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Some remarks on exact solution of Lidstone boundary value problem

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Dedicated to Professor Jerzy Gawinecki on his 60th birthday.

Abstract. In this paper we focus on the following form of the Lidstone BVP

$$\begin{cases} x^{(2k)}(t) - \sum_{i=1}^{k} \lambda^i x^{(2k-2i)}(t) = f(t, x(t), x''(t), \ldots, x^{(2k-2)}(t)) \\ x^{(2s)}(0) = x^{(2s)}(1) = 0, \qquad s = 0, \ldots, k-1 \end{cases}.$$

We examine correlation between parameters $\lambda^i \in \mathbb{R}$ and the kernel of differential opera-ror that corresponds to the right hand side of the considered problem. Next we present a method of inverting the mentioned operator. In consequence of this method, we obtain two exact formulas that describe the corresponding Green's functions and the form of a solution with its derivatives in the case when f depends only on t.
Keywords: Lidstone boundary value problem, eigenvalue

1. Introduction

In 1929 G.J. Lidstone introduced generalization of the Taylor's series approximating given function in neighborhood of not one but two points (see [4]). This caused that the natural question on a general class of function that can be represented as Lidstone's series arised. It turned out that the seeking class was the one of entire functions of exponential type at most π (see [1]). The mentioned functions were looked for with the help of differential equations with diverse boundary conditions. Throughout the years evolution of this considerations leads to the following BVP

$$\begin{cases} x^{(2k)}(t) = f\left(t, x(t), x''(t), \dots x^{(2k-2)}(t)\right) \\ x^{(2i)}(0) = x^{(2i)}(1) = 0, \qquad 0 \le i \le k-1 \end{cases}. \tag{1}$$

The above problem was examined by many authors (see [2], [3], [5] and references therein) and to honour the creator of its basic version it was called Lidstone BVP. Popularity of (1) is a consequence of its applicableness in many fields of science. For instance, if $k = 2$ then the problem describes deformation of an elastic beam whose two ends are simply supported.

We will study (1) in the case when its right hand side fall into linear and nonlinear part. Thus we will consider the BVP of the form

$$\begin{cases} x^{(2k)}(t) - \sum_{i=1}^{k} \lambda^i x^{(2k-2i)}(t) = f\left(t, x(t), x''(t), \dots, x^{(2k-2)}(t)\right) \\ x^{(2s)}(0) = x^{(2s)}(1) = 0, \qquad s = 0, \dots, k-1 \end{cases}, \tag{2}$$

where $\left(\lambda^1, \dots, \lambda^k\right) \in \mathbb{R}^k$. The main purpose of this paper is to present the method of inverting differential operator that corresponds to the left hand side of (2). We will present two formulas that describe clearly the form of Green's function. These formulas can be used to obtain a satisfying approximation of a solution to (2).

2. Preliminaries

Definition 2.1 Point $(\lambda^1, \dots \lambda^k) \in \mathbb{R}^k$ will be called k-dimensional eigenvalue iff the homogeneous problem

$$\begin{cases} x^{(2k)}(t) - \lambda^1 x^{(2k-2)}(t) - \lambda^2 x^{(2k-4)}(t) - \dots - \lambda^{k-1} x''(t) - \lambda^k x(t) = 0 \\ x^{(2s)}(0) = x^{(2s)}(1) = 0, \quad \text{for } s = 0 \dots k-1, \end{cases} \tag{3}$$

has a nonzero solution. The set of all such n-tuples will be denoted by σ^k

Let us fix $\lambda^1, \dots, \lambda^k \in \mathbb{R}$ and define the set

$$C_0^{2m}([0,1], \mathbb{R}) := \left\{ u \in C^{2m}([0,1], \mathbb{R}) \mid u^{(2s)}(0) = u^{(2s)}(1) = 0, s = 0, \dots, m-1 \right\}.$$

It is well-known that $C_0^{2m}([0,1],\mathbb{R})$ is a dense subset of $L^2([0,1],\mathbb{R})$. Let $T(\cdot;\lambda^1,\ldots,\lambda^k): C_0^{2k}([0,1],\mathbb{R}) \to L^2([0,1],\mathbb{R})$ be an operator given by the formula

$$T(u;\lambda^1,\ldots,\lambda^k) := D^k u - \lambda^1 D^{k-1}u - \lambda^2 D^{k-2}u - \ldots - \lambda^{k-1}Du - \lambda^k u,$$

where $(Dx)(t) = \frac{d^2}{dt^2}x(t)$.We note that each element of the family $\left\{T(\cdot;\lambda^1,\ldots,\lambda^k) \mid \lambda^1,\ldots,\lambda^k \in \mathbb{R}\right\}$ is correctly defined. Furthermore,

$$\sigma^k = \bigcup \left\{ (\lambda^1,\ldots,\lambda^k) \in \mathbb{R}^k \mid \ker T(\cdot;\lambda^1,\ldots,\lambda^k) \neq \{0\} \right\}.$$

