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Calculation of injury assessment for the chest wall velocity predictor by spline approximation

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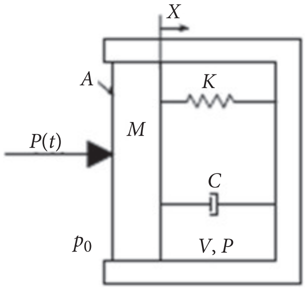
Abstract. The proposed method of calculation combines the spline approximation and precise solution of the ordinary differential equations. The solution is similar to the solution of the generalized Sturm-Liouville problem. The solution is sought in the form of power series with proper radius of convergence. The method is applied to the calculation of injury assessment prediction. The method may be used for problems when high precision of calculations is needed.
Keywords: Prediction of injury assessment, spline approximation, power series method, solution of ordinary differential equation, prediction of blast injury level

1. Introduction

Mines are significant threat for military vehicles and their occupants. Organs containing air are the most sensitive to overpressure caused by blast waves. Physical injury will take place if the biomechanical response is of such a nature that the biological system deforms beyond a tolerable limit resulting in damage to anatomical structures and/or alteration in normal function. In full-scale tests, the loads on the occupants are measured using instrumented anthropomorphic test devices (ATDs) — widely known as crash test dummies, for example, the Hybrid III 50th percentile male ATD [2]. The same is true for a person in building acted by blast waves.

Today, military standards describe the Axelsson & Yelverton model (Fig. 1) like the best model for non-auditory blast injury assessment occurring during an AV blast mine strike.



A: effective area;
M: effective mass;
V: initial gaseous volume of the lungs;
X: displacement;
C: damping factor;
K: elasticity constant;
P_0: ambient pressure;
$P(t)$: overpressure;
γ: polytropic exponent for gas in lungs.

Fig. 1. Thorax model — single chamber one lung model [Axelsson, 1996] [2]

$$M\frac{d^2x(t)}{dt^2}+C\frac{dx(t)}{dt}+Kx(t)=Ap_0\left\{p(t)+\frac{\left[\left(1-\frac{Ax(t)}{V}\right)^{\gamma}-1\right]}{\left(1-\frac{Ax(t)}{V}\right)^{\gamma}}\right\}. \tag{1.1}$$

Considering that there are no risk available, it was decided to use a conservative approach and take the no injury level (3.6 m/s) as the limit for the chest wall velocity predictor (CWVP).

The recommended model is simple but contains nonlinear interaction of air in lung against overpressure caused by blast. The velocity profile should be calculated for the injury assessment. Equation (1.1) is well known [2], [1] e.g. but contains nonlinear, polytropic reaction of the air in lung.

The function $p(t)$ represents the overpressure divided by ambient pressure. The overpressure was measured on ATD during real blast test. Usually, data are memorised at equal distance of time, but programs and formulas are prepared for unequal steps of time also.

Equation (1.1) may be solved by many methods. We propose the method containing two ideas: method of solution based at the definition of the solution to the initial problem and spline approximation applied to the results of measurement. The solution of the initial problem is defined as the power series satisfying the equation with proper radius of convergence and convergent to the initial conditions. The method is described in Ref. 5. The spline approximation method is described in Ref. 4 and in references cited there. The spline approximation fulfils the role of the specific filter [4]. The proposed method is too precise for the purpose but enables rejection of the charges about computational errors.

2. Mathematical model

Equation (2.1) is adequate to the process

$$\frac{d^2x}{dt^2}+2\beta\frac{dx}{dt}+\omega_0^2x=F_0\left\{p(t)+\frac{\left[\left(1-\frac{Ax}{V}\right)^{\gamma}-1\right]}{\left(1-\frac{Ax}{V}\right)^{\gamma}}\right\}, \tag{2.1}$$

where $\beta=\frac{C}{2M}$, $\omega_0=\sqrt{\frac{K}{M}}$, $F_0=p_0\frac{A}{M}$, $\gamma=1.2$.

