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On the determination of maps of fatigue damage under uniaxial stochastic loads

JANUSZ KOLENDA

Naval Academy, Faculty of Mechanical and Electrical Engineering,
81-103 Gdynia, Śmidowicza 69 Str., Poland

Abstract. The paper deals with fatigue damage accumulated in elements of machines and structures subjected to stresses in the high-cycle fatigue regime. Uniaxial stationary stochastic stresses are considered which are modelled by an equivalent stress in the form of a Gaussian process. The equivalence conditions are based on the theory of energy transformation systems. It is assumed that the mean value and the power spectral density of the original stress are known. The ratio of the required design life and the expected value of the time to failure is determined as the fatigue damage parameter the values of which can be used for visualization of fatigue damage of the material by means of maps.
Keywords: fatigue, stochastic stress, equivalent stress, fatigue damage parameter

Nomenclature

a — amplitude of the equivalent stress;
D — fatigue damage parameter;
$E\{\cdot\}$ — expected value;
j — imaginary unity;
k — natural number;
K — fatigue strength coefficient;
$K(\tau)$ — autocorrelation function;
m — fatigue strength exponent;
N — number of cycles to failure;
R_e — tensile yield strength;
s_e — standard deviation of the equivalent stress amplitude;
$S(\omega)$ — power spectral density;
T_d — required design life;

T_f — time to fatigue failure;
Γ — gamma function;
δ — Dirac's delta function;
$\tilde{\sigma}_e$ — equivalent stress;
$\tilde{\sigma}_x$ — original stress;
$\sigma_x(t)$ — stochastic stress process of zero mean value;
σ_{me} — mean value of the equivalent stress;
σ_{mx} — mean value of the original stress;
τ — time interval;
φ — phase angle of the equivalent stress;
ω — circular frequency;
ω_e — circular frequency of the equivalent stress;
$(\cdot)^*$ — complex conjugate.

1. Introduction

Damage by fatigue is one of the most common causes of failures that engineers must deal with. Some estimates are that over three-quarters of all failures of dynamically loaded elements are due to fatigue. The development of a fatigue crack is influenced by a number of factors associated with material properties, component designs and loading conditions. Machinery details, engine parts and vehicle elements are usually subjected to steady loading conditions resulting in constant, periodic and/or stationary stochastic stresses. For example, in marine environment, constant stresses are caused by dead-weight, hydrostatic pressure, wind and current, whereas those encountered in machinery systems are produced by torque, thrust, centrifugal force, etc. In static problems, the load capacity of metallic elements is referred to the yield strength or to the ultimate strength. In the case of multiaxial static load, a reduced uniaxial stress, equivalent to the original one in terms of effort of the material, is calculated. In design calculations of dynamically loaded machines and structures, the fatigue limits and/or *S-N* curves (Wöhler curves) are taken into account [1, 2]. In a multiaxial stress state, an equivalent uniaxial stress is also frequently determined by means of an adequate fatigue strength theory [3-5]. To facilitate the solution of fatigue problems, the Finite Element Method, fatigue-oriented software (ANSYS, WinLIFE, FEMFAT, etc.) and visualization of calculation results on maps can be used [5-7]. In [5], an algorithm for determination of fatigue damage maps, with the application of COMSOL and MATLAB modules [8, 9], is proposed. It deals with zero mean stochastic stresses which can be determined numerically in fatigue-critical regions of structures. Then, also the probability of fatigue damage accumulated in the material in these regions can be calculated and displayed on maps. The load cases discussed in [5] concern multiaxial stresses defined in the frequency domain.

In the present paper, uniaxial stationary stochastic stresses in the high-cycle fatigue regime are considered which are modelled by an equivalent stress in the form of a Gaussian process. The equivalence conditions are based on the theory of energy transformation systems [10] which is convenient for fatigue calculations both in the time domain and in the frequency domain [4]. Bearing in mind the effectiveness and application range of spectral methods, hereunder it is assumed that the mean value and the power spectral density of the original stress are known. As the fatigue damage parameter to be displayed on maps, the ratio of the required design life and the expected value of the time to fatigue failure is chosen.