By using the methods of the spectral analysis and the theory of self-disjoint and completely continuous operators, we prove that

$$T(u;\lambda^1,\ldots,\lambda^k) = \sum_{p=1}^{\infty}\left[\left(-p^2\pi^2\right)^k - \sum_{s=1}^{k}\lambda^s\left(-p^2\pi^2\right)^{k-s}\right]\langle e_p,u\rangle_{L^2} e_p,$$

where $e_p(t) = \sqrt{2}\sin(p\pi t)$,

Consequently, the uniqueness of coefficients of Fourier series gives us

$$\sigma^k = \bigcup_{p\in\mathbb{N}} H_p,$$

where

$$H_p := \left\{ (\lambda^1,\ldots,\lambda^k) \in \mathbb{R}^k \mid \left(-p^2\pi^2\right)^k - \sum_{s=1}^{k}\lambda^s\left(-p^2\pi^2\right)^{k-s} = 0 \right\}.$$

It is evident that for each $p \in \mathbb{N}$ the set H_p is a hyperplane in $\mathbb{R}^k$. Furthermore,

$$\det\begin{bmatrix} \left(-p_1^2\pi^2\right)^{k-1} & \left(-p_1^2\pi^2\right)^{k-2} & \ldots & -p_1^2\pi^2 & 1 \\ \left(-p_2^2\pi^2\right)^{k-1} & \left(-p_2^2\pi^2\right)^{k-2} & \ldots & -p_2^2\pi^2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \left(-p_k^2\pi^2\right)^{k-1} & \left(-p_k^2\pi^2\right)^{k-2} & \ldots & -p_k^2\pi^2 & 1 \end{bmatrix}$$

$$= (-1)^{\left[\frac{k}{2}\right]+\frac{k(k-1)}{2}} \cdot \pi^{k(k-1)} \cdot \det\begin{bmatrix} 1 & p_1^2 & \ldots & \left(p_1^2\right)^{k-2} & \left(p_1^2\right)^{k-1} \\ 1 & p_2^2 & \ldots & \left(p_2^2\right)^{k-2} & \left(p_2^2\right)^{k-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & p_k^2 & \ldots & \left(p_k^2\right)^{k-2} & \left(p_k^2\right)^{k-1} \end{bmatrix} \neq 0.$$

The last condition implies the theorem:

Theorem 2.1. *Let $s \in \mathbb{N}$ and $p_1, p_2, \ldots, p_s \in \mathbb{N}$ be such that $p_i \neq p_j$, $i \neq j$.*

(1) If $s \leq k$, then $W = \bigcap_{i=1}^{s} H_{p_i} \neq \emptyset$ is affine subspace in $\mathbb{R}^k$ and $\dim W = k - s$.
(2) If $s > k$, then $\bigcap_{i=1}^{s} H_{p_i} = \emptyset$.

Let us fix $\lambda = \left(\lambda^1, \ldots, \lambda^k\right) \in \sigma^k$. By the last theorem there exist $p_1, p_2, \ldots, p_l \in \mathbb{N}$ such that $\lambda \in H_p$, $p \in \{p_1, \ldots, p_l\}$, and $\lambda \notin H_p$, for $p \in \mathbb{N} \setminus \{p_1, \ldots, p_l\}$. Thus

$$\begin{aligned} \left(-p^2\pi^2\right)^k - \sum_{s=1}^{k} \lambda^s \left(-p^2\pi^2\right)^{k-s} &= 0, \quad \text{for } p \in \{p_1, \ldots, p_l\}, \\ \left(-p^2\pi^2\right)^k - \sum_{s=1}^{k} \lambda^s \left(-p^2\pi^2\right)^{k-s} &\neq 0, \quad \text{for } p \in \mathbb{N} \setminus \{p_1, \ldots, p_l\}. \end{aligned} \tag{4}$$

If we take $u \in \ker T\left(\cdot; \lambda^1, \ldots, \lambda^k\right)$, then we obtain

$$\sum_{p \in \mathbb{N}} \left[\left(-p^2\pi^2\right)^k - \sum_{s=1}^{k} \lambda^s \left(-p^2\pi^2\right)^{k-s}\right] \langle e_p, u\rangle_{L^2} e_p = 0.$$

This together with (4) imply

$$\langle e_p, u\rangle_{L^2} = 0 \quad \text{foreach } p \in \mathbb{N} \setminus \{p_1, \ldots, p_l\}.$$

Therefore,

$$u = \sum_{p \in \{p_1, \ldots, p_l\}} \langle e_p, u\rangle_{L^2} e_p. \tag{5}$$

It is easy to see that if u has a form (5) then $u \in \ker T\left(\cdot; \lambda^1, \ldots, \lambda^k\right)$. These results lead us to the following conclusion.