The pressure is measured and known as the set of pairs

$$\{t_i, p_i\},\ i=0,1,2,3,\dots,I. \tag{2.2}$$

The function $p(t)$ is approximated by the spline method (see [4] and references in Ref. 4)

$$p(t)=p_s(t). \tag{2.3}$$

At the time $t = 0$, the lungs are at rest

$$x(0)\ =\ 0;\ \frac{dx(t)}{dt}\Big|_{t=0}=0. \tag{2.4}$$

The relative displacement is used in calculations

$$\xi(t)=\frac{x(t)}{x_0} \tag{2.5}$$

and the dimensionless parameter is introduced $\varepsilon=\frac{Ax_0}{V}$. (2.6)

The range of solutions is $0\le\xi<\varepsilon^{-1}$, $\left\{0\le x(t)<\frac{V}{A}\right\}$.

The problem to solve consists of the initial conditions:

$$\xi(0)=0,\quad \frac{d\xi(0)}{dt}=0 \tag{2.7}$$

to the equation

$$\begin{gathered}\left[1+\sum_{n=1}^{N}w_n\varepsilon^n\xi^n(t)\right]\left[\frac{d^2\xi(t)}{dt^2}+2\beta\frac{d\xi(t)}{dt}+\omega_0^2\xi(t)\right]=\\ =W_0\left\{\left[1+\sum_{n=1}^{N}w_n\varepsilon^n\xi^n(t)\right]p_s(t)+\left[1+\sum_{n=1}^{N}w_n\varepsilon^n\xi^n(t)\right]-1\right\},\end{gathered} \tag{2.8}$$

where

$$W_0 = \frac{p_0 A}{M x_0} \text{ is for the known parameter,} \tag{2.9}$$

$$w_n = \frac{(-1)^n}{n!} \prod_{\nu=0}^{\nu=n-1} (\gamma - \nu), \;\; n = 1, 2, \ldots, N \tag{2.10}$$

are the coefficients of Taylor series of $(1-\varepsilon\xi)^\gamma$. The equivalent iterative formula is

$$w_{n+1} = w_n \frac{(-1)}{n+1}(\gamma - n), \quad w_1 = -\gamma. \tag{2.10'}$$

3. Solution to the problem

The solution is the compositions of the solutions in the intervals $(t_i, t_{i+1}]$; $i = 0, 1, 2, \ldots, I-1$. The intervals are taken from Eq. (2.2). The dimensionless time is used in each interval

$$s = \frac{t - t_i}{\Delta t_i}; \quad \Delta t_i = t_{i+1} - t_i; \quad s \in (0,1] \tag{3.1}$$

and the initial problem is solved

$$\begin{aligned} &\left[1 + \sum_{n=1}^{N} w_n \varepsilon^n \xi^n(s)\right]\left[\frac{d^2\xi(s)}{ds^2} + 2\beta\Delta t_i \frac{d\xi(s)}{ds} + \omega_0^2 (\Delta t_i)^2 \xi(s)\right] = \\ &= (\Delta t_i)^2 W_0 \left\{\left[1 + \sum_{n=1}^{N} w_n \varepsilon^n \xi^n(s)\right]\left[a_i s^3 + b_i s^2 + c_i s + d_i\right] + \left[\sum_{n=1}^{N} w_n \varepsilon^n \xi^n(t)\right]\right\}, \end{aligned} \tag{3.2}$$

$$\xi(0) = \xi_0, \quad \frac{d\xi(0)}{ds} = \upsilon_0. \tag{3.3}$$

Initial values (3.3) are taken from the solution in the previous interval. The function $x(t)$ and its first derivative $\frac{dx(t)}{dt}$ are continuous, so for interval i we have

$$\xi_0 = \xi(1); \quad \upsilon_0 = \frac{\Delta t_{i-1}}{\Delta t_i} \frac{d\xi(s)}{ds}\Big|_{s=1}. \tag{3.4}$$

The function $\xi(s)$ at the right side of (3.4) is taken from the previous interval i–1. For $i = 0$ initial conditions ξ_0, υ_0 are taken from initial conditions of the process (2.7). The problem may be considered as non-linear generalized Sturm-Liouville problem.

The solution to the initial problem (3.2), (3.3) is sought as the power series [1, 5] (and references in Ref. 1)

$$\xi(s) = \xi_0 + v_0 s + \sum_{m=2}^{m=M} h_m s^m; \quad h_0 = \xi_0; \quad h_1 = v_0. \tag{3.5}$$

Equating coefficients of the same powers s^q in equation (3.2) consecutive formulas for coefficients of the series (3.5) h_m, $m = 2, 3, \ldots, M$ are calculated.

