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On the determination of maps of fatigue safety under multiaxial stochastic loads

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Abstract. The paper deals with fatigue safety of machines and structures subjected to stresses below the fatigue limit of the material. Multiaxial stationary stochastic stresses are considered which are modelled by an equivalent stress in the form of a Gaussian process. The equivalence conditions are based on the Huber-Mises-Hencky strength theory and on the theory of energy transformation systems. It is assumed that the mean values, power spectral densities and cross power spectral densities of the stress processes are known. The formula for the expected value of fatigue safety margin is derived which can be used for visualization of fatigue safety level in selected parts of structural elements by means of maps.
Keywords: fatigue, stochastic stress, equivalent stress, fatigue safety margin

Nomenclature

a – amplitude of the equivalent stress;
$E\{\cdot\}$ – expected value;
f – fatigue safety factor;
j – imaginary unity;
k – natural number;
$K_{\sigma_i}(\tau)$ – autocorrelation function of the i-th stress process ($i = x, y, z, xy, yz, zx$);
$K_{\sigma_x\sigma_y}(\tau)$ – cross correlation function of the processes $\sigma_x(t)$ and $\sigma_y(t)$;
R_e – tensile yield strength;
s_e – standard deviation of the equivalent stress amplitude;
$S_{\sigma_i}(\omega)$ – power spectral density of the i-th stress process;
$S_{\sigma_x\sigma_y}(\omega)$ – cross power spectral density of the processes $\sigma_x(t)$ and $\sigma_y(t)$;
Z_{rc} – fatigue limit under fully reversed tension-compression;
Γ – gamma function;

δ – Dirac's delta function;
$\tilde{\sigma}_e(t)$ – equivalent stress;
$\tilde{\sigma}_i(t)$ – the i-th stress component;
$\sigma_i(t)$ – the i-th zero mean stochastic process;
σ_{me} – mean value of the equivalent stress;
σ_{mi} – mean value of the i-th stress component;
μ – relative fatigue safety margin;
τ – time interval;
ω – circular frequency;
ω_e – circular frequency of the equivalent stress;
$(\cdot)^*$ – complex conjugate.

1. Introduction

Under combined stochastic loads, the stress situation in any specific part of the structure may be extremely complex, with contribution from different load modes. For example, beam members of a structure can be subjected to simultaneous action of bending moments, axial forces and torsional loads of random character, which requires a multiaxial random fatigue model be applied [1, 2]. In the case of multiaxial stress with constant principal stress directions, the model based on the Huber-Mises-Hencky (H-M-H) distortion-energy strength theory is considered to be satisfactory [3]. However, if additionally the theory of energy transformation systems [4] with dissipative energy as a scalar parameter is utilized, then the requirement of invariance of the principal stress system can be avoided [5]. Such an approach is used in the present paper for determination of the fatigue safety margin in design for an infinite fatigue life. Analogously to maps of fatigue damage, accumulated in the material of elements operating in the high-cycle fatigue regime [6, 7], the fatigue safety margin, calculated in selected points of elements loaded below the fatigue limit, can be displayed on drawings of structural elements for visualization of their fatigue safety level.

Since the spectral methods lead to substantial savings of calculation time [8], the H-M-H theory and the theory of energy transformation systems will be hereunder implemented to time-varying stresses in the frequency domain.

2. Fatigue safety margin

The fatigue safety margin, at a given point of structural element loaded below the fatigue limit, will be here defined in deterministic approach by means of the uniaxial harmonic stress in tension-compression

$$\tilde{\sigma}(t) = \sigma_m + \sigma_a \sin(\omega t + \alpha), \tag{1}$$

where σ_a and σ_m are the amplitude and mean value of the stress. The maximum allowable stress amplitude A in design for an infinite fatigue life under asymmetric normal stress can be determined from empirical equations [9], such as Goodman equation

$$A = Z_{rc}\left(1 - \frac{\sigma_m}{R_m}\right), \tag{2}$$

for brittle materials, and Soderberg equation

$$A = Z_{rc}\left(1 - \frac{\sigma_m}{R_e}\right), \tag{3}$$

for ductile materials, where:

R_e — tensile yield strength;
R_m — ultimate tensile strength;
Z_{rc} — fatigue limit under fully reversed stress in tension-compression.