Corollary 2.1.
The collection $\{e_p \mid p \in \{p_1, \ldots, p_l\}\}$ is a basis of $\ker T\left(\cdot; \lambda^1, \ldots, \lambda^k\right)$ and $\dim \ker T\left(\cdot; \lambda^1, \ldots, \lambda^k\right) = l$.

The observation above means that the dimension of the kernel equals a number of hyperspaces that contain λ.

Let us fix $\left(\lambda^1, \ldots, \lambda^\kappa\right) \notin \sigma^\kappa$ and consider the following homogeneous linear differential equation

$$x^{(2\kappa)}(t) - \lambda^1 x^{(2\kappa-2)}(t) - \ldots - \lambda^{\kappa-1} x''(t) - \lambda^\kappa x(t) = 0. \tag{6}$$

We know that the set of solutions to the above equations is 2κ-dimensional linear space over $\mathbb{R}$. This space will be denoted by $D\left(\lambda^1,\ldots,\lambda^\kappa\right)$. Further consider the two IVPs connected with (6)

$$\begin{cases} x^{(2\kappa)}(t)-\lambda^1 x^{(2\kappa-2)}(t)-\ldots-\lambda^{\kappa-1}x''(t)-\lambda^\kappa x(t)=0 \\ x^{(2s)}(0)=0, \quad s=0,\ldots,\kappa-1 \end{cases}, \tag{7}$$

and

$$\begin{cases} x^{(2\kappa)}(t)-\lambda^1 x^{(2\kappa-2)}(t)-\ldots-\lambda^{\kappa-1}x''(t)-\lambda^\kappa x(t)=0 \\ x^{(2s)}(1)=0, \quad s=0,\ldots,\kappa-1 \end{cases}. \tag{8}$$

The sets of solutions to (7) and (8) are subspaces of $D\left(\lambda^1,\ldots,\lambda^\kappa\right)$. We will denote them by $D_0\left(\lambda^1,\ldots,\lambda^\kappa\right)$ and $D_1\left(\lambda^1,\ldots,\lambda^\kappa\right)$, respectively.

We start with the following basic and important lemma.

Lemma 2.1. $\dim D_0\left(\lambda^1,\ldots,\lambda^\kappa\right)=\dim D_1\left(\lambda^1,\ldots,\lambda^\kappa\right)=\kappa$, *furthermore* $D\left(\lambda^1,\ldots,\lambda^\kappa\right)=D_0\left(\lambda^1,\ldots,\lambda^\kappa\right)\oplus D_1\left(\lambda^1,\ldots,\lambda^\kappa\right)$.

Proof.
Since $\left(\lambda^1,\ldots,\lambda^\kappa\right)\notin\sigma^\kappa$, thus $D_0\left(\lambda^1,\ldots,\lambda^\kappa\right)\cap D_1\left(\lambda^1,\ldots,\lambda^\kappa\right)=\{0\}$. There exist numbers $a_i^j\in\mathbb{R}$, $i,j=1,\ldots,\kappa$ such that $v_s = \left[0,a_s^1,0,a_s^2,\ldots,0,a_s^\kappa\right]$, $s=1,\ldots,\kappa$ are linearly independent. Considering equivalent vector equation we obtain existence of solutions ϕ_s connected with (6) such that $\phi_s(0)=v_s$. Linearly independence of the set $\{v_s \mid s=1,\ldots,\kappa\}$ implies linearly independence of $\{\phi_s \mid s=1,\ldots,\kappa\}$, thus $\dim D_0\left(\lambda^1,\ldots,\lambda^\kappa\right)\geq\kappa$. Analogously, we show that $\dim D_1\left(\lambda^1,\ldots,\lambda^\kappa\right)\geq\kappa$. On the other hand, $D_0\left(\lambda^1,\ldots,\lambda^\kappa\right)\oplus D_1\left(\lambda^1,\ldots,\lambda^\kappa\right)\subseteq D\left(\lambda^1,\ldots,\lambda^\kappa\right)$ therefore $\dim D\left(\lambda^1,\ldots,\lambda^\kappa\right)=2\kappa$. □

Assume that $\widehat{f}\in L^1\left([0,1],\mathbb{R}\right)$ and let us consider the following BVP.

$$x^{(2\kappa)}(t)-\lambda_1 x^{(2\kappa-2)}(t)-\ldots-\lambda_{\kappa-1}x''(t)-\lambda_\kappa x(t)=\widehat{f}\,(t)\,, \tag{9}$$

$$x^{(2s)}(0)=x^{(2s)}(1)=0, \quad s=0,\ldots,\kappa-1. \tag{10}$$

It is special case of Lidstone BVP. Set $x_0(.):=x\,(.)$ and convert the above equation to the equivalent system of equations

$$\begin{gathered} x_0'(t)=x_1(t) \\ x_1'(t)=x_2(t) \\ \vdots \\ x_{2\kappa-2}'(t)=x_{2\kappa-1}(t) \\ x_{2\kappa-1}'(t)=\lambda_\kappa x_0\,(t)+\lambda_{\kappa-1}x_2(t)+\ldots+\lambda_1 x_{2\kappa-2}(t)+\widehat{f}(t). \end{gathered}$$