The algorithm of calculations contains stepping up for intervals $i = 0,1,2 \ldots, I - 1$. In each interval calculations begin at exponent $q = 0$ and step up with exponent filling all calculated symbols with proper values. The symbols are introduced:

$$I = \sum_{n=1}^{N} \left\{ w_n \varepsilon^n \left[\sum_{m=0}^{M} h_m s^m \right]^n \right\} = \sum_{\lambda=0}^{NM} \left\{ \left[\sum_{n=1}^{N} \left(w_n \varepsilon^n f_\lambda^{(n)} \right) \right] s^\lambda \right\}, \tag{3.6}$$

$$I_\lambda = \sum_{n=1}^{N} \left(w_n \varepsilon^n f_\lambda^{(n)} \right), \tag{3.7}$$

$$f_\lambda^{(1)} = h_\lambda,\ l = 0,1,2,\ldots, NM,\ h_l = 0 \ \text{for}\ l = M+1, M+2, \ldots NM,$$

$$f_0^{(2)} = h_0^2, \quad f_1^{(2)} = 2h_0 h_1; \quad f_\lambda^{(2)} = \sum_{\mu=0}^{\mu=\lambda} f_\mu^{(1)} h_{\lambda-\mu}, \quad \lambda = 2,3,\ldots,2M,$$

$$f_\lambda^{(2)} = 0, \quad \lambda = 2M+1, 2M+2, \ldots, NM,$$

$$f_0^{(n)} = h_0^n, \quad f_1^{(n)} = n h_0^{n-1} h_1; \quad f_\lambda^{(n)} = \sum_{\mu=0}^{\mu=\lambda} f_\mu^{(n-1)} h_{\lambda-\mu},$$

$$l = 2, 3, \ldots, nM;$$

$$f_\lambda^{(n)} = 0, \quad \lambda = nM+1, nM+2, \ldots, NM, \qquad n = 3,4,\ldots,N.$$

The convention is used in formulas: a^n means a to n (power), but $a^{(n)}$ is for vector with the upper index n.

$$\left[\frac{d^2\xi(s)}{ds^2} + 2\beta \Delta t_i \frac{d\xi(s)}{ds} + \omega_0^2 \left(\Delta t_i\right)^2 \xi(s) \right] = \sum_{\mu=0}^{\mu=M} J_\mu s^\mu,$$
$$J_\mu = h_{\mu+2}(\mu+2)(\mu+1) + 2\beta \Delta t_i (\mu+1) h_{u+1} + \omega_0^2 (\Delta t_i)^2 h_\mu. \tag{3.8}$$

Right side of Eq. (3.2) is

$$\boldsymbol{R} = (\Delta t_i)^2 W_0 \sum_{\lambda=0}^{M} R_\lambda s^\lambda, \tag{3.9}$$

$$R_0 = d_i + (1 + d_i)I_0,$$
$$R_1 = c_i(1 + I_0) + (1 + d_i)I_1,$$
$$R_2 = b_i(1 + I_0) + c_i I_1 + (1 + d_i)I_2,$$
$$R_3 = a_i(1 + I_0) + b_i I_1 + c_i I_2 + (1 + d_i)I_3,$$
$$R_\lambda = a_i I_{\lambda-3} + b_i I_{\lambda-2} + c_i I_{\lambda-1} + (1 + d_i)I_\lambda,\ \lambda = 4, 5, ..., M.$$

For $q = 0$ we get

$$2h_2 = -2\beta(\Delta t_i)h_1 - \omega_0^2(\Delta t_i)^2 h_0 + (\Delta t_i)^2 W_0 R_0 (1 + I_0)^{-1}. \tag{3.10}$$

For $q = 1$ we get

$$6h_3 = -2\beta(\Delta t_i)2h_2 - \omega_0^2(\Delta t_i)^2 h_1 + (\Delta t_i)^2 W_0 R_1 (1 + I_0)^{-1} - I_1 J_0 (1 + I_0)^{-1}. \tag{3.11}$$

For $q = 2, 3, \ldots, M$ we get

$$\begin{aligned}(q+2)(q+1)h_{q+2} = -2\beta(\Delta t_i)(q+1)h_{q+1} - \omega_0^2(\Delta t_i)^2 h_q + \\ + (\Delta t_i)^2 W_0 R_q (1 + I_0)^{-1} - (1 + I_0)^{-1}\sum_{\lambda=1}^{q} I_\lambda J_{q-\lambda}.\end{aligned} \tag{3.12}$$

Series are cut at the end of calculations, not at the intermediate steps. The problem of error estimation and convergence of the series is discussed with the final discussion of results.

