) = \langle \exp(i\omega X) \rangle \quad \omega \in R$$

The expansion of  $\varphi$  into a Taylor series reads

$$\varphi(\omega) = 1 + \underbrace{\sum_{j=1}^{\infty} \frac{m_j}{j!} (i\omega)^j}_{\equiv Z} \quad (2.1)$$

Because of  $m_j = d^j \varphi(0)/d\omega^j$  ( $j = 1, 2, \dots$ ),  $\varphi$  is also called the moment generating function. Next, consider the second characteristic function  $\psi$  defined by

$$\psi(\omega) = \ln \varphi(\omega) = \ln(1 + Z) = \sum_{k=1}^{\infty} \frac{\lambda_k}{k!} (i\omega)^k \quad (2.2)$$

where  $\lambda_k$  are called cumulants (semi-invariants). We expand  $\ln(1 + Z)$  into a Taylor series and compare the coefficients at the respective powers,  $(i\omega)^k$ , in Eq (2.2) so as to find the relations between the moments and the semi-invariants; the first ones are

$$\begin{aligned} \lambda_1 &= m_1 & \lambda_2 &= m_2 - m_1^2 = \mu_2 \\ \lambda_3 &= \mu_3 & \lambda_4 &= \mu_4 - 3\mu_2^2 \end{aligned}$$

Hence, substituting into Eq (2.2), we have the following cumulant expansion

$$\ln \varphi(\omega) = \sum_{j=1}^{\infty} \frac{\lambda_j}{j!} \omega^j = i m_1 \omega - \frac{1}{2} \mu_2 \omega^2 + \dots \tag{2.3}$$

It can be demonstrated that in the case of the Gaussian random variable, the expansion reduces to the two terms written above explicitly. Moreover, the name "semi-invariants" has its origin in the following property of the second characteristic function  $\psi$ . If we consider a "shifted" random variable  $Y$ , defined by the transformation  $Y = X + b$ , then it is easily shown that  $\psi_Y(\omega) = i b \omega + \psi_X(\omega)$ . This means that all cumulants, except the first, are invariants of the transformation.

### 2.2. Characteristic functional

Let  $Y(t)$  be a stochastic process (a random function of an argument  $t$ ); the characteristic (moment generating) functional of  $Y$  is introduced by

$$G[k] = \langle \exp\left(i \int_{-\infty}^{\infty} k(t) Y(t) dt\right) \rangle$$

where  $k$  is an arbitrary test function. The characteristic functional determines all moments and cumulants; it is a generalisation of the notion of the characteristic function in the sense that for  $k(t) = \omega \delta(t - t_0)$ ,  $G[k]$  becomes the characteristic function of a random variable  $Y(t_0)$ . Similarly, the cumulant expansion of  $G$  generalises Eq (2.3) and writes (cf Monin and Yaglom, 1971)

$$\ln G[k] = i \int k(t_1) \langle Y(t_1) \rangle dt_1 + \frac{i^2}{2!} \int k(t_1) k(t_2) \langle \langle Y(t_1) Y(t_2) \rangle \rangle dt_1 dt_2 + \dots \tag{2.4}$$

As in the definition of  $\mu$ , double angle brackets stand for the central moments; for example, the symbol

$$\langle \langle ab \rangle \rangle = \langle (a - \langle a \rangle)(b - \langle b \rangle) \rangle$$

denotes the covariance of two random variables  $a, b$ . We also note that the cumulant expansion (2.4) of a Gaussian random function is limited to the exactly two terms because all cumulants of the order three and higher disappear.

### 2.3. Stochastic differential equations

In general, a stochastic differential equation (SDE) can be written in the form

$$\frac{d\mathbf{u}}{dt} = \mathbf{F}(\mathbf{u}, t, \mathbf{Y}(t)) \quad (2.5)$$

where  $\mathbf{u}$ ,  $\mathbf{F}$  are vectors and  $\mathbf{Y}$  is a stochastic process of given statistical properties. A solution of the above SDE is also a stochastic process. For a particular realisation  $\mathbf{y}(t)$  of  $\mathbf{Y}(t)$ , Eq (2.5) becomes an ordinary differential equation (ODE) and its solution can be found. Solving an SDE usually means finding the stochastic properties of  $\mathbf{u}(t)$  like the mean value  $\langle \mathbf{u}(t) \rangle$  and the covariance matrix  $\langle \langle u_i(t) u_j(t) \rangle \rangle$  at every instant  $t$  as well as the autocorrelation  $\langle \langle u_i(t) u_j(t + \tau) \rangle \rangle$  of the process.