After this changes (9) has the form

$$X'(t) - \Lambda \cdot X(t) = F(t), \tag{11}$$

where

$$X(.) := \begin{bmatrix} x_0(.) \\ x_1(.) \\ \vdots \\ x_{2\kappa-2}(.) \\ x_{2\kappa-1}(.) \end{bmatrix}, \quad \Lambda := \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ \lambda_\kappa & 0 & \lambda_{\kappa-1} & \cdots & \lambda_1 & 0 \end{bmatrix},$$

$$F(.) := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \widehat{f}(.) \end{bmatrix}.$$

By lemma 2.1, there exists a collection of linearly independent solutions of the homogeneous equation (6) $\{\alpha_p \mid p = 1, \dots, 2\kappa\}$ such that

$$\alpha_{2p-1}^{(2s)}(0) = 0 \quad \text{for } p = 1, \dots, \kappa, s = 0, \dots, \kappa - 1, \tag{12}$$

and

$$\alpha_{2p}^{(2s)}(1) = 0 \quad \text{for } p = 1, \dots, \kappa, s = 0, \dots, \kappa - 1. \tag{13}$$

It obvious that

$$A(t) := \begin{bmatrix} \alpha_1(t) & \alpha_2(t) & \dots & \alpha_{2\kappa-1}(t) & \alpha_{2\kappa}(t) \\ \alpha_1'(t) & \alpha_2'(t) & \dots & \alpha_{2\kappa-1}'(t) & \alpha_{2\kappa}'(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_1^{(2\kappa-2)}(t) & \alpha_2^{(2\kappa-2)}(t) & \cdots & \alpha_{2\kappa-1}^{(2\kappa-2)}(t) & \alpha_{2\kappa}^{(2\kappa-2)}(t) \\ \alpha_1^{(2\kappa-1)}(t) & \alpha_2^{(2\kappa-1)}(t) & \dots & \alpha_{2\kappa-1}^{(2\kappa-1)}(t) & \alpha_{2\kappa}^{(2\kappa-1)}(t) \end{bmatrix}, \tag{14}$$

is a fundamental matrix for (11). Therefore, by elementary properties of the Wronskian determinant

$$W(t) := \det A(t) \neq 0, \quad \text{for } t \in [0,1]. \tag{15}$$

Furthermore, we have

$$X(t) = A(t) \cdot D + A(t) \int_0^t A^{-1}(s) F(s) ds, \tag{16}$$

where $D = [D_1, \ldots, D_{2\kappa}]^T$ is constant. Set

$$R(t) := [R_1(t), \ldots, R_{2\kappa}(t)]^T = D + \int_0^t A^{-1}(s)F(s)ds. \tag{17}$$

Then (16) can be written in the form

$$X(t) = A(t) \cdot R(t). \tag{18}$$

By (17) we get

$$A(t)R'(t) = F(t).$$

For each $t \in [0,1]$ the last equation is a linear vector one in variable $R'(t)$. Since $W(t) \neq 0$, thus Crammer's theorem implies that the system has exactly one solution of the form

$$R_i'(t) = (-1)^i \widehat{f}(t) \frac{1}{W(t)} \det A_i(t), \quad i = 1, \ldots, 2\kappa,$$

Matrix A_i arises from A by deleting the i-th column and the 2κ-th row. From the last equalities we have

$$R_{2j-1}(t) = \int_t^1 \frac{1}{W(s)} \det A_{2j-1}(s)\widehat{f}(s)ds + C_{2j-1}, \quad \text{for } j = 1, \ldots, \kappa$$

$$R_{2j}(t) = \int_0^t \frac{1}{W(s)} \det A_{2j}(s)\widehat{f}(s)ds + C_{2j} \quad \text{for } j = 1, \ldots, \kappa.$$

Condition (18) leads us to the conclusion that

$$\begin{aligned} x^{(p)}(t) =& x_p(t) = \sum_{j=1}^{\kappa} \alpha_{2j-1}^{(p)}(t) \int_t^1 \frac{1}{W(s)} \det A_{2j-1}(s)\widehat{f}(s)ds \\ &+ \sum_{j=1}^{\kappa} \alpha_{2j}^{(p)}(t) \int_0^t \frac{1}{W(s)} \det A_{2j}(s)\widehat{f}(s)ds \\ &+ \sum_{j=1}^{\kappa} \alpha_{2j-1}^{(p)}(t)C_{2j-1} + \sum_{j=1}^{\kappa} \alpha_{2j}^{(p)}(t)C_{2j}, \quad p = 0, \ldots, 2\kappa - 1. \end{aligned}$$

