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TIME-DEPENDENT PROBLEM FOR DETERMINATION OF EXHAUST CONCENTRATION IN URBAN TRANSPORT SYSTEM

Abstract

The paper proposes the 3-D mathematical model for analytic determination of exhaust concentration dynamics in the city regions under a priori information on airflow velocity. In this model it is provided that the turbulent and molecular diffusion coefficient changes depending on the vertical remoteness above the ground surface. The numerical example of problem solving is presented. The created model can be used for solutions of operative problems on urban traffic organization, of long-term planning of urban agglomeration development and new highway building.

Key words: Auto transport, urban ecology, time-dependent mathematical model, exhaust concentration, airflow velocity

INTRODUCTION

Environmental pollution, noise and vibration exploration of the urban traffic becomes more serious with every year. Evaluation problem of transport negative impact on urban ecology is the subject of many scientific researches. The evaluation of air pollution by automobile transport takes a special place in the analysis of town pollution level, since hazardous pollutants emission into the atmosphere is resulted by burning of fuel. According to the data [1] the environmental pollution from automobile transport is 60,6% of all the atmosphere pollution. The chemical constitution of emissions depends on the fuel type and quality, production technology, the type of fueling in the car engine, and its technical stage in future emission filtration. Among the harmful substances that are being emitted into the atmosphere, carbon dioxide, nitrogen oxides, hydrocarbons, aldehydes, sulfides and lead have to be pointed out in the first place. In this list carbon dioxide (CO2) has the main influence on the atmosphere. It exerts maximal influence on the glasshouse effect specifically. Besides, nitrogen oxides, hydrocarbons, aldehydes, sulfides, lead and other substances have negative impact on urban people health. The proof of that is the increase in quantity of urban people's oncological diseases, heart-vascular system and lungs diseases. As an example, cancer incidences rate in Switzerland is 9 times higher for people who live close to highways with the intensive traffic rather than for people who live in suburbs located 400 meters away from the highway.

Annual air pollution management costs are 100 million pounds sterling in Great Britain, 700 billion yen in Japan, and 1,5 billions dollars in the USA. There is an opinion in the USA that pure air could reduce the expenses on medicine by more than 2 billions dollars per year [1].

Issues of urban environment protection against transport pollution require development and environment protection actions plan implementation. The scheme of urban environment protection inter-relation problems caused by transport is overviewed by authors in the work [2].

The main subjects of authors' research are the air pollution status evaluation and dynamics forecasting in urban traffic circumstances; actions development for clearing and prevention of pollution in a town area, and foremost, living areas. The present article analyses the mathematical model for the evaluation of such factors as traffic and transport structure, urban agglomeration planning and meteorological conditions influence on air pollution in urban quarters. The introduced model can be used for long-term planning of urban agglomeration, new urban quarters designing, and new highway building. This model can also be useful in operative questions solutions on a traffic organization matter within a living area.

1. THE GENERAL DEFINITION OF THE PROBLEM

In general case the following definition of the problem takes place. Let the set $S(t)=\{s_1(t),s_2(t),...,s_n(t)\}$ be defining air condition in the given point of tree-dimensional space at the time t, where $s_j(t),\ j=1,2,...,n$ is determining the concentration of *j*-th substance.

It is assumed that at some moment of time t_e the condition of the air can be specified from the functional equation

$$S^*(t_e)=F\left[S(t_i),i=1,2,...,k;Y^*(t_1,...,t_k;t_e);Z^*(t_1,...,t_k;t_e)\right] \tag{1}$$

where F is some functional, $S(t_1),S(t_2),...,S(t_k)$ is the set specifying the condition of the air at the analyzed point at the moments of time $t_1,t_2,...,t_k$, that are anteceding to the moment t_e, i.e.; $Y^*\left(t_1,...,t_k;t_e\right)$ is the set of uncontrolled parameters, that are describing specification of external environment, influencing to the concentration of the substances in the air, meteorology in the first place (a temperature of the air, direction and speed of the wind) et al, $Z^*\left(t_1,...,t_k;t_e\right)$ is the set of partly or fully control external environment parameters, regulation of which influences on the concentration of the substances in the air, for example, a traffic intensity, in total and per a car type.

Note that depending on the problem observed, some indexes of an external environment can be uncontrolled parameters of set $Y^*\left(t_1,...,t_k;t_e\right)$ and controlled parameters of set $Z^*\left(t_1,...,t_k;t_e\right)$. For example, such interesting indexes as town quarters planning, a profile and specification of highways, specification of green space are related to the uncontrolled parameters of the set $Y^*\left(t_1,...,t_k;t_e\right)$, if problem of operational control of transport flow is considered, and to the controlled parameters of the set $Z^*\left(t_1,...,t_k;t_e\right)$, if the problem of urban development long-term planning and execution of ecology actions, for example, a green space implantation is considered. The tasks for operational control of transport flow, urban development long-term planning and building of new highways in general case are stated as follows: with the stated set $Y^*\left(t_1,...,t_k;t_e\right)$ and $Z^*\left(t_1,...,t_k;t_e\right)$ it is required to determinate a concentration of hazardous substances $S(t_e)$ in the examination area of town air space, and with the excess of *j*-th substance concentration allowed level $s_j^{\max}$ it is required to change the controlled parameters $Z^*\left(t_1,...,t_k;t_e\right)$ in the way to have eligible set elements $S^*(t_e)$, i.e. concentrations $s_j(t_e)\in S^*(t_e)$ would be within tolerance limits: $s_j(t_e)\le s_j^{\max}$, $j=1,2,...,n$.

