Lagrangian Relaxation and Linear Intersection Based QoS Routing Algorithm

Krzysztof Stachowiak and Piotr Zwierzykowski

Abstract—Due to the process of network convergence, the variety of types of traffic transmitted over a single medium increases steeply. This phenomenon can be handled by the existing networking structure although the protocols that are used and, especially, the underlying routing protocols need to be improved. The problem of finding the shortest path on the Internet can no longer be easily defined as there is an increasing number of different characteristics to describe a point-to-point link. The definition of the shortest path may differ for different traffic types. Therefore, in the mathematical models used to solve the modern routing problems multiple criteria must be taken into account. One of the interesting classes of the optimization problem is the problem of finding the solution that is minimized against one of the criteria under certain constraints with regard to the others. In this paper, two algorithms solving this kind of problems are presented and compared with a new solution proposed by the authors.

Keywords—Lagrangian relaxation, Quality of Service, routing.

I. INTRODUCTION

MODERN telecommunications networks tend to converge towards one single multi-purpose communication layer that conveys traffic of an increasing number of different network services, such as IPTV (IP Television) or VoIP (Voice over IP). The traffic classification aware routing (called the Quality of Service or QoS routing) may be easily presented with the means of the mathematical models for the multicriterial optimization [1], [2]. However, many of the emerging problems are computationally complex and, therefore, heuristic techniques for their solutions have been proposed [3]–[9].

In [2], an extensive comparison of different routing problems and the models associated with them has been presented. A differentiation between the link-, path- and tree- optimization is proposed, which refers respectively to the optimizing threshold criterion, or an additive criterion for the entire path or tree (i.e. a subgraph in general). The link optimization may be presented with the example of the flow optimization as the flow through a given path is defined as the minimal bandwidth over the set of its edges. The path optimization is a model suitable for the classical routing problem, where each of the edges of the result path additively builds up for the ultimate cost. The tree optimization reflects the multicast communication routing that consists in connecting a single

K. Stachowiak and P. Zwierzykowski are with Poznań University of Technology, Chair of Communication and Computer Networks, ul. Polanka 3, 60-965 Poznań, Poland (e-mail: krzysiek.stachowiak@gmail.com).

transmitter – the root of the tree, with multiple receivers – the tree's leaves.

When solving the routing problem, in both unicast and multicast cases, the cumulative cost of the result may be either optimized or constrained. For instance, we may consider a problem of minimizing the allocation cost of a given path. To give an example of the constrained problem, we may only want to obtain a route over which the end-to-end delay does not exceed a limit suitable for our needs. In general, we can describe each of the network's links with more than one property, in which case we introduce the multicriterial optimization domain.

In our research we are investigating the class of the "Path-Constrained Path-Optimization" (PCPO) [2] problems. We focus on the case of optimizing a single criterion, while constraining one or more others for the entire path. This kind of problem reflects the QoS routing well since constrained parameters are quite common in this domain. We can imagine that we model the link's bandwidth with one of the edge metrics that we minimize, and then we consider several constrained metrics such as: packet drops, bit error rate, jitter and delay. In such case, the QoS traffic class is defined as a set of the constraints for the aforementioned additional metrics. The metrics that we consider here, both the minimized and the constrained ones, are abstract – no QoS parameters have been associated with them because of the technique that has been chosen to determine the constraints for the simulated problems. Because networks are randomly generated, the values of the constraints depend only on the considered graph and the routing problem. It may, however, be easily adapted to a real life scenario if a real network is to be considered and the actual QoS constraints are applied.

One of the techniques used to solve PCPO problems is the Lagrangian relaxation (LR) [1]. It is worth noting that, apart from minimizing the minimized criterion, it does not just choose results that satisfy the constraints by just any measure. It rather tends to pick the solution with the highest values for the constrained metrics assuring at the same time that the respective constraints are satisfied. This results in providing the sharpest solutions with regard to applied constraints. One of the main drawbacks is that in order to utilize the LR, the Lagrangian dual problem (LD) [10] must be solved, which, for the shortest path problems, has no analytical solution. Numerous heuristic approximations have been proposed [3]–[5], [11]. Some of them provide better (i.e. sharper) solutions at the cost of poorer performance. On the other hand, the algorithms of the least computational complexity tend to provide solutions that are feasible, but farther from optimum.

In the paper, the new LR-based algorithm is proposed that enables obtaining better results than the above mentioned algorithms.

The paper is divided into the following parts. In Section II, the previous work in the field is briefly summarized. Section III presents the formal representation of the PCPO problem. Section IV describes the MLARAC algorithm emphasizing the path substitution problem – one of the core sub-procedures of the algorithm. The proposed simulation experiment is followed by a discussion on the results in Section V. Section VI concludes the paper.