2. Fatigue damage parameter

The proposed fatigue damage parameter can be defined in deterministic approach by means of the uniaxial normal harmonic stress

$$\tilde{\sigma}(t)=\sigma_m+\sigma_a \sin(\omega t+\alpha). \tag{1}$$

Its amplitude σ_a and the mean value σ_m are calculated from the two successive extrema $\sigma_{\max}$ and $\sigma_{\min}$ of the stress history as

$$\sigma_a=\frac{1}{2}(\sigma_{\max}-\sigma_{\min}), \quad \sigma_m=\frac{1}{2}(\sigma_{\max}+\sigma_{\min}). \tag{2}$$

The maximum allowable stress amplitude A in design for an infinite fatigue life under asymmetric normal stress can be determined from empirical equations [1, 2], such as Goodman equation

$$A=Z_0\left(1-\frac{\sigma_m}{R_m}\right), \tag{3}$$

for brittle materials, and Soderberg equation

$$A=Z_0\left(1-\frac{\sigma_m}{R_e}\right), \tag{4}$$

for ductile materials,

where: Z_0 is the fatigue limit under fully reversed stress, denoted Z_{go} at bending and Z_{rc} at tension-compression;
R_e is the tensile yield strength;
R_m is the ultimate tensile strength.

In what follows ductile materials are considered.

In the case of uniaxial shear stress, instead of Z_0 and R_e in Eq. (4) the fatigue limit $Z_{so} = Z_{go} / \sqrt{3}$ and the yield strength $R_{es} = R_e / \sqrt{3}$ (for steels) should be inserted.

The necessary modification of Eq. (4) to indicate finite fatigue life in the high-cycle regime reads [11]

$$\sigma_a = \sigma\left(1 - \frac{\sigma_m}{R_e}\right), \tag{5}$$

where: σ is the amplitude of the fully reversed stress at the given number of cycles to failure N, σ_a is the amplitude of the stress of Eq. (1) which will give that fatigue life.

For the relation between σ and N in the high-cycle fatigue regime, the following equation of the *S-N* curve is commonly accepted

$$N\sigma^m = K, \tag{6}$$

where K is the fatigue strength coefficient and m is the fatigue strength exponent, both chosen for each particular case on the basis of the relevant test data [2, 11]. Combining Eqs. (5) and (6) yields

$$N = K\left(\frac{1 - \sigma_m / R_e}{\sigma_a}\right)^m. \tag{7}$$

So, the time to fatigue failure under the stress, Eq. (1), is given by

$$T_f = \frac{2\pi K}{\omega}\left(\frac{1 - \sigma_m / R_e}{\sigma_a}\right)^m. \tag{8}$$

As the parameter of the fatigue damage, accumulated in the material during the required design life T_d, in the present paper the following quantity is proposed

$$D = \frac{T_d}{E\{T_f\}}, \tag{9}$$

where E{·} denotes the expected value.

3. Expected value of the time to fatigue failure

In the case of uniaxial stochastic stress $\tilde{\sigma}_x(t)$ it can be defined in the time domain by means of various expressions, in particular by

$$\tilde{\sigma}_x(t) = \sigma_x \sin(\omega t + \alpha), \tag{10}$$

$$\tilde{\sigma}_x(t) = \sigma_{mx} + \sigma_x \sin(\omega t + \alpha), \tag{11}$$

$$\tilde{\sigma}_x(t) = \sigma_x(t), \tag{12}$$

$$\tilde{\sigma}_x(t) = \sigma_{mx} + \sigma_x(t), \tag{13}$$

where: σ_x is the stress amplitude (random variable);
σ_{mx} is the mean stress value (deterministic quantity);
α is the stress phase angle (random variable);
$\sigma_x(t)$ is the stochastic process (with zero mean value);
ω is the circular frequency (deterministic quantity).

In this paper, it is assumed that $\sigma_x(t)$ is a stationary (in the wide sense) process.

In accordance with Eq. (8), the expected value of the time to fatigue failure under the stress, Eq. (11), is

$$E\{T_f\} = \frac{2\pi K}{\omega E\{\sigma_x^m\}} \left(1 - \frac{\sigma_{mx}}{R_e}\right)^m. \tag{14}$$

Simplicity of Eq. (14) suggests modelling of the stress, Eq. (13), with the equivalent stress of the form of Eq. (11), i.e.

$$\tilde{\sigma}_e(t) = \sigma_{me} + a\sin(\omega_e t + \varphi) = \sigma_{me} + a_1 \exp(\mathrm{j}\omega_e t) + a_{-1}\exp(-\mathrm{j}\omega_e t), \tag{15}$$

and calculating

$$E\{T_f\} = \frac{2\pi K}{\omega_e E\{a^m\}} \left(1 - \frac{\sigma_{me}}{R_e}\right)^m, \tag{16}$$

after determination of σ_{me}, ω_e, and $E\{a^m\}$. Here σ_{me} denotes the mean value of the equivalent stress, a and φ are the random variables representing amplitude and phase angle of the equivalent stress, ω_e is its constant circular frequency, and

$$a_1 = \frac{a}{2\mathrm{j}} \exp(\mathrm{j}\varphi), \quad a_{-1} = a_1^*, \tag{17}$$

$$E\{a_1\} = E\{a_{-1}\} = E\{a_1^* a_{-1}\} = E\{a_{-1}^* a_1\} = 0, \tag{18}$$

where: j is the imaginary unity;
$(\cdot)^*$ is the complex conjugate.