Note, that solving of problem (1) should be executed for different variants of external environment conditions $Y^*\left(t_1,...,t_k;t_e\right)$, besides, a corresponding managerial decision (the set of control impacts) will be defined for each variant. The set of these solutions allows to formulate the urban transport system management strategy and to find the proposals of its development.

For determination of the functional F in equation (1) different mathematical methods can be used, primarily, the methods that are based on mathematical description of physical processes, and statistical methods (for example, the methods of multiregression), that use the statistics which has been gathered in the process of ecological monitoring of the city. The main advantage of physical methods is the possibility to use them for each new situation, while the statistical methods are applied only after a large number of observations has been gathered. The physical models allow receiving more exact solutions, but in opposite to statistical models they are more complicated and require more sophisticated mathematical tools for their solution.

In this research the authors are building exactly the physical model, namely, under the functional F in the functional equation (1) a differential operator is being formed, which is describing the process of turbulent diffusion when mass transfer in the space is dictated by the turbulent motion of environment. In the present work the solution of the building model is analytically in closed formula. Besides, this research carries out numerical implementation of the obtained analytical solution with certain simplifying assumptions.

2. MATHEMATICAL MODEL

In present work the 3-D mathematical model for analytic determination of exhaust concentration dynamics in the city regions under a priori information on air flow velocity is proposed. In this model it is provided that the turbulent and molecular diffusion coefficient changes depending on the vertical remoteness above the ground surface, i.e. the condition of vertical layering of town air is designed-in the proposed mathematical model owing to long-term harmful substances accumulation therein produced by vehicles.

The mathematical statement of the problem formulated above is following: it is required to find the concentration $C^{\{n\}}(x_1,x_2,x_3,t)$ of n-th $\left(n=\overline{1,N}\right)$ harmful substance at any spatial point (x_1,x_2,x_3) in the bounded and closed domain $[0,l_1]\times[0,l_2]\times[0,l_3]$ at any moment of time $t\in[0,T]$ from the equation

$$\frac{\partial C^{\{n\}}(x,t)}{\partial t}=div\left(D\left(\vec{\vartheta}(x,t)\right)\cdot\overline{grad}\,C^{\{n\}}(x,t)\right)-\vec{\vartheta}(x,t)\cdot\overline{grad}\,C^{\{n\}}(x,t),\ t\ge 0,$$
$$x=(x_1,x_2,x_3):\ 0<x_i<l_i\ \left(i=\overline{1,3}\right); \tag{2}$$

from the initial condition

$$C^{\{n\}}(x,t)\Big|_{t=0}=C_0^{\{n\}}(x),\ x=(x_1,x_2,x_3):\ 0\le x_i\le l_i\ \left(i=\overline{1,3}\right); \tag{3}$$

from the boundary condition (for each fixed $j=\overline{0,M-1}$) (see [3-5])

$$\gamma_{i,1,j}^{\{n\}}\cdot\left.\frac{\partial C^{\{n\}}(x,t)}{\partial x_i}\right|_{x_i=a_{i,j}}-\gamma_{i,2,j}^{\{n\}}\cdot C^{\{n\}}(x,t)\Big|_{x_i=a_{i,j}}=C_{i,j}^{\{n\}}\left(x/\{x_i\},t\right),\ a_{i,j}\le x_i\le b_{i,j}\ \left(i=\overline{1,3}\right),\ t\ge 0, \tag{4}$$

$$\gamma_{i,3,j}^{\{n\}}\cdot\left.\frac{\partial C^{\{n\}}(x,t)}{\partial x_i}\right|_{x_i=b_{i,j}}+\gamma_{i,4,j}^{\{n\}}\cdot C^{\{n\}}(x,t)\Big|_{x_i=b_{i,j}}=C_{i+3,j}^{\{n\}}\left(x/\{x_i\},t\right),\ a_{i,j}\le x_i\le b_{i,j}\ \left(i=\overline{1,3}\right),\ t\ge 0, \tag{5}$$

from the matching condition

$$C^{\{n\}}(x,t)\Big|_{x_3=l_3^{\{j\}}-0}=C^{\{n\}}(x,t)\Big|_{x_3=l_3^{\{j\}}+0},\ j=\overline{1,M-1},\ 0\le x_i\le l_i\ \ (i=1,2), \tag{6}$$

$$D\left(\vec{\vartheta}(x,t)\right)\cdot\left.\frac{\partial C^{\{n\}}(x,t)}{\partial x_3}\right|_{x_3=l_3^{\{j\}}-0}=D\left(\vec{\vartheta}(x,t)\right)\cdot\left.\frac{\partial C^{\{n\}}(x,t)}{\partial x_3}\right|_{x_3=l_3^{\{j\}}+0},\ j=\overline{1,M-1},\ \ 0\le x_i\le l_i\ \ (i=1,2). \tag{7}$$