It remains to find C_i for that the above formula describes solution of (9) with boundary conditions (10). By (10), (12) and (13), we get two systems of equations

$$\begin{aligned} 0 =& x^{(2s)}(0) = \sum_{j=1}^{\kappa} \alpha_{2j}^{(2s)}(0) C_{2j}, \quad s = 0, \dots, \kappa - 1 \\ 0 =& x^{(2s)}(1) = \sum_{j=1}^{\kappa} \alpha_{2j-1}^{(2s)}(1) C_{2j-1}, \quad s = 0, \dots, \kappa - 1, \end{aligned}$$

that we can describe in the form

$$\begin{bmatrix} \alpha_2(0) & \alpha_4(0) & \dots & \alpha_{2\kappa-2}(0) & \alpha_{2\kappa}(0) \\ \alpha_2''(0) & \alpha_4''(0) & \dots & \alpha_{2\kappa-2}''(0) & \alpha_{2\kappa}''(0) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_2^{(2\kappa-2)}(0) & \alpha_4^{(2\kappa-2)}(0) & \cdots & \alpha_{2\kappa-2}^{(2\kappa-2)}(0) & \alpha_{2\kappa}^{(2\kappa-2)}(0) \end{bmatrix} \begin{bmatrix} C_2 \\ C_4 \\ \vdots \\ C_{2\kappa} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{19}$$

$$\begin{bmatrix} \alpha_1(1) & \alpha_3(1) & \dots & \alpha_{2\kappa-3}(1) & \alpha_{2\kappa-1}(1) \\ \alpha_1''(1) & \alpha_3''(1) & \dots & \alpha_{2\kappa-3}''(1) & \alpha_{2\kappa-1}''(1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_1^{(2\kappa-2)}(1) & \alpha_3^{(2\kappa-2)}(1) & \cdots & \alpha_{2\kappa-3}^{(2\kappa-2)}(1) & \alpha_{2\kappa-1}^{(2\kappa-2)}(1) \end{bmatrix} \begin{bmatrix} C_1 \\ C_3 \\ \vdots \\ C_{2\kappa-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{20}$$

Notice that both of the determinants corresponding to the above systems are nonzero. Indeed, by using (15) with $t = 0$, we obtain that

$$0 \neq W(0) = \begin{vmatrix} 0 & \alpha_2(0) & 0 & \dots & 0 & \alpha_{2\kappa}(0) \\ \alpha_1'(0) & \alpha_2'(0) & \alpha_3'(0) & \dots & \alpha_{2\kappa-1}'(0) & \alpha_{2\kappa}'(0) \\ 0 & \alpha_2''(0) & 0 & \dots & 0 & \alpha_{2\kappa}''(0) \\ \alpha_1^{(3)}(0) & \alpha_2^{(3)}(0) & \alpha_3^{(3)}(0) & \dots & \alpha_{2\kappa-1}^{(3)}(0) & \alpha_{2\kappa}^{(3)}(0) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \alpha_2^{(2\kappa-2)}(0) & 0 & \cdots & 0 & \alpha_{2\kappa}^{(2\kappa-2)}(0) \\ \alpha_1^{(2\kappa-1)}(0) & \alpha_2^{(2\kappa-1)}(0) & \alpha_3^{(2\kappa-1)}(0) & \dots & \alpha_{2\kappa-1}^{(2\kappa-1)}(0) & \alpha_{2\kappa}^{(2\kappa-1)}(0) \end{vmatrix}$$

$$= \operatorname{sgn} \sigma_1 \cdot \begin{vmatrix} \alpha_2(0) & \dots & \alpha_{2\kappa}(0) & 0 & \dots & 0 \\ \alpha_2'(0) & \dots & \alpha_{2\kappa}'(0) & \alpha_1'(0) & \dots & \alpha_{2\kappa-1}'(0) \\ \alpha_2''(0) & \dots & \alpha_{2\kappa}''(0) & 0 & \dots & 0 \\ \alpha_2^{(3)}(0) & \dots & \alpha_{2\kappa}^{(3)}(0) & \alpha_1^{(3)}(0) & \dots & \alpha_{2\kappa-1}^{(3)}(0) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \alpha_2^{(2\kappa-2)}(0) & \cdots & \alpha_{2\kappa}^{(2\kappa-2)}(0) & 0 & \cdots & 0 \\ \alpha_2^{(2\kappa-1)}(0) & \dots & \alpha_{2\kappa}^{(2\kappa-1)}(0) & \alpha_1^{(2\kappa-1)}(0) & \dots & \alpha_{2\kappa-1}^{(2\kappa-1)}(0) \end{vmatrix}$$