4. Results of numerical analysis

4.1. Convergence of the series

The splain approximation is performed by our "AS" program (Visual C++. NET 2003). Data are taken from the experiments at firing ground in form of Eq. (2.2) (or in the different forms). The overpressure divided by ambient pressure is approximated by the function $p_s(t)$, with the chosen deviation δ [4].

$$\begin{aligned} p_s(t) &= \left[a_i s^3 + b_i s^2 + c_i s + d_i \right], \\ s &= \frac{t - t_i}{\Delta t_i}; \quad \Delta t_i = t_{i+1} - t_i; \quad s \in (0,1]. \end{aligned} \tag{4.1}$$

The set $[a_i, b_i, c_i\, d_i, t_i, dt_i]$, $i = 0, 1, \ldots, I-1$, is taken form the file produced by "AS".

Convergence of the series $\left(1-\varepsilon\xi\right)^{\gamma}=\left[1+\sum_{n=1}^{N}w_n\varepsilon^n\xi^n(t)\right]$ (2.10) was investigated. For example, when $\varepsilon\xi = 0.901098901$, the difference between left and right side of equality (2.10) is as in Table 1.

Table 1

Convergence of $\left(1-\varepsilon\xi\right)^{\gamma}$ Taylor series

N	10	20	30	40	120
Δ	1.64×10^{-3}	1.74×10^{-4}	2.93×10^{-5}	0.6×10^{-6}	1.6×10^{-10}

The $N = 120$ elements of the sum of Eq. (2.10) where applied. Function (3.5) is the solution of the initial problem (3.3) for Eq. (3.2) if it is calculated in accordance with Eqs. (3.6)-(3.12) and its convergence radius is grater than zero [5]. The condition — the convergence radius is greater than zero — is the necessary condition in our problem. We are not able to prove mathematically this necessary condition, but we can easily recognise if the convergence radius R_c is greater than 1. If $R_c < 1$, the calculations of the series $[h_m]$ destabilizes abruptly. It is easily detected during calculations, especially for big numbers of elements in the series (3.5). We take the condition $R_c > 1$ as the sufficient condition of the existence of the solution (3.5). In Ref. 5, the method of analytic continuation is applied to the solution of the problem. Applying the method of analytic continuation we may get the solution when $0 < R_c < 1$ also. In the prepared computer program, the solutions can be found only for $R_c > 1$ for each step i, $t_i \leq t < t_{i+1}$. The problem (3.2 -3.12) is nonlinear, so the convergence radius depends on the external pressure (external force). The computer program verifies the condition $R_c > 1$ for each step i. Only sufficiently big numbers of the elements M (3.6) are applied, because the imprecisely calculated initial conditions at the interval i–1 influence the solution for interval i and next intervals. The influence of the number of elements in the series (3.5) on the results was investigated. The distance between the functions is measured by mean square deviation

$$MSD=\sqrt{\frac{1}{I-1}\sum_{i=1}^{I-1}\left[\xi_{M1}(t_i)-\xi_{M2}(t_i)\right]^2},$$

where $M1$ and $M2$ are the different numbers of the elements M.

Calculations were performed using $M = 145$. Calculations of MSD performed for $M = 92, 103, 114, 125, 136, 147$ disclose, that the differences MSD are of the order of computer 0. Generally, too many elements of the sums were used in calculations for this simple model with specific equation.

4.2. Comparison of spline interpolation and approximation

The results of spline interpolation applied to the experimental data are seen below. When spline interpolation is applied, the function $p_s(t)$ is equal to the measured values at $t = t_i$,