In the unsteady-state initial boundary value (2)-(7) problem the coefficient of turbulent and molecular diffusion is designated as $D\left(\vec{\vartheta}(x,t)\right)$, which is assumed as piecewise constant function in this work:

$$0<d_{\min}\le D\left(\vec{\vartheta}(x,t)\right)\overset{def}{\equiv}\begin{cases}D_1=const & if\quad 0=l_3^{\{0\}}\le x_3\le l_3^{\{1\}},\\ D_2=const & if\quad l_3^{\{1\}}\le x_3\le l_3^{\{2\}},\\ \dots\dots\dots\dots\dots\dots & \\ D_M=const & if\quad l_3^{\{M-1\}}\le x_3=l_3^{\{M\}},\end{cases}\quad \text{for } \forall x_i\in[0,l_i],\ (i=1,2),$$

where M is the quantity of layered domains on the vertical axis OX_3, i.e. these layered domains are parallel to the plane X_1OX_2; the vector–function $\vec{\vartheta}(x,t)$ signifies an airflow velocity, which it is purposed experimentally known and piecewise constant function in this work:

$$\vec{\vartheta}(x,t)\overset{def}{\equiv}\begin{cases}\vartheta_1^{\{av\}}=const & if\quad 0=l_3^{\{0\}}\le x_3\le l_3^{\{1\}},\\ \vartheta_2^{\{av\}}=const & if\quad l_3^{\{1\}}\le x_3\le l_3^{\{2\}},\\ \dots\dots\dots\dots\dots\dots & \\ \vartheta_M^{\{av\}}=const & if\quad l_3^{\{M-1\}}\le x_3=l_3^{\{M\}},\end{cases}\quad \text{for } \forall x_i\in[0,l_i],\ (i=1,2);$$

and finally, the points $a_{i,j}$ and $b_{i,j}$ mean the boundary points of layers, namely, the points $a_{i,j}$ and $b_{i,j}$ are definition as

$$a_{i,j}\overset{def}{\equiv}\begin{cases}0 & if\quad i=1,2,\\ l_3^{\{j\}} & if\quad i=3;\ j\ne 0,\\ l_3^{\{0\}} & if\quad i=3;\ j=0,\end{cases}\qquad b_{i,j}\overset{def}{\equiv}\begin{cases}l_i & if\quad i=1,2;\ \forall j,\\ l_3^{\{j+1\}} & if\quad i=3;\quad \forall j.\end{cases}$$

In the problem (2)-(7) we assume that the following constants are known initial constants: $N\in\mathbb{N}$, $M\in\mathbb{N}$, $T\in\mathbb{R}_+^1$, $D_j\in\mathbb{R}_+^1\ \left(j=\overline{1,M}\right)$, $l_i^{\{j\}}\in\mathbb{R}_+^1\left(i=\overline{1,3};\ j=\overline{0,M}\right)$, $\vartheta_j^{\{av\}}\ \left(j=\overline{1,M}\right)$, $\gamma_{i,k,j}^{\{n\}}\in\mathbb{R}_+^1\ \left(i=1,2;\ \ k=\overline{1,4};\ \ j=\overline{0,M-1}\right)$, and for $\forall n=\overline{1,N}$

$$\gamma_{i,k,j}^{\{n\}}=\begin{cases}1 & if\ j=0;\ i=1;\ k=1,3,\\ 0 & if\ j=0;\ i=1;\ k=2,4,\\ \gamma_{i,k,j}^{\{n\}}>0 & if\quad j\ne 0;\ \forall i,k.\end{cases}$$

Besides, in the problem (2)-(7) we suppose that the initial functions $C_0^{\{n\}}(x)\ \left(\forall n=\overline{1,N}\right)$, $C_{i,j}^{\{n\}}(\cdot,t)\ \left(i=\overline{1,6};\ j=\overline{0,M-1}\right)$ are given functions, and for $\forall n=\overline{1,N}$ it is

$$C_{i,j}^{\{n\}}(\cdot,t)=\begin{cases} C_{i,j}^{\{n\}}(\cdot,t) & if\ \{j=0\}\wedge\{i=3,6\}; \\ 0 & otherwise\ . \end{cases}$$

Finally, besides the conditions (1)-(7) in the mathematical statement of the considered problem we require the satisfaction of the corresponding consistency conditions, i.e. the initial function $C_0^{\{n\}}(x)$ and the boundary functions $C_{i,j}^{\{n\}}(\cdot,t)$ satisfy the consistency conditions at the corresponding conjugate points for each $n=\overline{1,N}$ harmful substance.

Using standard approach (see [6]) we can prove that the formulated problem (2)-(7) has an unique solution. In present paper we will suppose that the quantity of vertical layers, which are between the ground surface and the upper bound l_3, equals to four, i.e. $M=4$, and the bounds of each layer are given empirically.

Now we start to solve the formulated problem (2)-(7). For the purpose we will introduce the following new function

$$w^{\{n\}}(x,t)\overset{def}{\equiv}\begin{cases} w_1^{\{n\}}(x,t) & if\quad 0=l_3^{\{0\}}\le x_3\le l_3^{\{1\}}, \\ w_2^{\{n\}}(x,t) & if\quad l_3^{\{1\}}\le x_3\le l_3^{\{2\}}, \\ \dots\dots\dots\dots\dots\dots & \\ w_M^{\{n\}}(x,t) & if\quad l_3^{\{M-1\}}\le x_3=l_3^{\{M\}}, \end{cases}\quad \forall 0\le x_i\le l_i\ \left(i=1,2\right),$$

where

$$w_j^{\{n\}}(x,t)\overset{def}{\equiv}e^{\frac{3\cdot\left(\vartheta_j^{\{av\}}\right)^2}{4\cdot D_j}\cdot t-\frac{\vartheta_j^{\{av\}}}{2\cdot D_j}\cdot\sum_{i=1}^{3}x_i}\cdot C^{\{n\}}(x,t),\ t\ge 0,\ 0\le x_i\le l_i\ \left(i=1,2\right),$$
$$l_3^{\{j-1\}}\le x_3\le l_3^{\{j\}}\ \left(j=\overline{1,M};\ l_3^{\{0\}}=0,\ l_3^{\{M\}}=l_3\right). \tag{8}$$