II. PREVIOUS WORK

In the literature we can find algorithms solving the PCPO problem [3]–[9]. Some of them utilize the LR whereas the others rely on other multicriterial optimization means [3]–[5], [8].

In [3] the H_MCOP (Heuristic Multi-Constrained Optimal Path) algorithm, a non-linear variant of the Lagrangian relaxation has been presented. It has later been improved in [4] as the NLR_MCP (Nonlinear Lagrangian Relaxation Multipath Constrained Problem) algorithm. In both cases, a modified Dijkstra's algorithm [12] is applied to the original problem, once from the destination to the source and then from the source to the destination. The latter pass utilizes the data collected in the former one, thus allowing for a better prediction of the best routes towards the destination. The prediction is, however, a rough approximation that leads to obtaining results that are optimized less as compared to the other algorithms. While being very cheap with regard to the computational complexity, which is $\mathcal{O}(nlogn + knlog(kn) + (k^2 + 1)m)$, the H_MCOP algorithm does not provide the most optimal solutions.

An interesting deviation from the H_MCOP – the LBPSA (Lagrangian Branch-and-bound Path Selection Algorithm) algorithm – is presented in [5]. In this case, a two-step optimization has been proposed as well, however only the first pass is based upon a modified Dijkstra's algorithm. The second pass is a modified "breadth first search" algorithm that is interpreted as a "Branch and Bound" (B&B) procedure [13]. One of the important properties of the LR technique is that, despite providing a result that is close to the optimum, it also provides a relatively sharp lower bound for the optimized problem, which is then utilized for the pruning in the B&B phase by the LBPSA algorithm. The complexity of this algorithm is $\mathcal{O}(nlogn + m + n)$.

However, yet another approach to the PCPO problems can be used. The MLARAC (Multidimensional LAgrange Relaxation based Aggregated Cost) algorithm described in this paper allows obtaining well optimized solutions in terms of the obtained costs without a great increase in the computational complexity.

In [11] the LARAC (LAgrange Relaxation based Aggregated Cost) algorithm is presented. It utilizes the original, linear LR technique and proposes a simple iterative way of optimizing the LD problem. For the shortest path problems the LD becomes a maximization of a concave, piecewise linear function. The proposed solution is to repetitively intersect linear functions associated with the intermediate results in order to find better solutions at the acceptable cost of the complexity of $\mathcal{O}(n^2log^4n)$. The drawback of the LARAC algorithm is that it can only solve problems with only one constrained parameter.

The LARAC algorithm has been already used as a base for other routing algorithms, such as the MLRA presented in [14] and [15]. It is, however, a multicasting algorithm and, like LARAC, only considers a single constraint metric, i.e. it does not solve the PCPO problems that are the focus of this paper.

In [16] the MLARAC algorithm, an extension to the LARAC algorithm, is briefly introduced by the authors. In the MLARAC algorithm, the LARAC algorithm has been adapted to solving the multidimensional problem that arises as a result of the introduction of additional constraints. The complexity of this approach is $\mathcal{O}(m^3n^2log^4n)$. It is higher than that of the LARAC complexity by the m^3 factor, which comes from the necessity of solving a system of linear equations, though the m – the number of the considered metrics – will usually be small as only few of them are used to describe telecommunications networks' edges. The generalization of the LARAC algorithm introduced a new sub-problem [16] – "the path substitution". In this paper, the authors propose an extended path substitution study, whereas the comparison of the MLARAC algorithm with HMCOP and NLRMCP is presented for the first time.

III. FORMULATION OF THE PROBLEM

We model a communications network by an undirected graph $G(V, E)$, where V is a finite set of nodes and $E \subseteq \{(u, v) : u, v \in V\}$ is a set of edges that represent point-to-point links. Furthermore, we assume that every edge is assigned a set of M numeric properties (metrics). Metrics are real-valued functions ($m_i : E \to \mathbb{R}, i = 0, 1, ..., M-1$) and reflect the cost of a given edge. A path in the graph $G(V, E)$ is defined as a sequence of k non-repeated nodes $v_1, v_2, \ldots, v_k \in V$ such that for each $1 \leq i < k$ an edge $(v_i, v_{i+1}) \in E$ exists. For each of the metrics, except the first one, we define a maximum value (constraint) C_i, $i = 1, 2, ..., M-1$ that cannot be exceeded in any connection path. The connection path between a source s and terminal t nodes is defined as $p(s, t)$ that is a path $v_1, v_2, \ldots, v_k$, where $s \equiv v_1$, $t \equiv v_k$.