$$p_s(t_i) = p_i, \;\; i = 1, 2, 3, \ldots, I. \tag{4.2}$$

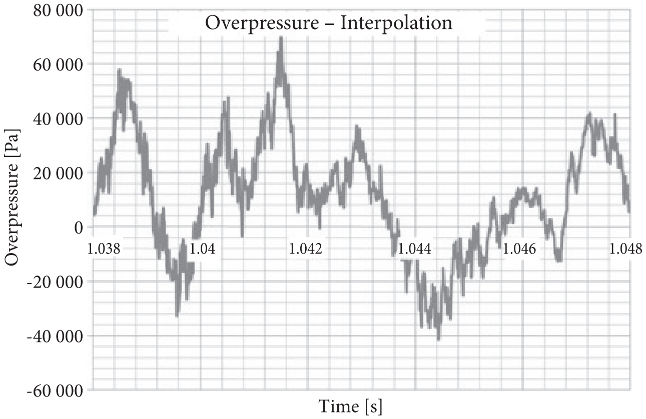


Fig. 2. Results of measurements interpolated by spline method

The results of spline interpolation were applied for calculations of displacement, velocity, acceleration by the method described in the previous chapter.

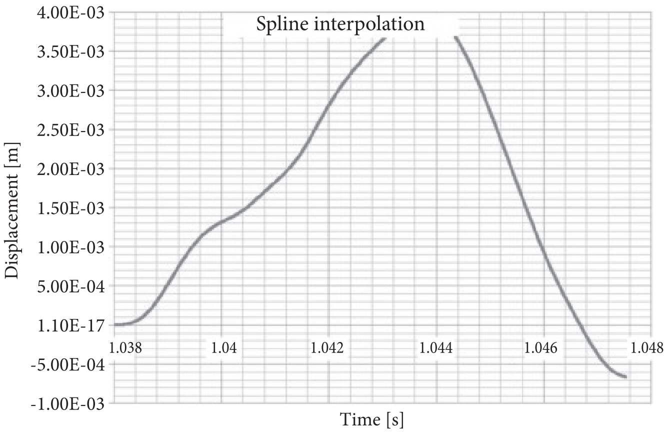


Fig. 3. The displacement of lungs calculated with spline interpolation

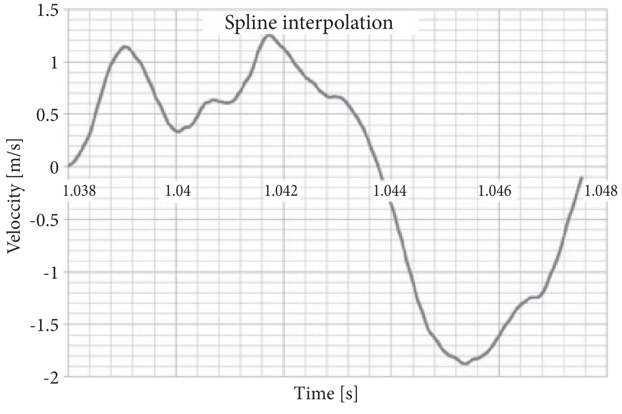


Fig. 4. The velocity of lungs calculated with spline interpolation

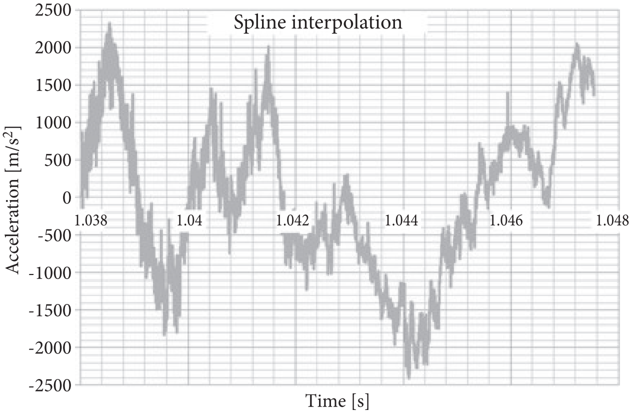


Fig. 5. The acceleration of lungs calculated with spline interpolation

The spline approximation minimizes functional

$$F\left[p\right]=u\int_0^{t_I}\left[\frac{d^2 p_s(t)}{dt^2}\right]^2 dt+\sum_{i=0}^{I} u_i\left[p_s(t_i)-p_i\right]^2, \tag{4.3}$$

where $u > 0$, and $u_i > 0$, $i=\overline{0,I}$. The theorem [3] determines, that if

$$p_s(t_i)+\frac{u}{u_i}\left[\frac{d^3 p_s(t)}{dt^3}\right]\Bigg|_{t=t_i}=p_i \tag{4.4}$$

then, functional (4.3) gets minimum.