### 2.4. Cumulant expansion of linear SDE

Let us consider a stochastic differential equation of the type

$$\frac{d\mathbf{u}}{dt} = [A_0 + \alpha A_1(t)]\mathbf{u} \quad (2.6)$$

where  $A_0$ ,  $A_1$  are linear operators (they can be thought of as matrices or differential operators);  $A_0$  is deterministic while  $A_1$  is random with a finite autocorrelation time  $\tau_c$ ;  $\alpha$  represents the level of fluctuations; it is supposed that  $\alpha\tau_c \ll 1$ ;  $\mathbf{u}$  is a vector. Using the substitution

$$\mathbf{u}(t) = e^{tA_0} \mathbf{v}(t)$$

our SDE is transformed to

$$\frac{d\mathbf{v}}{dt} = \alpha \underbrace{e^{-tA_0} A_1(t) e^{tA_0}}_{\equiv S(t)} \mathbf{v} \quad (2.7)$$

The symbol  $e^B$ , where  $B$  is an operator, is formally defined as the sum

$$e^B = \sum_{k=0}^{\infty} \frac{B^k}{k!}$$

From

$$d\mathbf{v} = \alpha S(t) \mathbf{v} dt$$

with  $v(0) = v_0$  we have by a direct substitution

$$\begin{aligned}
 v(t) &= v_0 + \alpha \int_0^t dt' S(t') v(t') = v_0 + \alpha \int_0^t dt' S(t') \left[ v_0 + \alpha \int_0^{t'} dt'' S(t'') v(t'') \right] = \\
 &= \left[ 1 + \alpha \int_0^t dt' S(t') + \alpha^2 \int_0^t dt' \int_0^{t'} dt'' S(t') S(t'') + \dots \right] v_0 = \quad (2.8) \\
 &= \left[ \exp \left( \alpha \int_0^t S(t') dt' \right) \right] v_0
 \end{aligned}$$

In the above equation the time-ordering symbol  $[\cdot]$  has been introduced. It rearranges all operators within the brackets in chronological order, from the right to the left. To better understand its meaning, we invoke the following two identities

$$\left[ \prod_{i=1}^n B(t_i) \right] = B(t_n) B(t_{n-1}) \dots B(t_1) \quad (2.9)$$

$$\frac{1}{2} \left[ \int_0^t \int_0^t B(t') B(t'') dt' dt'' \right] = \int_0^t dt' B(t') \int_0^{t'} dt'' B(t'')$$

where  $B$  is an operator and  $t_1 < t_2 < \dots < t_n$ . We take the ensemble average of Eq (2.8)

$$\langle v(t) \rangle = \left[ \left\langle \exp \left( \alpha \int_0^t S(t') dt' \right) \right\rangle \right] v_0 \quad (2.10)$$

Applying the cumulant expansion, Eq (2.4), with  $k(t) = -i$ , to the above equation, we obtain

$$\langle v(t) \rangle = \left[ \exp \left\{ \alpha \int_0^t \langle S(t') \rangle dt' + \frac{1}{2} \alpha^2 \int_0^t \int_0^t \langle \langle S(t') S(t'') \rangle \rangle dt' dt'' + \dots \right\} \right] v_0 \quad (2.11)$$

The expansion limited to the first order in  $\alpha$  writes

$$\langle v(t) \rangle = \left[ \exp \left\{ \alpha \int_0^t \langle S(t') \rangle dt' \right\} \right] v_0 \quad (2.12)$$

This represents the solution of the following deterministic ordinary differential equation (ODE) with no effect of random fluctuations

$$\frac{d\langle v \rangle}{dt} = \alpha \langle S(t) \rangle \langle v(t) \rangle \quad (2.13)$$