4. Equivalent stochastic stress

Comparing Eqs. (13) and (15) yields

$$\sigma_{me} = \sigma_{mx}, \tag{19}$$

which enables to proceed with the processes, Eq. (12), and

$$\tilde{\sigma}_e(t) = a\sin(\omega_e t + \varphi) = a_1\exp(\mathrm{j}\omega_e t) + a_{-1}\exp(-\mathrm{j}\omega_e t), \tag{20}$$

instead of the processes of Eqs. (13) and (15). In what follows it is assumed that $\tilde{\sigma}_e(t)$ represents a Gaussian process satisfying the condition

$$K_{\tilde{\sigma}_e}(\tau) = K_{\tilde{\sigma}_x}(\tau), \tag{21}$$

where: $K_{\tilde{\sigma}_e}(\tau)$ is the autocorrelation function of the stress, Eq. (20);
$K_{\tilde{\sigma}_x}(\tau)$ is the autocorrelation function of the stress, Eq. (12);
τ is the time interval.

With Eqs. (12) and (20), Eq. (21) leads to

$$\begin{aligned} &E\{[a_1^* \exp(-\mathrm{j}\omega_e t_1) + a_{-1}^* \exp(\mathrm{j}\omega_e t_1)][a_1 \exp(\mathrm{j}\omega_e t_2) + a_{-1} \exp(-\mathrm{j}\omega_e t_2)]\} = \\ &= E\{\sigma_x^*(t_1)\sigma_x(t_2)\}, \end{aligned} \tag{22}$$

which gives

$$\frac{1}{4}E\{a^2\}\left[\exp(\mathrm{j}\omega_e\tau) + \exp(-\mathrm{j}\omega_e\tau)\right] = K_{\sigma_x}(\tau), \tag{23}$$

where: $E\{a^2\}$ is the mean-square value of the equivalent stress amplitude;
$K_{\sigma_x}(\tau) = E\{\sigma_x^*(t_1)\sigma_x(t_2)\}$ is the autocorrelation function of the process $\sigma_x(t)$;
$\tau = t_2 - t_1$.

The power spectral density $S_{\sigma_x}(\omega)$ of the process $\sigma_x(t)$ is defined as

$$S_{\sigma_x}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} K_{\sigma_x}(\tau)\exp(-\mathrm{j}\omega\tau)d\tau. \tag{24}$$

After substitution of Eq. (23) in Eq. (24) one gets

$$\frac{1}{4}E\{a^2\}\left[\delta(\omega - \omega_e) + \delta(\omega + \omega_e)\right] = S_{\sigma_x}(\omega), \tag{25}$$

where δ is the Dirac's delta function.

To evaluate the mean-square value of the equivalent stress amplitude from Eq. (25), the theory of energy transformation systems [10] can be utilized. According to this theory, two stress states can be regarded as equivalent in terms of fatigue life of the material if during the service life the internally and externally dissipated energies per unit volume in these states are respectively equal. The counterpart of this statement in the frequency domain can be expressed as follows [4, 12].

Two stress states can be regarded as equivalent in terms of fatigue life of the material if over the whole frequency range the internally and externally dissipated powers per unit volume in these states are respectively equal.

In the considered case, the externally dissipated power is proportional to the integral of the power spectral density of the process $\sigma_x(t)$ over the actual frequency range. Thus, the following equivalence condition can be postulated

$$\frac{1}{4}E\{a^2\}\int_{-\infty}^{\infty}\left[\delta(\omega-\omega_e)+\delta(\omega+\omega_e)\right]d\omega=\int_{-\infty}^{\infty}S_{\sigma_x}(\omega)\,d\omega. \tag{26}$$

Hence,

$$E\{a^2\}=2\int_{-\infty}^{\infty}S_{\sigma_x}(\omega)\,d\omega. \tag{27}$$

Bearing in mind that the amplitude of the narrow-band Gaussian process of Eq. (20) follows Rayleigh distribution [13], its k-th statistical moment is [14]

$$E\{a^k\}=2^{k/2}\Gamma\left(1+\frac{k}{2}\right)s_e^k, \tag{28}$$

where: s_e is the standard deviation of the equivalent stress amplitude,
Γ is the gamma function.