$$= \operatorname{sgn}\sigma_2 \operatorname{sgn}\sigma_1 \cdot \begin{vmatrix} \alpha_2(0) & \dots & \alpha_{2\kappa}(0) & 0 & \dots & 0 \\ \alpha_2''(0) & \dots & \alpha_{2\kappa}''(0) & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \alpha_2^{(2\kappa-2)}(0) & \dots & \alpha_{2\kappa}^{(2\kappa-2)}(0) & 0 & \dots & 0 \\ \alpha_2'(0) & \dots & \alpha_{2\kappa}'(0) & \alpha_1'(0) & \dots & \alpha_{2\kappa-1}'(0) \\ \alpha_2^{(3)}(0) & \dots & \alpha_{2\kappa}^{(3)}(0) & \alpha_1^{(3)}(0) & \dots & \alpha_{2\kappa-1}^{(3)}(0) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \alpha_2^{(2\kappa-1)}(0) & \dots & \alpha_{2\kappa}^{(2\kappa-1)}(0) & \alpha_1^{(2\kappa-1)}(0) & \dots & \alpha_{2\kappa-1}^{(2\kappa-1)}(0) \end{vmatrix}$$

$$= \operatorname{sgn}(\sigma_2\sigma_1) \cdot \begin{vmatrix} \alpha_2(0) & \dots & \alpha_{2\kappa}(0) \\ \alpha_2''(0) & \dots & \alpha_{2\kappa}''(0) \\ \vdots & \ddots & \vdots \\ \alpha_2^{(2\kappa-2)}(0) & \dots & \alpha_{2\kappa}^{(2\kappa-2)}(0) \end{vmatrix} \cdot \begin{vmatrix} \alpha_1'(0) & \dots & \alpha_{2\kappa-1}'(0) \\ \alpha_1^{(3)}(0) & \dots & \alpha_{2\kappa-1}^{(3)}(0) \\ \vdots & \ddots & \vdots \\ \alpha_1^{(2\kappa-1)}(0) & \dots & \alpha_{2\kappa-1}^{(2\kappa-1)}(0) \end{vmatrix},$$

where σ_1 and σ_2 are permutations given by the formulas

$$\sigma_1(i) = \begin{cases} 2i & \text{for } i = 1, \dots \kappa \\ 2i - 1 - 2\kappa & \text{for } i = \kappa + 1, \dots, 2\kappa \end{cases}$$

and

$$\sigma_2(i) = \begin{cases} 2i - 1 & \text{for } i = 1, \dots \kappa \\ 2i - 2\kappa & \text{for } i = \kappa + 1, \dots, 2\kappa \end{cases}.$$

Therefore the matrix of system (19) is a nonsingular one. In the same way we show that the determinant of system (20) is also nonzero. Thus, it follows from Crammer's theorem that $C_i = 0$ for $i = 1, \dots, 2\kappa$.

We have proved the following corollary.

Corollary 2.2. *The p-th derivative, $p = 0, \dots, 2\kappa - 2$ of any solution to the problem (9)–(10) has the form*

$$x^{(p)}(t) = \sum_{j=1}^{\kappa} \alpha_{2j-1}^{(p)}(t) \int_t^1 \frac{1}{W(s)} \det A_{2j-1}(s)\widehat{f}(s)ds$$

$$+ \sum_{j=1}^{\kappa} \alpha_{2j}^{(p)}(t) \int_0^t \frac{1}{W(s)} \det A_{2j}(s)\widehat{f}(s)ds$$

$$= \int_0^1 \mathcal{H}_{\lambda^1 \dots \lambda^\kappa}^{(p),\kappa}(t,s)\, \widehat{f}(s)ds,$$

where $\mathcal{H}^{(p),\kappa}_{\lambda^1\ldots\lambda^\kappa} : [0,1]\times[0,1] \to R$ *is given by the formula*

$$\mathcal{H}^{(p),\kappa}_{\lambda^1\ldots\lambda^\kappa}(t,s) := \begin{cases} \dfrac{1}{W(s)}\displaystyle\sum_{j=1}^{\kappa} \alpha^{(p)}_{2j-1}(t)\det A_{2j-1}(s) & \textit{for } 0\le t\le s\le 1 \\ \dfrac{1}{W(s)}\displaystyle\sum_{j=1}^{\kappa} \alpha^{(p)}_{2j}(t)\det A_{2j}(s) & \textit{for } 0\le s\le t\le 1 \end{cases}. \quad (21)$$

Each function $\mathcal{H}^{(p),\kappa}_{\lambda^1\ldots\lambda^\kappa}$ will be called *a Green's function.*

Lemma 2.2. *Let* $(\lambda^1,\ldots,\lambda^\kappa)\notin\sigma^\kappa$*. Then for each* $p=0,\ldots,2\kappa-2$ *series*

$$\sum_{n=1}^{\infty}\frac{(-1)^{\lfloor\frac{p}{2}\rfloor}(n\pi)^p}{(-n^2\pi^2)^\kappa-\sum_{\alpha=1}^{\kappa}\lambda^\alpha(-n^2\pi^2)^{\kappa-\alpha}}$$

is absolute convergent[1]*.*

Proof. Set $a_n := (-n^2\pi^2)^\kappa-\sum_{\alpha=1}^{\kappa}\lambda^\alpha(-n^2\pi^2)^{\kappa-\alpha}$. Then

$$|a_n| = n^{2\kappa}\pi^{2\kappa}\left|1-\sum_{\alpha=1}^{\kappa}\frac{\lambda^\alpha}{(-n^2\pi^2)^\alpha}\right|.$$