The cost of the path p is defined as:

$$c_p = \sum_{e \in p} m_0(e), \tag{1}$$

the problem of the multi-constrained path optimization (MCOP) is reduced then to finding a path $p^*(s, t)$ such that:

$$\forall_{p \in P(s,t)} c_{p^*} \leq c_p, \tag{2}$$

where $P(s, t)$ is a set feasible solutions, i.e. the set of all the paths in graph G between nodes s and t that fulfil the following condition:

$$\forall_{i \in \{1,2,...,M-1\}} \sum_{e \in p(s,t)} m_i(e) \leq C_i. \tag{3}$$

IV. MLARAC ALGORITHM

A. Lagrangian Relaxation Technique

The MLARAC algorithm is based upon the general purpose optimization technique of the Lagrangian relaxation. The use of the LR in telecommunications has been presented in [1].

In the LR the constraints from Formula (3) are moved into the target function by means of the linear combination. It is assumed that if an appropriate set of weights $\lambda_1, \lambda_2, ..., \lambda_{M-1}$ is applied to the constrained metrics in the modified target function, a good suboptimal solution may be found.

When applying the LR to the path finding problem, the following formula is used to associate the linear function with any path for a given vector of lambdas:

$$c_p(\lambda) = \sum_{e \in p} \{ m_0(e) + \sum_{i=1}^{M-1} \lambda_i [m_i(e) - C_i] \}. \quad (4)$$

The key to the LR is to obtain a set of optimal λ factors, which is done by maximizing the following function:

$$L(\lambda) = min\{ c_p(\lambda) : p \in P(s,t) \}, \quad (5)$$

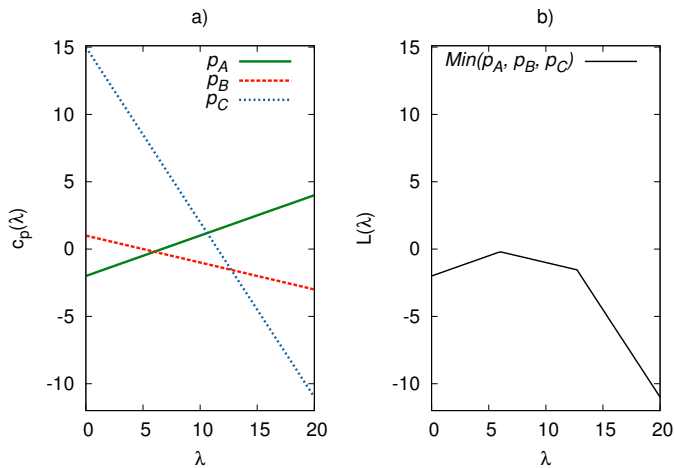which is the merit of the LD problem.

Fig. 1. The overview of the one constraint Lagrangian Dual Problem.

In Fig. 1, an example case considering only one constraint of the LR oriented analysis is presented. In Fig. 1(a), the (4) functions for certain paths are presented as p_A, p_B and p_C. In Fig. 1(b), only the part obtained with the (5) function is visible, which becomes the function $Min(p_A, p_B, p_C)$.

When multiple constraints are considered, each of them adds another dimension to the problem, therefore, instead of lines, planes or in general – hyperplanes are assigned to the paths. Nevertheless, the $L(\lambda)$ remains a concave piecewise linear function and the same algebraic tools may be used in its analysis. In an attempt at depicting the problem for a case with two constraints determined, Figure 2 is presented in which the increase of the complexity may be observed. In Fig. 2(a) the (4) functions form planes p_A, p_B and p_C in this case, and in Fig. 2(b) the (5) function becomes a concave piecewise linear surface $Min(p_A, p_B, p_C)$.
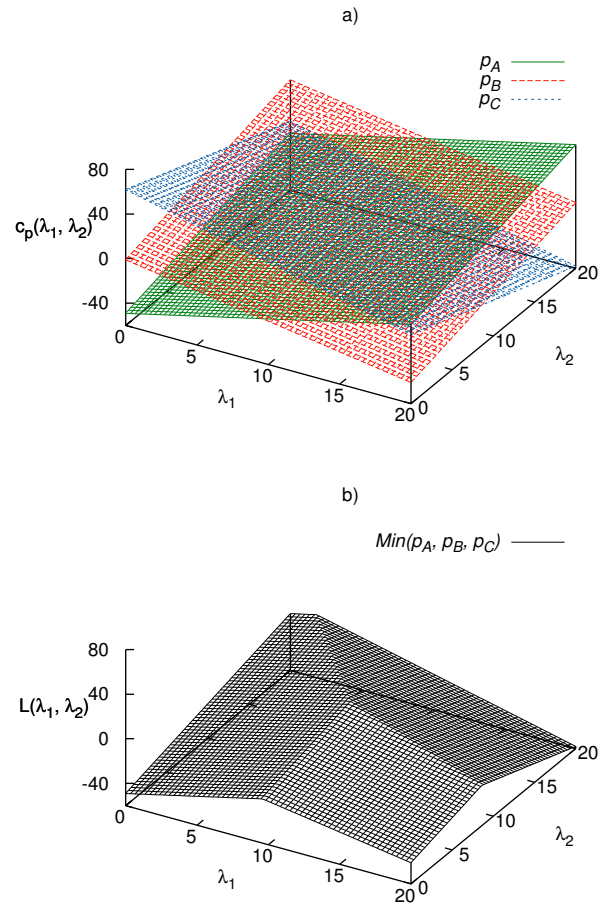
Fig. 2. The overview of the two constraints Lagrangian Dual Problem.