Taking into account the notation (8) in the equation (2) and in the initial condition (3), we have the following new problem: it is necessary to find the functions $w_j^{\{n\}}(x,t)\ \left(j=1,2\right)$ from the equation

$$\frac{\partial w_j^{\{n\}}(x,t)}{\partial t}=D_j\cdot\sum_{i=1}^{3}\frac{\partial^2 w_j^{\{n\}}(x,t)}{\partial x_i^2},\ t>0,\ 0<x_i<l_i\ \left(i=1,2\right),$$
$$l_3^{\{j-1\}}<x_3<l_3^{\{j\}}\ \left(j=\overline{1,M};\ l_3^{\{0\}}=0,\ l_3^{\{M\}}=l_3\right), \tag{9}$$

from the initial condition

$$w_j^{\{n\}}(x,t)\Big|_{t=0}=w_0(x),\ 0<x_i<l_i\ \left(i=1,2\right),\ l_3^{\{j-1\}}<x_3<l_3^{\{j\}}\ \left(j=\overline{1,M};\ l_3^{\{0\}}=0,\ l_3^{\{M\}}=l_3\right), \tag{10}$$

where

$$w_0(x)\overset{def}{\equiv}e^{-\frac{\vartheta_j^{\{av\}}}{2\cdot D_j}\cdot\sum_{i=1}^{3}x_i}\cdot C_0^{\{n\}}(x),$$

and from the corresponding boundary and matching conditions.

Thus, with the help of the nondegenerate transformation (8) the original problem (2)-(7) with inhomogeneous equation (2) is reduced to the problem (9)-(10) with homogeneous equation (9). The obtained problem is simpler than the original problem (1)-(7). Now we will find the solution of the received problem (9)-(10) in the form (see [6])

$$w^{\{n\}}(x_1,x_2,x_3,t)=\overline{W}(x_1)\cdot\overline{\overline{W}}(x_2)\cdot\overline{\overline{\overline{W}}}(x_3)\cdot T(t),\quad 0\le x_i\le l_i\ \left(i=\overline{1,3}\right). \tag{11}$$

Substituting (11) into (9)-(10) and having some not difficult transformations we have the solution of the reduced problem (9)-(10):

$$w^{\{n\}}\left(x_1,x_2,x_3,t\right)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}A_{i,j,k}^{\{n\}}\cdot W_{i,j,k}^{\{n\}}\left(x_1,x_2,x_3\right)\cdot e^{-\left(\overline{\lambda}_i^{\{n\}}\right)^2\cdot\left(\overline{\overline{\lambda}}_j^{\{n\}}\right)^2\cdot\left(\overline{\overline{\overline{\lambda}}}_k^{\{n\}}\right)^2\cdot t}, \tag{12}$$

where

$$A_{i,j,k}^{\{n\}}=\frac{\int_0^{l_1}dx_1\int_0^{l_2}dx_2\int_0^{l_3}w_0\left(x_1,x_2,x_3\right)\cdot W_{i,j,k}^{\{n\}}\left(x_1,x_2,x_3\right)dx_3}{\int_0^{l_1}dx_1\int_0^{l_2}dx_2\int_0^{l_3}\left\{W_{i,j,k}^{\{n\}}\left(x_1,x_2,x_3\right)\right\}^2dx_3}-\frac{\int_0^{T}dt\int_0^{l_1}dx_1\int_0^{l_2}w_{3,0}\left(x_1,x_2,t\right)\cdot W_{i,j,k}^{\{n\}}\left(x_1,x_2,x_3\right)dx_2}{\int_0^{l_1}dx_1\int_0^{l_2}dx_2\int_0^{l_3}\left\{W_{i,j,k}^{\{n\}}\left(x_1,x_2,x_3\right)\right\}^2dx_3}+$$

$$+\frac{\int_0^{T}dt\int_0^{l_1}dx_1\int_0^{l_2}w_{6,0}\left(x_1,x_2,t\right)\cdot W_{i,j,k}^{\{n\}}\left(x_1,x_2,x_3\right)dx_2}{\int_0^{l_1}dx_1\int_0^{l_2}dx_2\int_0^{l_3}\left\{W_{i,j,k}^{\{n\}}\left(x_1,x_2,x_3\right)\right\}^2dx_3},$$

$\overline{\lambda}_i^{\{n\}},\ \overline{\overline{\lambda}}_j^{\{n\}},\ \overline{\overline{\overline{\lambda}}}_k^{\{n\}}\quad\left(i,j,k\in\mathbb{N}\right)$ are roots of transcendental equation, respectively

$$tg\left(\overline{\lambda}^{\{n\}}\cdot l_1\right)=\overline{\lambda}^{\{n\}}\cdot\left\{\frac{\dfrac{\gamma_{1,2,1}^{\{n\}}}{\gamma_{1,1,1}^{\{n\}}}+\dfrac{\gamma_{1,4,1}^{\{n\}}}{\gamma_{1,3,1}^{\{n\}}}}{\overline{\lambda}^{\{n\}}-\dfrac{\gamma_{1,2,1}^{\{n\}}}{\gamma_{1,1,1}^{\{n\}}}\cdot\dfrac{\gamma_{1,4,1}^{\{n\}}}{\gamma_{1,3,1}^{\{n\}}}}+\frac{\dfrac{\gamma_{1,2,2}^{\{n\}}}{\gamma_{1,1,2}^{\{n\}}}+\dfrac{\gamma_{1,4,2}^{\{n\}}}{\gamma_{1,3,2}^{\{n\}}}}{\overline{\lambda}^{\{n\}}-\dfrac{\gamma_{1,2,2}^{\{n\}}}{\gamma_{1,1,2}^{\{n\}}}\cdot\dfrac{\gamma_{1,4,2}^{\{n\}}}{\gamma_{1,3,2}^{\{n\}}}}+\frac{\dfrac{\gamma_{1,2,3}^{\{n\}}}{\gamma_{1,1,3}^{\{n\}}}+\dfrac{\gamma_{1,4,3}^{\{n\}}}{\gamma_{1,3,3}^{\{n\}}}}{\overline{\lambda}^{\{n\}}-\dfrac{\gamma_{1,2,3}^{\{n\}}}{\gamma_{1,1,3}^{\{n\}}}\cdot\dfrac{\gamma_{1,4,3}^{\{n\}}}{\gamma_{1,3,3}^{\{n\}}}}+\frac{\dfrac{\vartheta_1^{\{average\}}}{D_1}}{\overline{\lambda}^{\{n\}}-\left(\dfrac{\vartheta_1^{\{average\}}}{D_1}\right)^2}\right\},$$