More interesting is the second-order expansion

$$\begin{aligned} \langle v(t) \rangle &= \left[ \exp \left\{ \alpha \int_0^t \langle S(t') \rangle dt' + \alpha^2 \int_0^t dt' \underbrace{\int_0^{t'} \langle \langle S(t') S(t'') \rangle \rangle}_{\equiv K(t')} dt'' \right\} \right] v_0 = \\ &= \left[ \exp \left\{ \int_0^t \left[ \alpha \langle S(t') \rangle + \alpha^2 K(t') \right] dt' \right\} \right] v_0 \end{aligned} \quad (2.14)$$

It is the solution of

$$\frac{d\langle v \rangle}{dt} = \left\{ \alpha \langle S(t) \rangle + \alpha^2 K(t) \right\} \langle v(t) \rangle \quad (2.15)$$

The above can be thought of as a "renormalized" form of the initial equation, Eq (2.7), where the effect of fluctuations appears in the form of an additional deterministic operator; see Eq (2.13) for comparison. The original representation in terms of  $\mathbf{u}$  writes

$$\frac{d\langle \mathbf{u}(t) \rangle}{dt} = \left[ A_0 + \alpha \langle A_1(t) \rangle + \alpha^2 \int_0^t \langle \langle A_1(t) e^{\tau A_0} A_1(t-\tau) \rangle \rangle e^{-\tau A_0} d\tau \right] \langle \mathbf{u}(t) \rangle \quad (2.16)$$

## 2.5. Nonlinear SDE

Let the function  $\mathbf{y}(t)$  be a single realisation of a stationary stochastic process  $\mathbf{Y}(t)$ . Consider the deterministic ODE

$$\frac{du_i}{dt} = F_i(\mathbf{u}, t, \mathbf{y}(t)) \quad (2.17)$$

where the subscript  $i$  denotes any component of the vector  $\mathbf{u}$ . The equation gives the trajectory of a particular realisation of the process  $\mathbf{u}$  in the phase space. Density of the flow in  $\mathbf{u}$ -space satisfies the Liouville equation

$$\frac{\partial \rho(\mathbf{u}, t)}{\partial t} = - \frac{\partial [F_i(\mathbf{u}, t, \mathbf{y}) \rho]}{\partial u_i} \quad (2.18)$$



where summation over  $i$  is implied. When  $\mathbf{Y}$  is substituted for  $\mathbf{y}$ , the above ODE becomes a linear SDE for  $\rho$ . At any value of  $t$ , the density of the flow in the phase space of  $\mathbf{u}$ , averaged over all possible realisations  $\mathbf{y}(t)$ , is equal (cf Van Kampen, 1992) to the probability density of  $\mathbf{u}$

$$\langle \rho(\mathbf{u}, t) \rangle = P(\mathbf{u}, t) \tag{2.19}$$

Moreover, the flow density in  $\mathbf{u}$ -space verifies

$$\rho(\mathbf{u}, t) = \rho(\mathbf{u}^{-t}, 0) \frac{D\mathbf{u}^{-t}}{D\mathbf{u}} \tag{2.20}$$

where  $D\mathbf{u}^\tau/D\mathbf{u}$  stands for the Jacobian of the transformation  $\mathbf{u} = \mathbf{u}(t) - \mathbf{u}^\tau = \mathbf{u}(t + \tau)$ .

The superscript notation represents a rather important point and is used as follows: if  $\mathbf{u}$  denotes a value (at time  $t$ ) of a particular realisation of the stochastic process, then  $\mathbf{u}^\tau$  stands for its value at the instant  $t + \tau$ . In particular,  $\mathbf{u}^{-t}$  stands for the value at the initial time. More generally, suppose that  $f$  is any function defined along the trajectory of  $\mathbf{u}$ . Then  $f(\mathbf{u}^\tau, t + \tau)$  denotes a value of the function at time  $t + \tau$  on the trajectory that passed by a particular value of  $\mathbf{u}$  at  $t$ . In the LHDIA notation, we would write

$$f(\mathbf{u}^\tau, t + \tau) \equiv f(\mathbf{u}, t | t + \tau) \tag{2.21}$$

$t$  is called the labeling time and  $t + \tau$  the measuring time. This generalized notation contains both the Eulerian (for  $\tau = 0$ ) and Lagrangian descriptions.