B. Pseudo-code of the MLARAC Algorithm

The operation of the MLARAC algorithm is presented in Algorithm 1. In this paper we focus on the *Substitution* procedure which will be described in detail in the following subsection.

C. Definition of the Path Substitution Problem

In the iterations loop (lines 15-22) we manage a set of paths that are called "candidate paths". Each of them is initially obtained by solving the routing problem minimizing each of the metrics individually (lines 2 and 6-8). Therefore for M metrics we obtain M candidate paths. One of them is special because it has been obtained by means of optimization against the metric that is to be minimized, whereas the remaining candidate paths have been found with regard to the metrics that are constrained.

The first path is called the "exceeding path" as it may exceed up to all of the constraints as it is only optimized against the non-constrained criterion. It may actually exceed none of the constraints, in which case it is chosen as an immediate optimal solution without even entering the approximation loop (lines 3-4). The remaining paths are called "non-exceeding paths" as each of them fulfills at least one constraint, in particular

Algorithm 1 Multidimensional LARAC

```
 1: procedure MLARAC(s, t, M, C)
 2:     ExPath ← ShortPath(M[0], s, t)
 3:     if SatisfiesAll(ExPath) = True then
 4:         ReturnExPath
 5:     end if
 6:     for i := 1 to M do
 7:         UnPaths[i] ← ShortPath(M[i], s, t)
 8:     end for
 9:     for i := 1 to M do
10:         if CostForMetric(UnPaths[i], i) > M[i] then
11:             ReturnFailure
12:         end if
13:     end for
14:     BestUn ← Failure
15:     repeat
16:         Lambdas ← Inter(ExPath, UnPaths)
17:         Path ← ShortPath(AggrM(Lambdas), s, t)
18:         if SatisfiesAll(Path) = True then
19:             BestUn ← Path
20:         end if
21:         Substitute(Path, ExPath, UnPaths)
22:     until AllAgrCostsEqual(ExPath, UnPaths)
23:     ReturnBestUn
24: end procedure
```

the one for which it has been obtained. If any of the non-exceeding paths actually breaks its respective constraint it is certain that no feasible solution exists (lines 9-13). Otherwise, the set of the candidates is ready and valid and, therefore, the approximation loop commences.

In every approximating iteration, another candidate path is obtained by means of the intersection of the linear hyperplanes associated with all of the current candidate paths (line 16), which boils down to solving the following system of linear equations [16]:

$$
\begin{aligned}
\sum_{i=1}^{M} \lambda_i (m_{iu0} - m_{ie}) &= m_{0e} - m_{0u0} \\
\sum_{i=1}^{M} \lambda_i (m_{iu1} - m_{ie}) &= m_{0e} - m_{0u1} \\
&\cdots \\
\sum_{i=1}^{M} \lambda_i (m_{iuM} - m_{ie}) &= m_{0e} - m_{0uM}
\end{aligned}
\tag{6}
$$

where

- λ_i is the *i-th* element of the λ vector,
- m_{ie} means the *i-th* metric of the "exceeding path",
- m_{iuj} means the *i-th* metric of the *j-th* path from the "non-exceeding" paths set.

Because of the initialization technique used for the candidates, it is very likely that the intersection approximates the optimal λ vector as each of the hyperplanes occupies a slope associated with a different dimension, therefore they all surround the peak of the "hyper-hill" that is being analyzed. However, the first intersection may not find the optimal, or even a feasible solution, therefore a new candidate is used in the further iterations in place of one of the candidate paths used prior to its determination.