$$tg\left(\overline{\overline{\lambda}}^{\{n\}}\cdot l_2\right)=\overline{\overline{\lambda}}^{\{n\}}\cdot\left\{\frac{\dfrac{\gamma_{2,2,0}^{\{n\}}}{\gamma_{2,1,0}^{\{n\}}}+\dfrac{\gamma_{2,4,0}^{\{n\}}}{\gamma_{2,3,0}^{\{n\}}}}{\overline{\overline{\lambda}}^{\{n\}}-\dfrac{\gamma_{2,2,0}^{\{n\}}}{\gamma_{2,1,0}^{\{n\}}}\cdot\dfrac{\gamma_{2,4,0}^{\{n\}}}{\gamma_{2,3,0}^{\{n\}}}}+\frac{\dfrac{\gamma_{2,2,1}^{\{n\}}}{\gamma_{2,1,1}^{\{n\}}}+\dfrac{\gamma_{2,4,1}^{\{n\}}}{\gamma_{2,3,1}^{\{n\}}}}{\overline{\overline{\lambda}}^{\{n\}}-\dfrac{\gamma_{2,2,1}^{\{n\}}}{\gamma_{2,1,1}^{\{n\}}}\cdot\dfrac{\gamma_{2,4,1}^{\{n\}}}{\gamma_{2,3,1}^{\{n\}}}}+\frac{\dfrac{\gamma_{2,2,2}^{\{n\}}}{\gamma_{2,1,2}^{\{n\}}}+\dfrac{\gamma_{2,4,2}^{\{n\}}}{\gamma_{2,3,2}^{\{n\}}}}{\overline{\overline{\lambda}}^{\{n\}}-\dfrac{\gamma_{2,2,2}^{\{n\}}}{\gamma_{2,1,2}^{\{n\}}}\cdot\dfrac{\gamma_{2,2,2}^{\{n\}}}{\gamma_{2,1,2}^{\{n\}}}}+\frac{\dfrac{\gamma_{2,2,3}^{\{n\}}}{\gamma_{2,1,3}^{\{n\}}}+\dfrac{\gamma_{2,4,3}^{\{n\}}}{\gamma_{2,3,3}^{\{n\}}}}{\overline{\overline{\lambda}}^{\{n\}}-\dfrac{\gamma_{2,2,3}^{\{n\}}}{\gamma_{2,1,3}^{\{n\}}}\cdot\dfrac{\gamma_{2,4,3}^{\{n\}}}{\gamma_{2,3,3}^{\{n\}}}}\right\},$$

$$tg\left(\overline{\overline{\overline{\lambda}}}^{\{n\}}\cdot l_2\right)=\overline{\overline{\overline{\lambda}}}^{\{n\}}\cdot\left\{\frac{\dfrac{\gamma_{3,2,0}^{\{n\}}}{\gamma_{3,1,0}^{\{n\}}}+\dfrac{\gamma_{3,4,0}^{\{n\}}}{\gamma_{3,3,0}^{\{n\}}}}{\overline{\overline{\overline{\lambda}}}^{\{n\}}-\dfrac{\gamma_{3,2,0}^{\{n\}}}{\gamma_{3,1,0}^{\{n\}}}\cdot\dfrac{\gamma_{3,4,0}^{\{n\}}}{\gamma_{3,3,0}^{\{n\}}}}+\frac{\dfrac{\gamma_{3,2,1}^{\{n\}}}{\gamma_{3,1,1}^{\{n\}}}+\dfrac{\gamma_{3,4,1}^{\{n\}}}{\gamma_{3,3,1}^{\{n\}}}}{\overline{\overline{\overline{\lambda}}}^{\{n\}}-\dfrac{\gamma_{3,2,1}^{\{n\}}}{\gamma_{3,1,1}^{\{n\}}}\cdot\dfrac{\gamma_{3,4,1}^{\{n\}}}{\gamma_{3,3,1}^{\{n\}}}}+\frac{\dfrac{\gamma_{3,2,2}^{\{n\}}}{\gamma_{3,1,2}^{\{n\}}}+\dfrac{\gamma_{3,4,2}^{\{n\}}}{\gamma_{3,3,2}^{\{n\}}}}{\overline{\overline{\overline{\lambda}}}^{\{n\}}-\dfrac{\gamma_{3,2,2}^{\{n\}}}{\gamma_{3,1,2}^{\{n\}}}\cdot\dfrac{\gamma_{3,4,2}^{\{n\}}}{\gamma_{3,3,2}^{\{n\}}}}+\frac{\dfrac{\gamma_{3,2,3}^{\{n\}}}{\gamma_{3,1,3}^{\{n\}}}+\dfrac{\gamma_{3,2,3}^{\{n\}}}{\gamma_{3,1,3}^{\{n\}}}}{\overline{\overline{\overline{\lambda}}}^{\{n\}}-\dfrac{\gamma_{3,2,3}^{\{n\}}}{\gamma_{3,1,3}^{\{n\}}}\cdot\dfrac{\gamma_{3,2,3}^{\{n\}}}{\gamma_{3,1,3}^{\{n\}}}}\right\},$$