Let us suppose that  $F_i$  can be split into two parts

$$F_i(\mathbf{u}, t, \mathbf{Y}) = F_i^0(\mathbf{u}) + \alpha F_i^1(\mathbf{u}, t) \tag{2.22}$$

where  $F_i^0$  is stationary and not stochastic while  $F_i^1$  is random and of zero mean; the explicit dependence of  $F_i^1$  on  $\mathbf{Y}(t)$  will henceforth be skipped in the notation. In this case, the Liouville equation reads

$$\frac{\partial \rho(\mathbf{u}, t)}{\partial t} = \left[ -\frac{\partial F_i^0(\mathbf{u})}{\partial u_i} - \alpha \frac{\partial F_i^1(\mathbf{u}, t)}{\partial u_i} \right] \rho = \left[ A_0(\mathbf{u}) + \alpha A_1(\mathbf{u}, t) \right] \rho \tag{2.23}$$

where the operators  $A_0$  and  $A_1$  are introduced as

$$A_0 = -\frac{\partial F_i^0(\cdot)}{\partial u_i} \qquad A_1 = -\frac{\partial F_i^1(\cdot)}{\partial u_i}$$

The form of Eq (2.23) is identical to that of Eq (2.6) with  $u$  replaced by  $\rho$ . Thus, we can write an equivalent of Eq (2.16) as

$$\frac{\partial \langle \rho \rangle}{\partial t} = -\frac{\partial [F_i^0 \langle \rho \rangle]}{\partial u_i} + \left[ \alpha^2 \int_0^t \left\langle \frac{\partial F_i^1(t)}{\partial u_i} e^{\tau A_0} \frac{\partial F_j^1(t-\tau)}{\partial u_j^{-\tau}} \right\rangle e^{-\tau A_0} d\tau \right] \langle \rho \rangle \quad (2.24)$$

Now, consider the unperturbed Liouville equation, i.e. Eq (2.23) with  $\alpha = 0$ . It is easily verified that its solution is given by  $e^{tA_0} f(u)$  where  $f(u)$  is any function. Then, the following identity is obtained from Eq (2.20)

$$e^{tA_0} f(u) = f(u^{-t}) \frac{Du^{-t}}{Du} \quad (2.25)$$

We also note in passing that, in particular, one can make the substitution  $\exp(\tau A_0) = Du^{-\tau}/Du$  in Eq (2.24). By virtue of Eq (2.25), we have

$$e^{-\tau A_0} \rho(u, t) = \rho(u^\tau, t) \frac{Du^\tau}{Du} \quad (2.26)$$

and, with the use of Eq (2.19)

$$e^{-\tau A_0} \langle \rho(u, t) \rangle = P(u^\tau, t) \frac{Du^\tau}{Du} \quad (2.27)$$

Substituting this into Eq (2.24) and using the identity (2.25) with

$$f(u) = \frac{\partial F_j^1(u, t-\tau)}{\partial u_j} e^{-\tau A_0} \langle \rho(u, t) \rangle = \frac{\partial F_j^1(u, t-\tau)}{\partial u_j} P(u^\tau, t) \frac{Du^\tau}{Du}$$

the final form of the transport equation for the probability density function

$$\begin{aligned} \frac{\partial P(u, t)}{\partial t} = & -\frac{\partial [F_i^0 P(u, t)]}{\partial u_i} + \\ & + \alpha^2 \frac{\partial}{\partial u_i} \int_0^t d\tau \langle F_i^1(u, t) \frac{Du^{-\tau}}{Du} \frac{\partial}{\partial u_j^{-\tau}} F_j^1(u^{-\tau}, t-\tau) \rangle \frac{Du}{Du^{-\tau}} P(u, t) \end{aligned} \quad (2.28)$$

is obtained. The equation deserves some comment. The first expression on the right-hand side represents the transport of the PDF by a purely deterministic operator  $F^0$  while the second corresponds to the influence of the stochastic term which depends on the correlations of a random component  $F^1$ .