In order to find an $M - 1$ dimensional intersection, $M - 1$ equations are needed and the M candidate paths are enough

to build such a linear system by putting the linear functions associated with the non-exceeding paths on the left sides of the equations and the function associated with the exceeding path on the right side of every equation. After obtaining a new candidate, one of the current paths from the set must be discarded (line 21). If the new candidate exceeds all the constraints, it is assumed that it must replace the exceeding candidate, however if it fulfills some of the constraints then it is not trivial to clearly associate it with any particular metric.

In order to find the best substitution technique, the following strategies have been proposed and analyzed.

- **The most expensive non blocking criterion (ENB)**
  The most expensive non blocking substitution variant first finds all the metrics for which the constraints are not broken by the candidate path. Then, from all of them one that identifies the candidate best is chosen, i.e. the one that associates it best with a given metric. This approach stems from the assumption that all of the candidate paths must be associated with a given criterion that should, in general, secure an advantageous surround of the analyzed "hyper-hill's" peak. Choosing the metric of the greatest value gives us the least steepness with regard to a given dimension, therefore it is chosen to be the most valuable for the approximation. However, if during the intersection multiple paths are found that are suitable only for a narrow range of the criteria, better and better results for them shall be continuously discarded while maintaining possibly poorly approximating paths for the other criteria. Although this was the first approach developed, others have been examined in order to clarify any doubts that the application of the ENB technique may have raised.

- **Minimal sum of gradients (MSG)**
  This strategy is similar to ENB but with the assumption of the association between the paths and the metrics dropped. Every time a new candidate enforces discarding one of the current approximating paths, the one is chosen that has the lowest sum of the gradients. Since the slopes only depend on the respective metrics, this is equivalent to finding the one with the least sum of the metrics.
  This strategy in general discards the paths associated with the steepest functions which, because the function is concave, should be considered as the farthest from the peak.

- **The random selection (RND)**
  In order to easily determine if the substitution has any notable impact for the solution, the random selection variant has been introduced. Every time a new candidate must be placed in the current candidates' set, a random index is picked for the candidate path that is to be discarded. The new candidate is then put in its place.
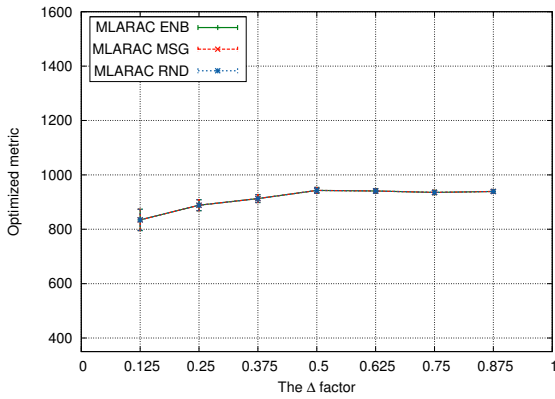
(a) 100 nodes, the optimized metric
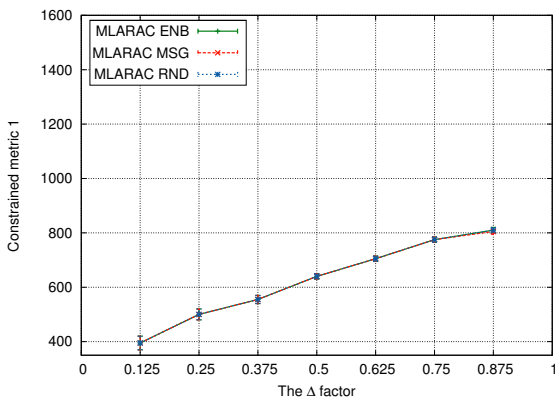
(b) 100 nodes, the constrained metric 1
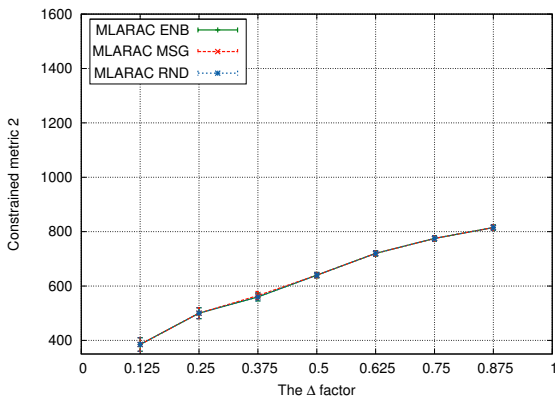
(c) 100 nodes, the constrained metric 2

Fig. 3. The comparison of the different MLARAC algorithm variants for 100 nodes.