For each layer the function $W_{i,j,k}^{\{n\}}\left(x_1,x_2,x_3\right)$ from the formula (12) can be defined as following:

– ***the first layer*** $0\le x_1\le l_1,\ 0\le x_2\le l_2,\ 0\le x_3\le l_3^{\{1\}}$ is circumterraneous layer, and in this lowermost layer the function $W_{i,j,k}^{\{n\}}\left(x_1,x_2,x_3\right)$ has the form

$$W_{i,j,k}^{\{n\}}\left(x_1,x_2,x_3\right)=\frac{\cos\left(\theta_{i,j,k}^{\{n\},\{1\}}x_3\right)}{\cos\left(\theta_{i,j,k}^{\{n\},\{1\}}l_3^{\{1\}}\right)}\left[\cos\left(\overline{\beta}_i^{\{n\}}x_1\right)-\frac{\vartheta_1^{\{av\}}}{2D_1\overline{\beta}_i^{\{n\}}}\sin\left(\overline{\beta}_i^{\{n\}}x_1\right)\right]\left[\cos\left(\overline{\overline{\beta}}_i^{\{n\}}x_2\right)-\frac{2\cdot\gamma_{2,2,0}^{\{n\}}D_1-\vartheta_1^{\{av\}}}{2\cdot D_1\gamma_{2,1,0}^{\{n\}}\overline{\overline{\beta}}_i^{\{n\}}}\sin\left(\overline{\overline{\beta}}_i^{\{n\}}x_2\right)\right];$$

– in ***the second layer*** $0\le x_1\le l_1,\ 0\le x_2\le l_2,\ l_3^{\{1\}}\le x_3\le l_3^{\{2\}}$ the function $W_{i,j,k}^{\{n\}}\left(x_1,x_2,x_3\right)$ has the form

$$W_{i,j,k}^{\{n\}}\left(x_1,x_2,x_3\right)=\frac{\cos\left(\theta_{i,j,k}^{\{n\},\{2\}}\left(l_3^{\{2\}}-x_3\right)\right)}{\cos\left(\theta_{i,j,k}^{\{n\},\{2\}}\left(l_3^{\{2\}}-l_3^{\{1\}}\right)\right)}\left[\cos\left(\overline{\beta}_i^{\{n\}}x_1\right)+\frac{\vartheta_2^{\{av\}}-2\gamma_{1,2,1}^{\{n\}}D_2}{2D_2\gamma_{1,1,1}^{\{n\}}\overline{\beta}_i^{\{n\}}}\sin\left(\overline{\beta}_i^{\{n\}}x_1\right)\right]\left[\frac{\vartheta_2^{\{av\}}-2\gamma_{2,2,1}^{\{n\}}D_2}{2D_2\gamma_{2,1,1}^{\{n\}}\overline{\overline{\beta}}_i^{\{n\}}}\sin\left(\overline{\overline{\beta}}_i^{\{n\}}x_2\right)+\right.$$

$$\left.+\cos\left(\overline{\overline{\beta}}_i^{\{n\}}x_2\right)\right];$$

– in ***the third layer*** $0 \le x_1 \le l_1,\ 0 \le x_2 \le l_2,\ l_3^{\{2\}} \le x_3 \le l_3^{\{3\}}$ the function $W_{i,j,k}^{\{n\}}(x_1,x_2,x_3)$ has the form

$$W_{i,j,k}^{\{n\}}(x_1,x_2,x_3) = \frac{\cos\left(\theta_{i,j,k}^{\{n\},\{3\}}\left(l_3^{\{3\}}-x_3\right)\right)}{\cos\left(\theta_{i,j,k}^{\{n\},\{3\}}\left(l_3^{\{3\}}-l_3^{\{2\}}\right)\right)}\left[\cos\left(\overline{\overline{\beta}}_i^{\{n\}}x_2\right)+\frac{\vartheta_3^{\{av\}}-2\gamma_{1,2,2}^{\{n\}}D_3}{2D_3\gamma_{1,1,2}^{\{n\}}\overline{\beta}_i^{\{n\}}}\sin\left(\overline{\beta}_i^{\{n\}}x_1\right)\right]\left[\frac{\vartheta_3^{\{av\}}-2\cdot\gamma_{2,2,2}^{\{n\}}D_3}{2D_3\gamma_{2,1,2}^{\{n\}}\overline{\overline{\beta}}_i^{\{n\}}}\sin\left(\overline{\overline{\beta}}_i^{\{n\}}x_2\right)+\right.$$
$$\left.+\cos\left(\overline{\overline{\beta}}_i^{\{n\}}x_2\right)\right];$$