### 3. Derivation of the kinetic equation

Now, the problem of the particle motion in turbulent fluid, considered by Reeks, is reconsidered. The governing equations write

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= -\beta\mathbf{v} + \beta\langle U_f \rangle + \mathbf{F}_E + \mathbf{f} \end{aligned} \tag{3.1}$$

with  $\mathbf{x}$  and  $\mathbf{v}$  being the particle position and velocity, respectively.  $\beta$  is the particle inertia parameter (the inverse of the aerodynamic relaxation time  $\tau_p$ );  $\langle U_f \rangle$  stands for the mean fluid velocity and  $\mathbf{F}_E$  for an external force field.  $\mathbf{f}$  is a random force exerted by the turbulent fluid;  $\mathbf{f} = \beta\mathbf{u}_f$  where  $\mathbf{u}_f$  is the fluid fluctuating velocity.

The method described in the preceding section is applied as follows. Eq (3.1) is equivalent to Eq (2.17),  $\dot{\mathbf{u}}_i = \mathbf{F}_i$ , with

$$\begin{aligned} \mathbf{u} &= \begin{bmatrix} \mathbf{x} \\ \mathbf{v} \end{bmatrix} & \mathbf{F}^{(0)} &= \begin{bmatrix} \mathbf{v} \\ -\beta\mathbf{v} + \underbrace{\beta\langle U_f \rangle + \mathbf{F}_E}_{\equiv G} \end{bmatrix} \\ \mathbf{F}^{(1)} &= \begin{bmatrix} \mathbf{0} \\ \mathbf{f} \end{bmatrix} & \mathbf{F} &= \mathbf{F}^{(0)} + \mathbf{F}^{(1)} \end{aligned} \tag{3.2}$$

The unperturbed equation for a deterministic (non-stochastic) system  $\mathbf{u}_{(d)}$  writes  $\dot{\mathbf{u}}_{(d)} = \mathbf{F}^{(0)}$  or, explicitly

$$\dot{\mathbf{x}}_{(d)} = \mathbf{v}_{(d)} \qquad \dot{\mathbf{v}}_{(d)} = -\beta\mathbf{v}_{(d)} + \mathbf{G} \tag{3.3}$$

The subscript  $(d)$  will now be skipped; as before, we note

$$\mathbf{v} = \mathbf{v}(t) \qquad \mathbf{v}^{-\tau} = \mathbf{v}(t - \tau)$$

For  $G = 0$  we obtain

$$\begin{cases} \mathbf{v}^\tau = \mathbf{v}e^{-\beta\tau} \\ \mathbf{x}^\tau = \mathbf{x} + \frac{\mathbf{v}}{\beta}(1 - e^{-\beta\tau}) \end{cases} \quad \text{or} \quad \begin{cases} \mathbf{v} = \mathbf{v}^\tau e^{\beta\tau} \\ \mathbf{x} = \mathbf{x}^\tau - \frac{\mathbf{v}^\tau}{\beta}(e^{\beta\tau} - 1) \end{cases} \tag{3.4}$$

For any function

$$h = h(\mathbf{x}(\mathbf{x}^{-\tau}, \mathbf{v}^{-\tau}), \mathbf{v}(\mathbf{x}^{-\tau}, \mathbf{v}^{-\tau}))$$

we have

$$\frac{\partial h}{\partial v_i^{-\tau}} = \frac{\partial h}{\partial x_j} \frac{\partial x_j}{\partial v_i^{-\tau}} + \frac{\partial h}{\partial v_j} \frac{\partial v_j}{\partial v_i^{-\tau}} \quad (3.5)$$

Thus, using Eq (3.4)

$$\frac{\partial}{\partial v_i^{-\tau}} = \frac{1}{\beta} (1 - e^{-\beta\tau}) \frac{\partial}{\partial x_i} + e^{-\beta\tau} \frac{\partial}{\partial v_i} \quad (3.6)$$