V. NUMERIC RESULTS

A. The Proposed Experiment

In order to evaluate the proposed algorithm numerous simulation experiments have been conducted. The subject of the evaluation were the average costs of the paths generated by the algorithms for all of the three metrics. The costs of each of the metrics were considered independently. The experiments have been performed for a set of graphs representing networks consisting of 100, 150 and 200 nodes, generated using the Waxman's algorithm [17]. The parameters $\alpha = 0.15$ and $\beta = 0.2$ were assumed for the algorithm, and the additional metrics, the constrained ones, were randomly drawn from a uniform distribution on the interval $(1, 500)$.

For 500 graphs, 1000 paths were searched for in order to achieve a satisfactory confidence intervals, i.e. two orders of magnitude smaller than the obtained mean values.

Each routing task was chosen as a pair of nodes to be connected. The method of choosing the input parameters – the path's constraints – was based on [4]. However, in this case more than one metric was constrained, therefore the original technique was generalized by the authors in order to handle an arbitrary number of constrained metrics. The modified technique is as follows. For each of the constrained metrics a minimal and maximal applicable value are found. The minimal value is obtained by performing the Dijkstra's shortest path finding algorithm with regard to a given metric m_i resulting in a path p_{m_i}. Based on this method, a lower bound for the given metrics' constraint is defined. The maximal value is obtained from the Dijkstra's shortest path finding algorithm performed with regard to the minimized metric (cost), resulting in a path p_c. The constrained metrics of the result are interpreted as the upper applicable bounds for the constraints as for greater constraints the path minimizing the base metric would be the optimal solution. The constraints are linearly scaled within their respective ranges between the lower and the upper bounds with a scalar value Δ. Therefore, the formula for the *i-th* constraint C_i is the following:

$$C_i(\Delta) = m_i(p_{m_i}) + \Delta(m_i(p_c) - m_i(p_{m_i})). \qquad (7)$$

The simulations were performed for a set of Δ factors picked from the range between 0 and 1.

B. The Experiment Results

The results presentation has been broken into two parts. In the first part only the different variations of the MLARAC algorithm have been compared. In the second part the MLARAC algorithm has been compared with the other mentioned algorithms: the H_MCOP and the LBPSA in order to illustrate the thesis of the paper. Apart from the results the confidence intervals were marked in the figures. For that purpose the Student's distribution was used with the confidence level of 95%. It may be observed that the intervals are of a size comparable to the size of the symbols of the given results and that they don't overlap for the most of the cases.