Consequently, for $k = 2$ one obtains

$$E\{a^2\}=2s_e^2. \tag{29}$$

Equating the right-hand sides of Eqs. (27) and (29) gives

$$s_e=\left[\int_{-\infty}^{\infty}S_{\sigma_x}(\omega)d\omega\right]^{1/2}, \tag{30}$$

so that

$$E\{a^m\}=\Gamma\left(1+\frac{m}{2}\right)\left[2\int_{-\infty}^{\infty}S_{\sigma_x}(\omega)\,d\omega\right]^{m/2}. \tag{31}$$

As to the frequency ω_e in Eq. (16), the theory of energy transformation systems can be used which links the lifetime of dynamically loaded systems also with the internally dissipated energy.

In the high-cycle fatigue regime, the deformations are small and can be characterized by a material's elastic behaviour [15]. Therefore, in order to reflect also the dissipative capacity of ductile materials under dynamic loads below the yield point, we assume that the Kelvin-Voigt's models [16, 17]

$$\tilde{\sigma} = E\tilde{\varepsilon} + \eta\dot{\tilde{\varepsilon}}, \tag{32}$$

$$\tilde{\tau} = G\tilde{\gamma} + \lambda\dot{\tilde{\gamma}}, \tag{33}$$

are applicable, where

$\tilde{\sigma}, \tilde{\varepsilon}$ are the normal stress and strain;
$\tilde{\tau}, \tilde{\gamma}$ are the shear stress and strain;
E, G are the Young and shear moduli;
η, λ are the coefficients of internal viscous damping of the material.

Following the results obtained in [4] with the aid of the theory of energy transformation systems and of the model of Eq. (32), we have the equivalence condition

$$E\{\dot{\tilde{\sigma}}_e^2(t)\} = E\{\dot{\tilde{\sigma}}_x^2(t)\}. \tag{34}$$

To Eq. (34), the correlation theory of stochastic processes can be applied. For this purpose, we assume that the process $\tilde{\sigma}_x(t) = \sigma_x(t)$ is differentiable in the mean-square sense [13]. Accordingly, there exists the second order time derivative of its autocorrelation function $K_{\sigma_x}(\tau)$, so that

$$K_{\dot{\sigma}_x}(\tau) = E\{\dot{\sigma}_x^*(t_1)\dot{\sigma}_x(t_2)\} = -\frac{d^2}{d\tau^2}K_{\sigma_x}(\tau). \tag{35}$$

In order to obtain the frequency domain formulation of Eq. (34), we rewrite it in terms of the autocorrelation functions as follows

$$E\{\dot{\tilde{\sigma}}_e^*(t_1)\dot{\tilde{\sigma}}_e(t_2)\} = E\{\dot{\tilde{\sigma}}_x^*(t_1)\dot{\tilde{\sigma}}_x(t_2)\}. \tag{36}$$

By Eqs. (12), (17), (18), (20), and (35), one gets from Eq. (36)

$$\frac{1}{4}\omega_e^2 E\{a^2\}\left[\exp(j\omega_e\tau) + \exp(-j\omega_e\tau)\right] = -\frac{d^2}{d\tau^2}K_{\sigma_x}(\tau). \tag{37}$$

Fourier transformation of Eq. (37) gives

$$\frac{1}{4}\omega_e^2 E\{a^2\}\left[\delta(\omega-\omega_e) + \delta(\omega+\omega_e)\right] = \omega^2 S_{\sigma_x}(\omega). \tag{38}$$

To determine the circular frequency of the equivalent stress, Eq. (38) must be integrated over the whole frequency range [compare Eqs. (25) and (26)] which yields

$$\frac{1}{2}\omega_e^2 E\{a^2\} = \int_{-\infty}^{\infty} \omega^2 S_{\sigma_x}(\omega)\, d\omega. \tag{39}$$

Combining Eqs. (27) and (39) leads to

$$\omega_e = \left[\frac{\int_{-\infty}^{\infty} \omega^2 S_{\sigma_x}(\omega)\, d\omega}{\int_{-\infty}^{\infty} S_{\sigma_x}(\omega)\, d\omega} \right]^{1/2}. \tag{40}$$

Finally, the values of the proposed fatigue damage parameter to be displayed on fatigue damage maps of structural elements are given by

$$D = \frac{2^{(m-2)/2} T_d \Gamma(1+m/2)}{\pi K (1-\sigma_{mx}/R_e)^m} \left[\int_{-\infty}^{\infty} \omega^2 S_{\sigma_x}(\omega)\, d\omega \right]^{1/2} \left[\int_{-\infty}^{\infty} S_{\sigma_x}(\omega)\, d\omega \right]^{(m-1)/2}. \tag{41}$$

It is seen that the fatigue damage parameter D is fully determined by the mean value and the power spectral density of the original stress as well as by the *S-N* curve and the tensile (or shear) yield strength of the material, and depends linearly on the required design life.