Substitution into Eq (2.28) results in

$$\begin{aligned} \frac{\partial P(\mathbf{x}, \mathbf{v}, t)}{\partial t} &= -\frac{\partial}{\partial x_i} (v_i P) - \frac{\partial}{\partial v_i} (-\beta v_i P) + \\ &+ \frac{\partial}{\partial v_i} \left\{ \int_0^t \langle f_i(\mathbf{u}, t) \left[ \frac{1}{\beta} (1 - e^{-\beta\tau}) \frac{\partial}{\partial x_j} + e^{-\beta\tau} \frac{\partial}{\partial v_j} \right] f_j(\mathbf{u}^{-\tau}, t - \tau) \rangle d\tau \right\} P(\mathbf{x}, \mathbf{v}, t) \end{aligned} \quad (3.7)$$

In the notation of Reeks we obtain

$$\begin{aligned} \frac{\partial P(\mathbf{x}, \mathbf{v}, t)}{\partial t} + \frac{\partial}{\partial x_i} (v_i P) + \frac{\partial}{\partial v_i} (-\beta v_i P) &= \\ = \frac{\partial}{\partial v_i} \left\{ \int_0^t \langle f_i(\mathbf{u}, t) \frac{\partial}{\partial v_j} f_j(\mathbf{u}^{-\tau}, t - \tau) \rangle e^{-\beta\tau} d\tau + \right. & (3.8) \\ \left. + \int_0^t \langle f_i(\mathbf{u}, t) \frac{\partial}{\partial x_j} f_j(\mathbf{u}^{-\tau}, t - \tau) \rangle \frac{1}{\beta} (1 - e^{-\beta\tau}) d\tau \right\} P(\mathbf{x}, \mathbf{v}, t) \end{aligned}$$

After substitution for  $s = t - \tau$  the expression in curly brackets on the right-hand side of the above equation writes

$$\begin{aligned} &\int_0^t e^{\beta(s-t)} \langle f_i(\mathbf{u}, t) \frac{\partial}{\partial v_j} f_j(\mathbf{u}^{s-t}, s) \rangle ds + \\ &+ \int_0^t \frac{1}{\beta} (1 - e^{\beta(s-t)}) \langle f_i(\mathbf{u}, t) \frac{\partial}{\partial x_j} f_j(\mathbf{u}^{s-t}, s) \rangle ds \end{aligned}$$

We recall that in the LHDIA notation, cf Eq (2.21)

$$f(\mathbf{u}^{s-t}, s) \equiv f(\mathbf{x}, \mathbf{v}, t|s)$$

and  $f$  should be meant as the generalised Lagrangian force acting at time  $s$  on the particle that passes through  $\mathbf{x}$  with a velocity  $\mathbf{v}$  at some labelling

time  $t$ . The force depends on the particle trajectory; this somehow explains the words "Lagrangian History". If  $t$  is considered as the initial time,  $f$  is the "classical" Lagrangian force. On the other hand, for  $s = t$  we identify  $f$  as the Eulerian force and note

$$f(\mathbf{u}^0, t) \equiv f(\mathbf{x}, t)$$

Thus Eq (3.8) becomes identical with Eq (41) in the paper of Reeks (1992)

$$\begin{aligned} & \left( \frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i} - \frac{\partial}{\partial v_i} \beta v_i \right) P(\mathbf{x}, \mathbf{v}, t) = \\ & = \frac{\partial}{\partial v_i} \left\{ \int_0^t e^{\beta(s-t)} \langle f_i(\mathbf{x}, t) \frac{\partial}{\partial v_j} f_j(\mathbf{x}, \mathbf{v}, t|s) \rangle ds + \right. \\ & \left. + \int_0^t \frac{1}{\beta} (1 - e^{\beta(s-t)}) \langle f_i(\mathbf{x}, t) \frac{\partial}{\partial x_j} f_j(\mathbf{x}, \mathbf{v}, t|s) \rangle ds \right\} P(\mathbf{x}, \mathbf{v}, t) \end{aligned} \tag{3.9}$$

In a general case of non-uniform flows, the unperturbed equation takes the form

$$\begin{aligned} \dot{\mathbf{x}}_{(d)} &= \mathbf{v}_{(d)} \\ \dot{\mathbf{v}}_{(d)} &= -\beta(\mathbf{x}_{(d)}, t) \mathbf{v}_{(d)} + \mathbf{G}(\mathbf{x}_{(d)}, t) \end{aligned} \tag{3.10}$$