The simulation results are presented in Figs. 3, 4, 5 and 6. Figures 3(a), 3(b) and 3(c) present the results for the different variants of the MLARAC algorithm, whereas Figs. 4, 5 and 6 present the comparison of the MLARAC algorithm with the LBPSA and the H_MCOP algorithms.
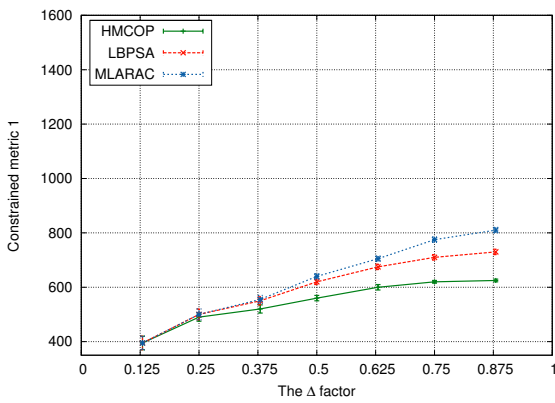
1) Performance evaluation of MLARAC variants: It turned out that the chosen substitution technique had no significant impact on the quality of the results. The RND variant was
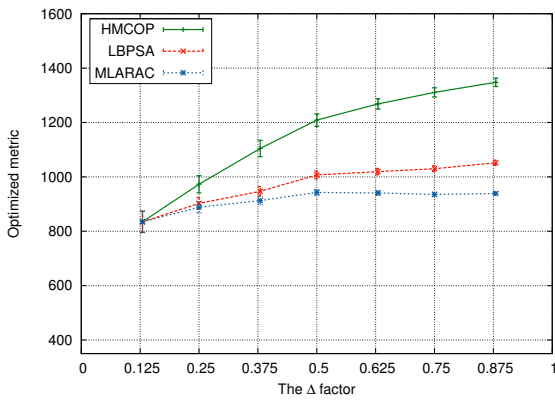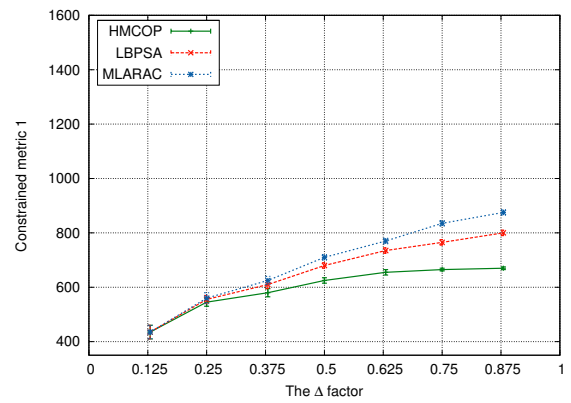
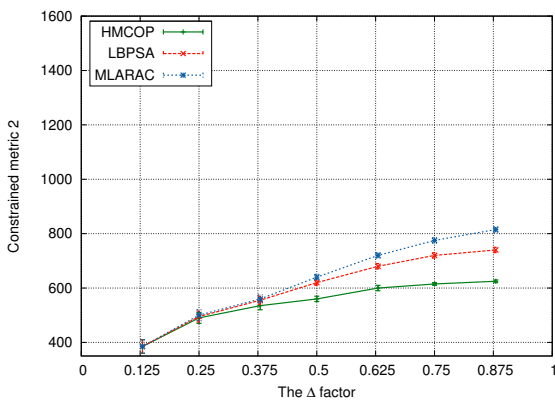(a) 100 nodes, the optimized metric
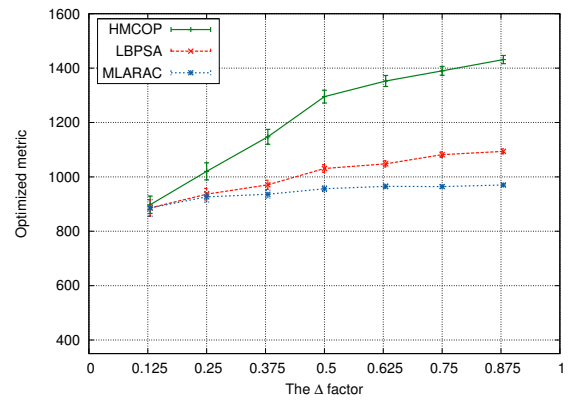
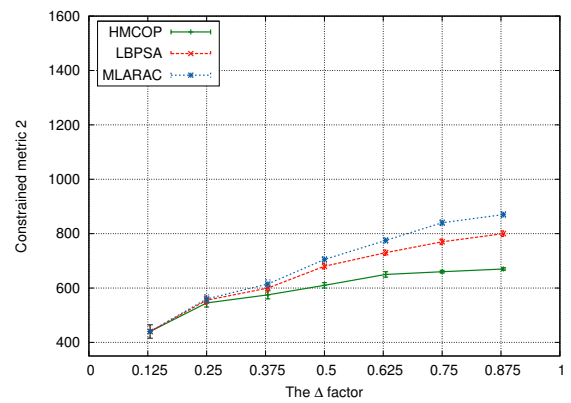(a) 150 nodes, the optimized metric

(b) 100 nodes, the constrained metric 1

(b) 150 nodes, the constrained metric 1

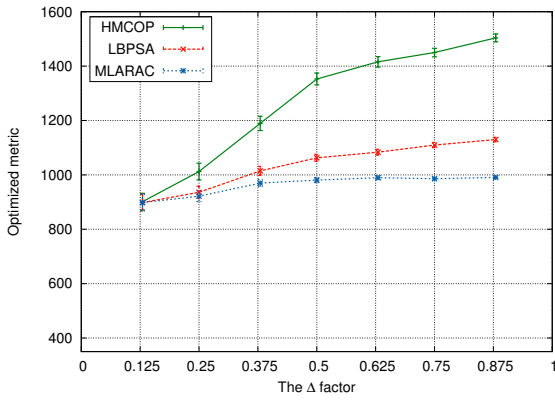(c) 100 nodes, the constrained metric 2

(c) 150 nodes, the constrained metric 2

Fig. 4. The comparison of the MLARAC algorithm to others for 100 nodes.
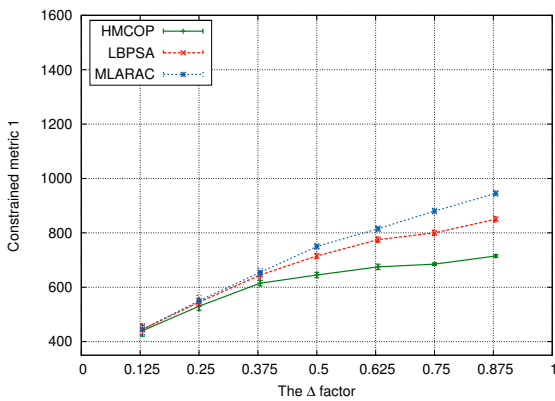
Fig. 5. The comparison of the MLARAC algorithm to others for 150 nodes.