If during the service life different loading conditions are envisaged, an appropriate fatigue damage cumulation model must be taken into account. Among many cumulation models described in the literature, that referred to as the Palmgren-Miner rule [2, 11] is dominantly used. Its application for the sequence of stationary load states is straightforward.

5. Summary

The paper deals with the fatigue damage accumulated in the material of engineering elements under random loading conditions. Uniaxial stationary stress described in the frequency domain is considered. In order to make use of the *S-N* curve (Wöhler's curve), an equivalent stress in the form of a Gaussian process is determined. As a result, the formula for the expected value of the time to fatique failure is derived. A fatique damage parameter related to the required design life is defined, the values of which can be displayed on drawings of elements by means of maps. Such a visualization of damage distribution in selected parts of a structure may be useful in the design process.

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REFERENCES

[1] S. Kocańda, *Fatigue failure of metals*, WNT, Warszawa, 1985 (in Polish).

[2] S. Kocańda, J. Szala, *Fundamentals of fatigue calculations*, PWN, Warszawa, 1997 (in Polish).

[3] I. V. Papadopoulos, P. Davoli, C. Gorla, M. Filippini, A. Bernasconi, A *comparative study of multiaxial high -cycle fatigue criteria for metals*, Int. J. Fatigue, 3, 1997, 219-235.

[4] J. Kolenda, *On fatigue safety of metallic elements under static and dynamic loads*, Gdańsk University of Technology, 2004.

[5] A. Niesłony, *Determination of fatigue damage maps with spectral methods*, Opole University of Technology, 2008 (in Polish).

[6] R. Browell, *Calculating and displaying fatigue results*, ANSYS, Inc., 2006.

[7] U. Kocabicak, M. firat, A *simple approach for multiaxial fatigue damage prediction based on FEM post-processing*, Materials and Design, 25, 2004, 73-82.

[8] Comsol Multiphysics, *Structural mechanics module users guide*, Version 3.3, COMSOL AB, Stockholm, 2006.

[9] Matlab User Guide, *Release 14 with Service Pack 2*, Version V, The MathWorks, Inc., 2005.

[10] C. Cempel, *Theory of energy transformation systems and their application in diagnostics of operating systems*, Applied Math. and Computer Sciences, 2, 1993, 533-538.

[11] A. Almar-Naess (ed.), *Fatigue handbook. Offshore steel structures*, Tapir Publishers, Trondheim, 1985.

[12] J. Kolenda, *Spectral criterion of infinite fatigue life of beams under combined random load*, Polish Maritime Research, 2, 2002, 3-7.

[13] A. Preumont, *Vibrations aléatoires et analyse spectrale*, Presses Polytechniques et Universitaires Romandes, CH-1015, Lausanne, 1990.

[14] A. Pacut, *Probability, Theory, Probabilistic modelling in technology*, WNT, Warszawa, 1985 (in Polish).

[15] A. Blake, (ed.), *Handbook of mechanics, materials and structures*, J. Wiley & Sons, New York, 1985.

[16] J. Awrejcewicz, W. A. Krysko, *Vibrations of continuous systems*, WNT, Warszawa, 2000 (in Polish).

[17] J. Kolenda, *Lectures on vibrations*, Polish Naval Academy, Gdynia, 2009.

J. KOLENDA

O wyznaczaniu warstwic uszkodzeń zmęczeniowych przy jednoosiowych obciążeniach stochastycznych

Streszczenie. Praca dotyczy uszkodzeń zmęczeniowych elementów maszyn i konstrukcji poddanych naprężeniom w zakresie ograniczonej wytrzymałości zmęczeniowej wysokocyklowej. Rozpatrywane jest jednoosiowe naprężenie stacjonarne, które zamodelowano naprężeniem równoważnym w postaci procesu Gaussa. Warunki równoważności oparto na teorii systemów transformacji energii. Założono znajomość wartości średniej i gęstości widmowej mocy naprężenia. Iloraz założonego czasu eksploatacji i wartości oczekiwanej czasu do chwili pęknięcia przyjęto jako parametr uszkodzenia zmęczeniowego, którego wartości mogą być użyte do wizualizacji stopnia uszkodzenia zmęczeniowego materiału za pomocą warstwic.

Słowa kluczowe: zmęczenie, naprężenie stochastyczne, naprężenie równoważne, parametr uszkodzenia zmęczeniowego