It is easy to see that there exists N such that for $n>N$, we have $\left|1-\sum_{\alpha=1}^{\kappa}\lambda^\alpha(-n^2\pi^2)^{-\alpha}\right|\ge 1/2$. Thus

$$|a_n|\ge\frac{1}{2}n^{2\kappa}\pi^{2\kappa}.$$

This implies that

$$\left|\frac{(-1)^{\lfloor\frac{p}{2}\rfloor}(n\pi)^p}{(-n^2\pi^2)^\kappa-\sum_{\alpha=1}^{\kappa}\lambda^\alpha(-n^2\pi^2)^{\kappa-\alpha}}\right|\le\frac{2}{(n\pi)^{2\kappa-p}}.$$

Lemma is proved.

Hereinafter it will be assumed that $\widehat{f}\in L^2([0,1],\mathbb{R})$. □

[1] $\lfloor x\rfloor := \max\{n\in\mathbb{Z}\mid n\le x\}$.

Theorem 2.2. *The p-th derivative, $p = 0, \ldots, 2\kappa - 2$, of any solution to the problem (9)–(10) has the form*

$$x^{(p)}(t) = \int_0^1 \sum_{n=1}^{\infty} \frac{(-1)^{\lfloor \frac{p}{2} \rfloor} (n\pi)^p}{(-n^2\pi^2)^\kappa - \sum_{\alpha=1}^{\kappa} \lambda^\alpha (-n^2\pi^2)^{\kappa-\alpha}} \varphi_p(t) \sin(n\pi s) \widehat{f}(s)\, ds,$$

Where $\varphi_p(t) = \begin{cases} \sin(n\pi t) & \text{for } p \text{ even} \\ \cos(n\pi t) & \text{for } p \text{ odd} \end{cases}$. *Furthermore, the Green's function* $\mathcal{H}^{(p),\kappa}_{\lambda^1 \ldots \lambda^\kappa}$ *is given by the formula*

$$\mathcal{H}^{(p),\kappa}_{\lambda^1 \ldots \lambda^\kappa}(t, s) = \sum_{n=1}^{\infty} \frac{(-1)^{\lfloor \frac{p}{2} \rfloor} (n\pi)^p}{(-n^2\pi^2)^\kappa - \sum_{\alpha=1}^{\kappa} \lambda^\alpha (-n^2\pi^2)^{\kappa-\alpha}} \varphi_p(t) \sin(n\pi s).$$

Both of these series is uniformly convergent.

Proof. Let us fix $(\lambda^1, \ldots, \lambda^\kappa) \notin \sigma^\kappa$. If x is a solution of the problem (9)–(10), then $T(x; \lambda^1, \ldots, \lambda^\kappa) = \widehat{f}$. If we reason in the same way as in Preliminaries, we get

$$x = \sum_{n=1}^{\infty} \left[(-n^2\pi^2)^\kappa - \sum_{\alpha=1}^{\kappa} \lambda^\alpha (-n^2\pi^2)^{\kappa-\alpha} \right]^{-1} \left\langle e_n, \widehat{f} \right\rangle_{L^2} e_n,$$

thus

$$x(t) = \sum_{n=1}^{\infty} \int_0^1 \left[(-n^2\pi^2)^\kappa - \sum_{\alpha=1}^{\kappa} \lambda^\alpha (-n^2\pi^2)^{\kappa-\alpha} \right]^{-1} \sin(n\pi t) \sin(n\pi s) \widehat{f}(s) ds.$$

By lemma 2.2, series

$$\sum_{n=1}^{\infty} \left[(-n^2\pi^2)^\kappa - \sum_{\alpha=1}^{\kappa} \lambda^\alpha (-n^2\pi^2)^{\kappa-\alpha} \right]^{-1} \sin(n\pi t) \sin(n\pi s) \widehat{f}(s)$$

is uniformly convergent. This implies that

$$x(t) = \int_0^1 \sum_{n=1}^{\infty} \left[(-n^2\pi^2)^\kappa - \sum_{\alpha=1}^{\kappa} \lambda^\alpha (-n^2\pi^2)^{\kappa-\alpha} \right]^{-1} \sin(n\pi t) \sin(n\pi s) \widehat{f}(s) ds.$$