$$\left[\frac{d^3 p_s(t)}{dt^3}\right]\Bigg|_{t=t_i} = \frac{d^3 p_s(t)}{dt^3}\Bigg|_{t=t_i+0} - \frac{d^3 p_s(t)}{dt^3}\Bigg|_{t=t_i-0},$$
$$\frac{d^3 p_s(t)}{dt^3}\Bigg|_{t=t_0-0} = 0,\ \frac{d^3 p_s(t)}{dt^3}\Bigg|_{t=t_I+0} = 0. \tag{4.4'}$$

The set of conditions is closed by "natural boundary conditions"

$$\frac{d^2 p_s(t)}{dt^2}\Bigg|_{t=t_0} = 0 \ \text{and}\ \frac{d^2 p_s(t)}{dt^2}\Bigg|_{t=t_I} = 0. \tag{4.5}$$

If the errors of measurements are known, then it is possible to choose the greater numbers u_i for the times t_i when the error is smaller. The exemplary data considered in the paper where obtained at experiments without possibility of errors estimation for each sample p_i. So, for all t_i, $u_i = 1$ is assumed. The rate of approximation is estimated by the number ε

$$\varepsilon = \frac{1}{\sigma}\sqrt{\frac{1}{I+1}\sum_{i=0}^{i=I} u_i \left[p_i - p_s(t_i)\right]^2}. \tag{4.6}$$

The divisor

$$\sigma = \sqrt{\frac{1}{I+1}\sum_{i=0}^{i=I} u_i \left[p_i\right]^2}, \tag{4.7}$$

removes the influence of constant multiplier at the rate of approximation. It means, that the units do not influence ε. For the chosen rate of approximation, the parameter u (4.3) is calculated by the computer program "AS". The influence of the rate ε is seen in Fig. 2 ($\varepsilon = 0$), in Fig. 6 ($\varepsilon = 0.05$), and in Fig. 7 ($\varepsilon = 0.10$).

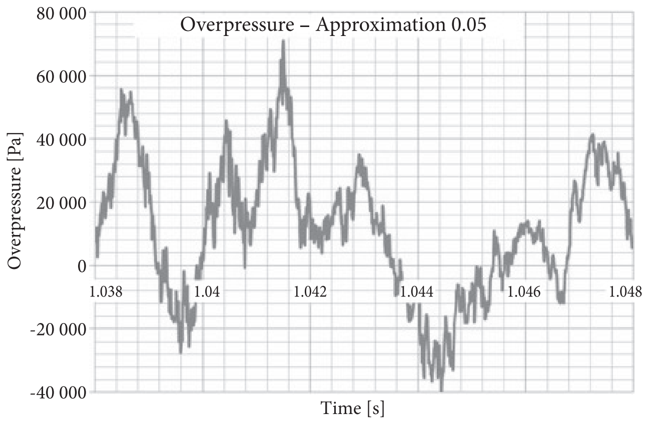


Fig. 6. Results of measurements approximated by spline, $\varepsilon = 0.05$

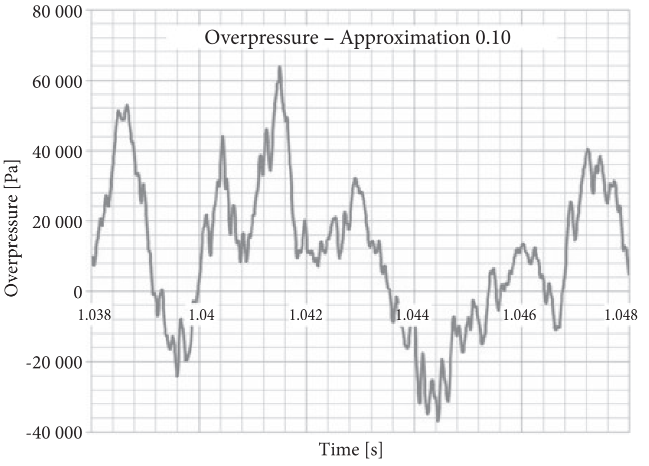


Fig. 7. Results of measurements approximated by spline, $\varepsilon = 0.10$

The chaotic vibrations with frequency about 5 kHz easily seen in Fig. 2 (interpolation) and also seen in Fig. 6 [approximation ($\varepsilon = 0.05$)] nearly disappear for approximation ($\varepsilon = 0.10$) in Fig. 7. This process of filtration is seen in spectrum [4]. At the same time, the filtration with the small rate ε does not influence displacement, velocity and even acceleration considerably. It is illustrated in exemplary Figs. 3, and 8; 4, and 9; 5, and 10.