In what follows, ductile materials are considered.
Introducing the fatigue safety factor

$$f = \frac{A}{\sigma_a}, \tag{4}$$

and partial safety factors

$$f_d = \frac{Z_{rc}}{\sigma_a}, \quad f_s = \frac{R_e}{\sigma_m}, \tag{5}$$

one gets

$$f = f_d\left(1 - f_s^{-1}\right). \tag{6}$$

It means that the following safety margins can be defined:

— the absolute safety margins

$$M = A - \sigma_a, \quad M_d = Z_{rc} - \sigma_a, \quad M_s = R_e - \sigma_m, \tag{7}$$

— the dimensionless safety margins

$$m = \frac{M}{\sigma_a} = f - 1, \quad m_d = \frac{M_d}{\sigma_a} = f_d - 1, \quad m_s = \frac{M_s}{\sigma_m} = f_s - 1, \tag{8}$$

— the relative fatigue safety margin

$$\mu = \frac{M}{A} = \frac{m}{f} = 1 - f^{-1}. \tag{9}$$

As the fatigue safety parameter to be displayed by means of maps, in the present paper, the expected value of the relative fatigue safety margin is proposed. For this purpose, a reduced uniaxial stress, equivalent in terms of fatigue safety to the original multiaxial stress, must be determined.

3. Equivalent stress

In the general case of multiaxial stochastic loads, the Cartesian stress components are given as

$$\tilde{\sigma}_i(t) = \sigma_{mi} + \sigma_i(t); \quad i = x, y, z, xy, yz, zx, \tag{10}$$

where: σ_{mi} — mean value of the i-th stress component;
$\sigma_i(t)$ — the i-th zero mean stochastic process.

It is assumed that $\sigma_i(t)$ are stationary (in the wide sense) and stationary correlated processes of known power spectral densities and cross power spectral densities, and that σ_{mi} are the deterministic quantities.

Similarly as in [10], the equivalent stress will be sought in the form of a periodic (in the mean-square sense) stress in tension-compression as a Gaussian process

$$\tilde{\sigma}_e(t) = \sigma_{me} + a \sin(\omega_e t + \varphi) = \sigma_{me} + a_1 \exp(\mathrm{j}\omega_e t) + a_{-1} \exp(-\mathrm{j}\omega_e t), \tag{11}$$

where: a — random amplitude;
σ_{me} — mean value (deterministic quantity);
φ — random phase angle;
ω_e — circular frequency (deterministic quantity),

and

$$\begin{aligned} &a_1 = \frac{a}{2\mathrm{j}} \exp(\mathrm{j}\varphi), \quad a_{-1} = a_1^* \\ &E\{a_1\} = E\{a_{-1}\} = E\{a_1^* a_{-1}\} = E\{a_{-1}^* a_1\} = 0, \end{aligned} \tag{12}$$

$E\{\cdot\}$ denotes the expected value, j — the imaginary unity and $(\cdot)^*$ — the complex conjugate.

Consequently, the expected value of the relative fatigue safety margin can be calculated by means of the formula

$$E\{\mu\} = 1 - \frac{E\{a\}}{Z_{rc}\left(1 - \sigma_{me} / R_e\right)}, \tag{13}$$

if $E\{a\}$ and σ_{me} are known.

Adaptation of the H-M-H theory to the stress components, Eq. (10), leads to equation

$$\tilde{\sigma}_e^2 = \tilde{\sigma}_x^2 + \tilde{\sigma}_y^2 + \tilde{\sigma}_z^2 - \tilde{\sigma}_x \tilde{\sigma}_y - \tilde{\sigma}_y \tilde{\sigma}_z - \tilde{\sigma}_z \tilde{\sigma}_x + 3\left(\sigma_{xy}^2 + \tilde{\sigma}_{yz}^2 + \tilde{\sigma}_{zx}^2\right). \tag{14}$$

For the sake of brevity, the stress components $\tilde{\sigma}_z, \tilde{\sigma}_{yz}$ and $\tilde{\sigma}_{zx}$ will be dropped.

The time-domain relationship, Eq. (14), is not convenient for evaluation of parameters of the equivalent stress from spectral data. Therefore this relationship will be transformed into the frequency domain. For this purpose, the correlation theory of stochastic processes [11] will be used and Eq. (14) will be rewritten in terms of correlation functions of the processes, Eq. (10), as follows [5]:

$$\begin{aligned} &E\left\{\left[\sigma_{me} + a_1^* \exp\left(-\mathrm{j}\omega_e t_1\right) + a_{-1}^{\bullet} \exp\left(\mathrm{j}\omega_e t_1\right)\right]\left[\sigma_{me} + a_1 \exp\left(\mathrm{j}\omega_e t_2\right) + a_{-1} \exp\left(-\mathrm{j}\omega_e t_2\right)\right]\right\} = \\ &= E\left\{\left[\sigma_{mx} + \sigma_x^*(t_1)\right]\left[\sigma_{mx} + \sigma_x(t_2)\right]\right\} + E\left\{\left[\sigma_{my} + \sigma_y^*(t_1)\right]\left[\sigma_{my} + \sigma_y(t_2)\right]\right\} - \\ &- E\left\{\left[\sigma_{mx} + \sigma_x^*(t_1)\right]\left[\sigma_{my} + \sigma_y(t_2)\right]\right\} + 3E\left\{\left[\sigma_{mxy} + \sigma_{xy}^*(t_1)\right]\left[\sigma_{mxy} + \sigma_{xy}(t_2)\right]\right\}. \end{aligned} \tag{15}$$

Hence, in accordance with Eqs. (12),

$$\begin{aligned} &\sigma_{me}^2 + \frac{1}{4} E\left\{a^2\right\}\left[\exp\left(\mathrm{j}\omega_e \tau\right) + \exp\left(-\mathrm{j}\omega_e \tau\right)\right] = \sigma_{mx}^2 + \sigma_{my}^2 - \sigma_{mx}\sigma_{my} + \\ &+ 3\sigma_{mxy}^2 + K_{\sigma_x}(\tau) + K_{\sigma_y}(\tau) - K_{\sigma_x \sigma_y}(\tau) + 3K_{\sigma_{xy}}(\tau), \end{aligned} \tag{16}$$

where

$$\begin{aligned} K_{\sigma_x}(\tau) &= E\left\{\sigma_x^*(t_1)\sigma_x(t_2)\right\} \\ K_{\sigma_y}(\tau) &= E\left\{\sigma_y^*(t_1)\sigma_y(t_2)\right\} \\ K_{\sigma_{xy}}(\tau) &= E\left\{\sigma_{xy}^*(t_1)\sigma_{xy}(t_2)\right\}, \end{aligned} \tag{17}$$

are the autocorrelation functions of the processes $\sigma_x(t), \sigma_y(t), \sigma_{xy}(t)$ and

$$K_{\sigma_x \sigma_y}(\tau) = E\left\{\sigma_x^*(t_1)\sigma_y(t_2)\right\}, \tag{18}$$

is the cross correlation function of the processes $\sigma_x(t)$ and $\sigma_y(t)$.

Here, $\tau = t_2 - t_1$ is the time interval.
Fourier transformation of Eq. (16) yields

$$\begin{aligned}&\sigma_{me}^2\delta(\omega)+\frac{1}{4}E\left\{a^2\right\}\left[\delta(\omega-\omega_e)+\delta(\omega+\omega_e)\right]=\\&=\left(\sigma_{mx}^2+\sigma_{my}^2-\sigma_{mx}\sigma_{my}+3\sigma_{mxy}^2\right)\delta(\omega)+\\&+S_{\sigma_x}(\omega)+S_{\sigma_y}(\omega)-S_{\sigma_x\sigma_y}(\omega)+3S_{\sigma_{xy}}(\omega),\end{aligned} \tag{19}$$

where: $S_{\sigma_x}(\omega)$, $S_{\sigma_y}(\omega)$, $S_{\sigma_{xy}}(\omega)$ — power spectral densities of the processes $\sigma_x(t)$, $\sigma_y(t)$ and $\sigma_{xy}(t)$;
$S_{\sigma_x\sigma_y}(\omega)$ — cross power spectral density of the processes $\sigma_x(t)$ and $\sigma_y(t)$;
δ — Dirac's delta function.

In order to make use of the theory of energy transformation systems, Eq. (19) must be integrated over the whole frequency range which gives [5]

$$\begin{aligned}&\sigma_{me}^2+\frac{1}{2}E\left\{a^2\right\}=\sigma_{mx}^2+\sigma_{my}^2-\sigma_{mx}\sigma_{my}+3\sigma_{mxy}^2+\\&+\int_{-\infty}^{\infty}S_{\sigma_x}(\omega)d\omega+\int_{-\infty}^{\infty}S_{\sigma_y}(\omega)d\omega-\int_{-\infty}^{\infty}S_{\sigma_x\sigma_y}(\omega)d\omega+3\int_{-\infty}^{\infty}S_{\sigma_{xy}}(\omega)d\omega.\end{aligned} \tag{20}$$