proposed as a model for the extremely neutral with regard to the constrained metrics. As it has been shown in Fig. 3, it gives very similar results to the others that attempt at utilizing the information obtained by means of the new candidate's analysis. For the cases of the bigger networks the absolute

values of the obtained average costs were higher, however, the differences between them were as insignificant as in Fig. 3. Therefore, the plots for the bigger networks have been omitted. It is worth emphasizing that the average number of the approximating iterations for a given category of problems

(a) 200 nodes, the optimized metric

(b) 200 nodes, the constrained metric 1
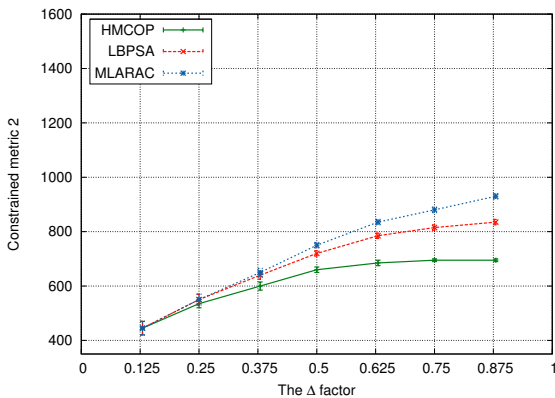
(c) 200 nodes, the constrained metric 2

Fig. 6. The comparison of the MLARAC algorithm to others for 200 nodes.

is usually small, which is in general favorable, but at the same time does not allow for any refined techniques to expose their advantages.

2) Comparison of MLARAC, LBPSA and H_MCOP algorithms: In Fig. 4 it can be seen that LBPSA and MLARAC perform better than the H_MCOP algorithm. They give paths

of the lower optimized cost and higher constrained costs, i.e. their resource consumption is closer to the optimum as they do not eagerly optimize the metrics that should only be constrained. Of the two, MLARAC performs slightly better, especially for greater values of the Δ factor. All of the algorithms result in paths of similar quality for the lower Δ coefficient, which presents the range of the most difficult (i.e. tighter constrained) problems.

When analyzing Figs. 5 and 6 it can be observed that a trend of the overall increase in the costs of the resulting paths emerges. The increase stems from the ability of finding longer paths as the constraints get lower, thus the average cost of the results increases. However, the relation between the results of the particular algorithms and the proportion of the results remains unchanged.

For both H_MCOP and LBPSA algorithms' performance gain with regard to the constrained metrics decreases for the higher Δ faster than for the MLARAC algorithm. It is also observable in Fig. 6(a) that for the bigger networks the average optimized cost obtained with the LBPSA algorithm increases towards very high values of Δ, whereas the performance of the MLARAC improves linearly which allows for obtaining better results for the problems for which the assumed constraints are relatively close to the metrics of the path minimized against the base metric.

VI. CONCLUSION

It has been shown in this paper that the method of finding the intersection of the linear functions shown in [11] may be generalized to the cases when more than one constraint is defined. The difficulty encountered in [16] has been given special attention and the conclusion has been drawn that the choice of the substitution method has no significant impact on the results. This may be due to the fact that the approximation loop is not performed many times, therefore not enough repetitions were performed to expose any underlying differences. The refined variant of the MLARAC algorithm, in comparison to other similar ones, provides solutions of the best quality. Firstly, the general optimality of the results was analyzed by a comparison of the optimized costs of the paths found with different algorithms. Secondly, the costs with regard to the constrained metrics were expected to be as high as possible since the higher the constrained costs the smaller the unnecessary drain of the network resources. In both cases the MLARAC algorithm led to the best results: the lowest optimized costs and the highest constrained costs were obtained.

Since the MLARAC algorithm is based upon a relatively simple mathematical model and procedures, it may be considered an interesting choice for the base for the QoS-oriented routing algorithms.

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APPENDIX A
ABBREVIATIONS (IN ORDER OF APPEARANCE)

- **IPTV** – Internet Protocol TeleVision
- **VoIP** – Voice over Internet Protocol
- **PCPO** – Path Constrained Path Optimization
- **LR** – Lagrangian Relaxation
- **LD** – Lagrangian Dual
- **MLARAC** – Multidimensional LARAC
- **LARAC** – LAgrange Relaxation based Aggregated Cost
- **H_MCOP** – Heuristic Multi-Constrained Optimal Path
- **NLR_MCP** – Nonlinear Lagrangian Relaxation Multi-path Constrained Problem
- **LBPSA** – Lagrangian Branch-and-bound Path Selection Algorithm
- **B&B** – Branch & Bound
- **ENB** – Most Expensive Non Blocking
- **MSG** – Minimal Sum of Gradients
- **RND** – RaNDom selection