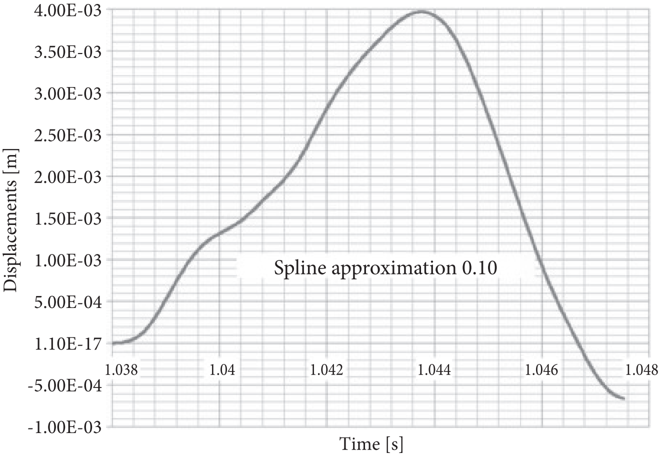


Fig. 8. Displacements of the lungs, spline approximation $\varepsilon = 0.10$

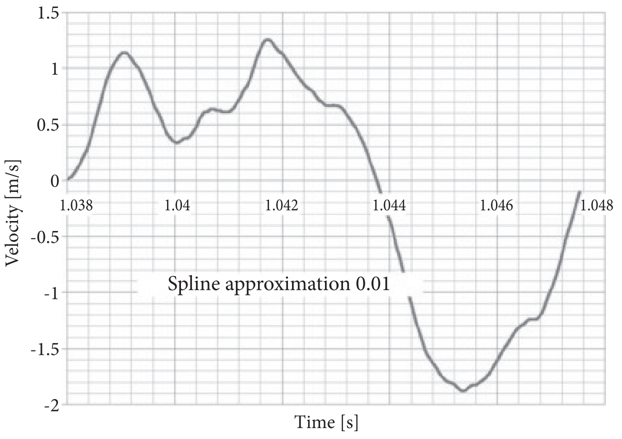


Fig. 9. Velocity of the lungs, spline approximation ε = 0.10

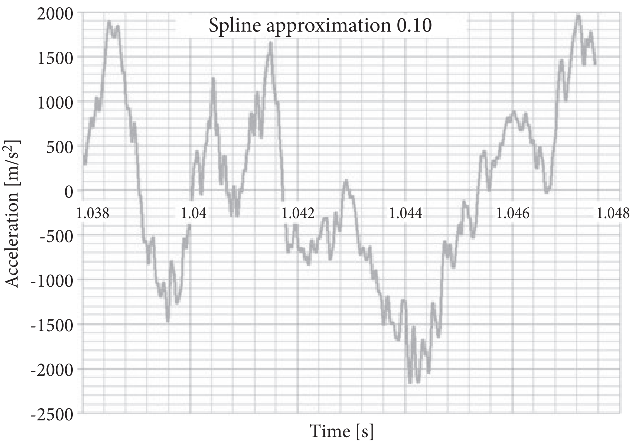


Fig. 10. Acceleration of the lungs, spline approximation ε = 0.10

The computer programs are ready and validated but they are not user friendly. The methods are worth to be known because they may be the only one for the similar problem when the high precision would be necessary.

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M. SZUDROWICZ, L. SOLARZ

Obliczenia prawdopodobieństwa obrażeń płuc spowodowanych falą podmuchu

Streszczenie. Proponowana metoda obliczeń łączy aproksymację funkcjami sklejanymi z precyzyjną metodą rozwiązania równań różniczkowych zwyczajnych przez szeregi potęgowe o dostatecznie dużym promieniu zbieżności. Metoda ta jest podobna do metody rozwiązania zagadnienia początkowego w uogólnionym problemie Sturma-Liouvilla. Metoda została zastosowana do obliczenia współczynnika szacującego możliwość urazu płuc poprzez oddziaływanie fali uderzeniowej wybuchu.

Słowa kluczowe: szacowanie urazu, oddziaływanie wybuchu na płuca, aproksymacja funkcjami sklejanymi, metoda szeregów potęgowych