– ***the last layer*** $0 \le x_1 \le l_1,\ 0 \le x_2 \le l_2,\ l_3^{\{3\}} \le x_3 \le l_3$ is the most removed from a ground surface layer in considering bounded and closed parallelepiped $[0,l_1]\times[0,l_2]\times[0,l_3]$. In this uppermost layer the function $W_{i,j,k}^{\{n\}}(x_1,x_2,x_3)$ has the form

$$W_{i,j,k}^{\{n\}}(x_1,x_2,x_3) = \frac{\cos\left(\theta_{i,j,k}^{\{n\},\{4\}}\left(l_3-x_3\right)\right)}{\cos\left(\theta_{i,j,k}^{\{n\},\{4\}}\left(l_3-l_3^{\{3\}}\right)\right)}\left[\cos\left(\overline{\beta}_i^{\{n\}}x_1\right)+\frac{\vartheta_4^{\{av\}}-2\gamma_{1,2,3}^{\{n\}}D_4}{2D_4\gamma_{1,1,3}^{\{n\}}\overline{\beta}_i^{\{n\}}}\sin\left(\overline{\beta}_i^{\{n\}}x_1\right)\right]\left[\frac{\vartheta_4^{\{av\}}-2\gamma_{2,2,3}^{\{n\}}D_4}{2D_4\gamma_{2,1,3}^{\{n\}}\overline{\overline{\beta}}_i^{\{n\}}}\sin\left(\overline{\overline{\beta}}_i^{\{n\}}x_2\right)+\right.$$
$$\left.+\cos\left(\overline{\overline{\beta}}_i^{\{n\}}x_2\right)\right].$$

Here

$$\theta_{i,j,k}^{\{n\},\{m\}} \overset{def}{\equiv} \frac{\overline{\lambda}_i^{\{n\}}\cdot\overline{\overline{\lambda}}_j^{\{n\}}\overline{\overline{\overline{\lambda}}}_k^{\{n\}}}{D_m}-\frac{i^2\cdot\pi^2}{l_1^2}-\frac{j^2\cdot\pi^2}{l_2^2}\quad\left(m=\overline{1,M=1,4}\right),$$

$\overline{\beta}_i^{\{n\}}$ and $\overline{\overline{\beta}}_i^{\{n\}}$ are positive roots of the transcendental equation, respectively

$$D_1\cdot\sqrt{\frac{1}{D_1}-\frac{\pi^2}{l_1^2}}\cdot ctg\sqrt{\frac{1}{D_1}-\frac{\pi^2}{l_1^2}} = D_2\cdot\sqrt{\frac{1}{D_2}-\frac{\pi^2}{l_1^2}}\cdot ctg\sqrt{\frac{1}{D_2}-\frac{\pi^2}{l_1^2}},$$

$$D_1\cdot\sqrt{\frac{1}{D_1}-\frac{\pi^2}{l_2^2}}\cdot ctg\sqrt{\frac{1}{D_1}-\frac{\pi^2}{l_2^2}} = D_2\cdot\sqrt{\frac{1}{D_2}-\frac{\pi^2}{l_2^2}}\cdot ctg\sqrt{\frac{1}{D_2}-\frac{\pi^2}{l_2^2}}.$$

Thus, the formula (12) is the solution of the reduced problem. Then the solution of the original problem (1)-(7) can be found by the nondegenerate inverse transformation:

$$C^{\{n\}}(x_1,x_2,x_3,t) = e^{\frac{\vartheta_j^{\{average\}}}{2\cdot D_j}\cdot\sum_{i=1}^{3}x_i-\frac{3\cdot\left(\vartheta_j^{\{average\}}\right)^2}{4\cdot D_j}\cdot t}\cdot w^{\{n\}}(x_1,x_2,x_3,t),\ t\ge 0,\ 0\le x_i\le l_i\ \left(i=\overline{1,3}\right). \tag{13}$$

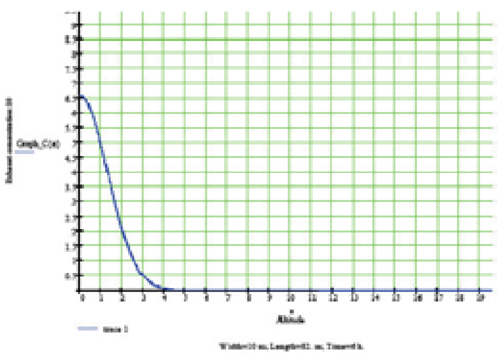
The formula (13) lets to determine required concentration $C^{\{n\}}(x_1,x_2,x_3,t)$ of the n-th harmful substance at any moment of time $t\in[0,T]$ in any spatial point (x_1,x_2,x_3) for each of N harmful substances.

3. EXAMPLE OF RESULTS

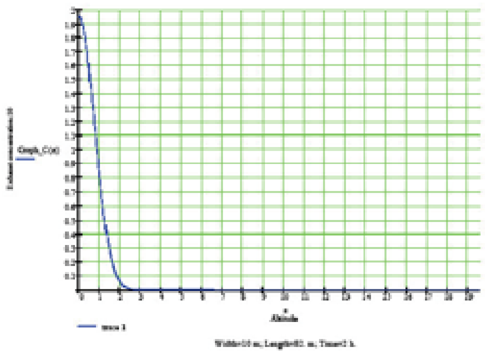
Let's consider the example of calculation of concentration change of harmful substance on a different height above the road area at the postinitiation fixed time moments 1, 2, 6 and 12 hours, respectively. We consider the road section with the width of 21 m and the length of 165 m. It is supposed that there are multistory buildings in both sides of considered road, at that the average height of buildings is 20 m. Besides, we assume that the number of cars driving through the road section per 12 hours is known and equals to 11000 units, i.e. the traffic flow rate in the considered road is accepted as equidistributed, namely, it equals to 917 cars/hour approximately. Other initial data are following: depending on the altitude wind velocity on layers changes from 4 m/s till 1 m/s; the coefficient of turbulent diffusion depends on altitude and it changes from 0.13 (highest) to 0.16 (lowest); average concentration of the

investigated harmful substance (as investigated material is taken CO_2 particularly) is assumed as 179 g/km; exhaust speed near cars is 60-100 m/s.
Computations will be performed for an "imaginary vertical column", the foundation of which is exactly in the middle of the considered road and it is determined for the point $(x_1 = 10.5\ \text{m};\ x_2 = 82.5\ \text{m})$. Numerical implementation of considered mathematical model has been realized by the packaged MathCAD. The results of calculations for the different moments of time, passing after the beginning of turbulent diffusion process, are presented in Figure 1.