So, unperturbed particle trajectories are more complicated than those given explicitly in Eq (3.4). Following our standard notation, for  $\tau = t - s$  they are symbolically written as

$$\mathbf{x}^\tau = \mathbf{x}_{(d)}(\mathbf{x}, \mathbf{v}, t|s) \qquad \mathbf{v}^\tau = \mathbf{v}_{(d)}(\mathbf{x}, \mathbf{v}, t|s) \tag{3.11}$$

and should be read as: the position (velocity, respectively) at time  $s$  of a particle that passes through  $(\mathbf{x}, \mathbf{v})$  at time  $t$ . We define

$$g_{ij}(s|t) = \frac{\partial x_{(d)j}}{\partial v_i^{-\tau}} \qquad \dot{g}_{ij}(s|t) = \frac{d}{dt} g_{ij}(s|t) \tag{3.12}$$

So, instead of Eq (3.6) we have

$$\frac{\partial}{\partial v_i^{s-t}} = g_{ij}(s|t) \frac{\partial}{\partial x_j} + \dot{g}_{ij}(s|t) \frac{\partial}{\partial v_j} \tag{3.13}$$

Then, we follow the same steps as in the derivation of the PDF transport equation in the previous case, Eq (3.9). Finally, the corresponding PDF equation for dispersed particles in general non-uniform flows writes

$$\begin{aligned}
 & \left( \frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i} - \frac{\partial}{\partial v_i} \beta v_i \right) P(\mathbf{x}, \mathbf{v}, t) = \\
 & = \frac{\partial}{\partial v_i} \left\{ \int_0^t \dot{g}_{ik}(s|t) \langle f_k(\mathbf{x}, t) \frac{\partial}{\partial v_j} f_j(\mathbf{x}, \mathbf{v}, t|s) \rangle ds + \right. \\
 & \left. + \int_0^t g_{ik}(s|t) \langle f_k(\mathbf{x}, t) \frac{\partial}{\partial x_j} f_j(\mathbf{x}, \mathbf{v}, t|s) \rangle ds \right\} P(\mathbf{x}, \mathbf{v}, t)
 \end{aligned} \tag{3.14}$$

It is identical to Eq 88 in Reeks (1992). Comparing the above equation with the shorthand form Eq (1.2), the explicit expressions for diffusion tensors  $\mu$ ,  $\lambda$  and for  $\gamma$  can be easily found.

#### 4. Conclusion

In the paper, we have proposed a different way to derive formally the kinetic equation for particles in a turbulent flow. As seen above, the equation is not closed, because of the unknown form of the correlation of fluid velocity along particle trajectories. There are some recent proposals to overcome this inherent key difficulty (Reeks, 1993; Simonin et al., 1993; Simonin, 1996). Moreover, using the Lagrangian approach, a valid model for particle dispersion in homogeneous isotropic turbulence can be obtained (Pozorski et al., 1993). Once a general model for the correlations of fluid velocity "seen" by the particles is proposed and validated, it can be expressed in the formalism presented above and the governing Eulerian (i.e. two-fluid) equations for the two-phase flow can be derived from the kinetic equation.

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## Wyprowadzenie równania kinetycznego dla cząstek w turbulentnym przepływie dwufazowym

### Streszczenie

W pracy przedstawiono alternatywny sposób wyprowadzenia równania kinetycznego dla cząstek w turbulentnym przepływie dwufazowym. Wykorzystano metody rozwinięte w ramach teorii stochastycznych równań różniczkowych. Przedstawiono je w zarysie, łącznie z rozwinięciem liniowych i nieliniowych równań stochastycznych w szereg pólmiezienników (kumulantów). Następnie, to ogólne podejście zastosowano do równania ruchu małych cząstek w przepływie turbulentnym i wyprowadzono odpowiadające mu równanie kinetyczne. Zawiera ono korelację fluktuacji prędkości płynu wzdłuż trajektorii cząstek.

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