Equation (20) corresponds to the condition that the powers dissipated externally per unit volume in the equivalent and original stress states are equal. From this equation one obtains

$$\sigma_{me}^2=\sigma_{mx}^2+\sigma_{my}^2-\sigma_{mx}\sigma_{my}+3\sigma_{mxy}^2, \tag{21}$$

$$E\left\{a^2\right\}=2\left[\int_{-\infty}^{\infty}S_{\sigma_x}(\omega)d\omega+\int_{-\infty}^{\infty}S_{\sigma_y}(\omega)d\omega-\int_{-\infty}^{\infty}S_{\sigma_x\sigma_y}(\omega)d\omega+3\int_{-\infty}^{\infty}S_{\sigma_{xy}}(\omega)d\omega\right]. \tag{22}$$

The amplitude of the Gaussian process, Eq. (11), follows Rayleigh distribution [11], so that

$$E\left\{a^k\right\}=2^{k/2}\Gamma\left(1+\frac{k}{2}\right)s_e^k;\quad k=1,2,..., \tag{23}$$

where: s_e — standard deviation of the equivalent stress amplitude;
Γ — gamma function.

In particular [12],

$$E\{a\} = (0.5\pi)^{1/2} s_e, \tag{24}$$

$$E\{a^2\} = 2s_e^2. \tag{25}$$

Equating the right-hand sides of Eqs. (22) and (25) gives

$$s_e = \left[\int_{-\infty}^{\infty} S_{\sigma_x}(\omega)d\omega + \int_{-\infty}^{\infty} S_{\sigma_y}(\omega)d\omega - \int_{-\infty}^{\infty} S_{\sigma_x\sigma_y}(\omega)d\omega + 3\int_{-\infty}^{\infty} S_{\sigma_{xy}}(\omega)d\omega \right]^{1/2}. \tag{26}$$

4. Expected value of safety margin

The considered reduced stress, equivalent in terms of fatigue safety to the original multiaxial stress, is fully determined by its mean value σ_{me}, the standard deviation s_e of its amplitude and by its circular frequency ω_e. The quantity ω_e can be also obtained with the aid of the theory of energy transformation systems [5]. However, the knowledge of ω_e is not necessary for visualization of fatigue safety level in design for an infinite fatigue life because then, as it follows from Eqs. (13), (21), (24), and (26), the expected values of the relative fatigue safety margins in selected points of structural elements to be displayed on drawings of these elements can be calculated as

$$E\{\mu\} = 1 - \frac{R_e}{Z_{rc}\left[R_e - \left(\sigma_{mx}^2 + \sigma_{my}^2 - \sigma_{mx}\sigma_{my} + 3\sigma_{mxy}^2\right)^{1/2}\right]} \left\{ 0.5\pi \left[\int_{-\infty}^{\infty} S_{\sigma_x}(\omega)d\omega + \int_{-\infty}^{\infty} S_{\sigma_y}(\omega)d\omega - \int_{-\infty}^{\infty} S_{\sigma_x\sigma_y}(\omega)d\omega + 3\int_{-\infty}^{\infty} S_{\sigma_{xy}}(\omega)d\omega \right] \right\}^{1/2}. \tag{27}$$

5. Summary

The paper deals with the fatigue safety of engineering elements under random loading conditions. Multiaxial stationary stress, described in the frequency domain, is considered. In order to determine the fatigue safety of elements, an equivalent uniaxial stress is defined. As a result, the formula for the fatigue safety margin is derived. Expected values of the safety margin can be displayed on drawings of elements by means of maps. Such a visualization of fatigue safety level in selected parts of a structure may be useful in the design process.

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O wyznaczaniu warstwic bezpieczeństwa zmęczeniowego przy wieloosiowych obciążeniach stochastycznych

Streszczenie. Praca dotyczy bezpieczeństwa zmęczeniowego maszyn i konstrukcji poddanych naprężeniom poniżej granicy zmęczenia. Rozpatrywane jest wieloosiowe naprężenie stacjonarne, które zamodelowano jednoosiowym naprężeniem równoważnym w postaci procesu Gaussa. Warunki równoważności oparto na hipotezie wytężenia Hubera-Misesa-Hencky'ego i na teorii systemów transformacji energii. Założono znajomość wartości średnich, gęstości widmowych mocy i wzajemnych gęstości widmowych mocy naprężeń składowych. Wyprowadzono wzór na wartość oczekiwaną marginesu bezpieczeństwa zmęczeniowego, która może być użyta do wizualizacji poziomu bezpieczeństwa zmęczeniowego wybranych części konstrukcji za pomocą warstwic.

Słowa kluczowe: zmęczenie, naprężenie stochastyczne, naprężenie równoważne, margines bezpieczeństwa zmęczeniowego