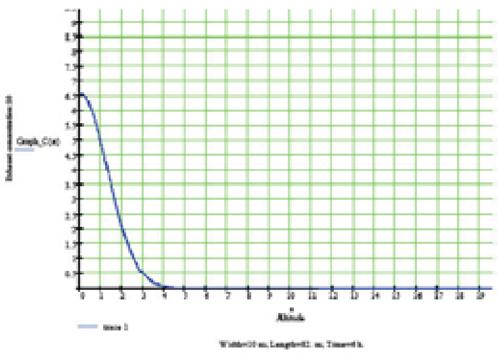
a)



b)



c)



d)

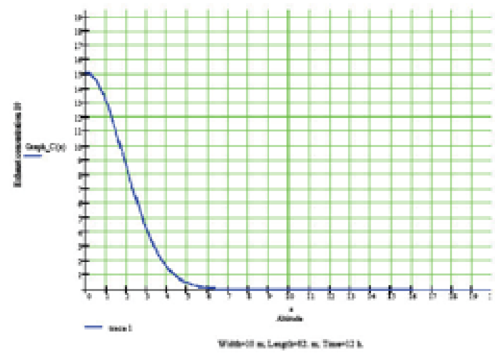


Figure 1. Change of the concentration of harmful matter on a different height above a road area

Figure 1*a* shows a change of concentration $C(x_1 = 10.5, x_2 = 82.5, x_2, t = 1)$ depending on the variable x_3, i.e. the constructed curve reflects a change of harmful substance concentration depending on a height x_3 in 1 hour after the beginning of process of supervision of harmful substance turbulent diffusion in the fixed point of the road area $(x_1 = 10.5\ \text{m};\ x_2 = 82.5\ \text{m})$.
Changes of concentration depending on a height in the same point at the moments 2, 6 and 12 hours after the beginning of turbulent diffusion process are presented in Figure 1*b*, 1*c* and 1*d*, respectively. Note that in Figure 1*a* scale of ordinates is compressed 10^2 times less, and in the other figures scales of ordinates are taken 10 times less, i.e. there is the graph of function $C(x_3) = 10^{-2} \cdot C(x_1, x_2, x_3, t)\big|_{x_1=10.5; x_2=82.5;\ t=1}$ in Figure 1*a*, and there are the graphs of functions $10^{-1} \cdot C(x_1, x_2, x_3, t)\big|_{x_1=10.5; x_2=82.5;\ t=2}$, $10^{-1} \cdot C(x_1, x_2, x_3, t)\big|_{x_1=10.5; x_2=82.5;\ t=6}$ and $10^{-1} \cdot C(x_1, x_2, x_3, t)\big|_{x_1=10.5; x_2=82.5;\ t=12}$ in Figures 1*b*, 1*c* and 1*d*, respectively.

Let's note that this example considered by authors has illustrative character, because the solving of the offered mathematical model with respect to the considered example has been executed under some simplifying assumptions. For wide application of the offered mathematical model in practical questions it is necessary to develop more complex program using the high-level language.

4. DISCUSSION

As can be seen from the mathematical statement (2)-(7) of the investigated problem, there is assumption to the effect that a priori information on airflow velocity $\vec{\vartheta}$ and its direction, at that this vector velocity is considered the known piecewise constant function of variable x_3 only and consequently this is independent of other two spatial variables x_1, x_2 and temporary variable t, i.e. $\vec{\vartheta} = \vec{\vartheta}(x_3)$. In addition, in present work it is assumed that the coefficient of turbulent and molecular diffusion $D(\vec{\vartheta})$ is also piecewise constant function, which changes depending on a vertical remoteness from the ground surface. Obviously, these assumptions simplify the equation (2) and the matching conditions (6)-(7). In general, both the vector velocity $\vec{\vartheta}$ and the coefficient of turbulent and molecular diffusion $D(\vec{\vartheta})$ are not of necessity to be piecewise constant functions. Moreover, these functions cannot be considered a priori known. In the near future, the authors of present work intend to investigate a more general case preliminarily relinquishing above-mentioned strict requirement.

In the process of this investigation the program system must be develop allowing computations of atmospheric air pollution in dwelling zones of city in view of manifold initial data. As evident from the example of results, these computations are intricate and laborious problem.

CONCLUSIONS

1. The present work proposes the 3-D mathematical model for analytic determination of exhaust concentration dynamics in the city under a priori information on airflow velocity. In this model it is provided that the turbulent and molecular diffusion coefficient changes depending on the vertical remoteness above the ground surface.
2. With the help of the nondegenerate transformation an original problem, where in equation has summands answering to the turbulent phenomena, is reduced to the equivalent problem, where already in equation has no terms in an explicit form answering to the turbulence effect.
3. In the research it has been proved that the reduced problem has unique solution. Consequently, with the help of the nondegenerate transformation the formulated original mathematical model has unique solution also. This solution is determined analytically in the closed form.
4. It can draw on results of these researches at the decision of operative questions on organization of urban traffic within the limits of dwellings boroughs. The offered model can be used for the perspective planning of urban agglomeration and building new motorways.

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