Lemma 2.2 and fact that the integral of a sum is the sum of the integrals gives us the following equality

$$x^{(p)}(t) = \int_0^1 \sum_{n=1}^{\infty} \frac{(-1)^{\lfloor \frac{p}{2} \rfloor} (n\pi)^p}{(-n^2\pi^2)^{\kappa} - \sum_{\alpha=1}^{\kappa} \lambda^{\alpha} (-n^2\pi^2)^{\kappa-\alpha}} \varphi_p(t) \sin(n\pi s) \widehat{f}(s)\, ds,$$

where $\varphi_p(t) = \begin{cases} \sin(n\pi t) & \text{for } p \text{ even} \\ \cos(n\pi t) & \text{for } p \text{ odd} \end{cases}$. Finally, corollary 2.2 implies that

$$\mathcal{H}^{(p),\kappa}_{\lambda^1 \ldots \lambda^{\kappa}}(t,s) = \sum_{n=1}^{\infty} \frac{(-1)^{\lfloor \frac{p}{2} \rfloor} (n\pi)^p}{(-n^2\pi^2)^{\kappa} - \sum_{\alpha=1}^{\kappa} \lambda^{\alpha} (-n^2\pi^2)^{\kappa-\alpha}} \varphi_p(t) \sin(n\pi s).$$

This finishes the proof. □

Corollary 2.3. *Function $\mathcal{H}^{(p),\kappa}_{\lambda^1 \ldots \lambda^{\kappa}}$ is continuous for each $p = 0, \ldots, 2\kappa - 2$.*

The analysis given above shows that in order to determine the function $\mathcal{H}^{(p),\kappa}_{\lambda^1 \ldots \lambda^{\kappa}}$ it is necessary to find the matrix A, what is equivalent to the problem of finding a linearly independent solutions of (6) satisfying the conditions (12)–(13).

Example 2.1 Let us consider the following BVP

$$\begin{aligned} &x^{(2\kappa)}(t) = \widehat{f}(t) \\ &x^{(2s)}(0) = x^{(2s)}(1) = 0, \quad s = 0, \ldots, \kappa - 1 \end{aligned} \qquad 21$$

According to the above considerations we have to find solutions of the following equation

$$X'(t) = \Lambda \cdot X(t), \qquad (23)$$

corresponding to (11) and satisfying conditions (12)–(13). Then we have

$$\Lambda = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

The well known theory of linear differential equations follows that it is sufficient to find the matrix exponential for Λ. To do that notice that

$\Lambda = [a_{ij}]_{1\le i,j\le 2\kappa} = [\delta_{i,j+1}]_{1\le i,j\le 2\kappa}$. Furthermore, for each $p = 2,3,\ldots$ we get

$$\Lambda^p := \Lambda^{p-1}\Lambda = \left[a_{ij}^{(p)}\right]_{1\le i,j\le 2\kappa} = [\delta_{i+p,j}]_{1\le i,j\le 2\kappa}\,. \qquad (24)$$

Indeed, we have $\Lambda^2 := \left[a_{ij}^{(2)}\right]_{1\le i,j\le 2\kappa}$, where $a_{ij}^{(2)} = \sum_{l=1}^{2\kappa} a_{il}a_{lj} = \sum_{l=1}^{2\kappa} \delta_{i+1,1}\delta_{l+1,j}$. It is easy to see that $a_{ij}^{(2)}$ may be either zero or one. The second condition is satisfied if $i+1 = l$ and $l+1 = j$. This implies that $i+2 = j$. Thus $\Lambda^2 := [\delta_{i+2,j}]_{1\le i,j\le 2\kappa}$. Let us fix p and suppose that $\Lambda^{p-1} = \left[a_{ij}^{(p-1)}\right]_{1\le i,j\le 2\kappa} = [\delta_{i+p-1,j}]_{1\le i,j\le 2\kappa}$. Then we have $a_{ij}^{(p)} = \sum_{l=1}^{2\kappa} a_{il}^{(p-1)}a_{lj} = \sum_{l=1}^{2\kappa} \delta_{i+p-1,l}\delta_{l+1,j}$. Furthermore $a_{ij}^{(p)} \in \{0,1\}$ and $a_{ij}^{(p)} = 1$, if $i+p-1 = l$ and $l+1 = j$, therefore $i+p = j$. This implies that $\Lambda^p = [\delta_{i+p,j}]_{1\le i,j\le 2\kappa}$, and prove (24). Because of natural restriction $1 \le i,j \le 2\kappa$, it follows from formula (24) that the matrix Λ^p is the zero one for $p \ge 2\kappa$. Therefore

$$\exp\left[(t-t_0)\,\Lambda\right] := \sum_{l=0}^{\infty} \frac{1}{l!}\,(t-t_0)^l\,\Lambda^l = [e_{ij}]_{1\le i,j\le 2\kappa}\,,$$

where

$$e_{ij} = \delta_{i,j} + (t-t_0)\,\delta_{i,j+1} + \frac{(t-t_0)^2}{2}\delta_{i,j+2} + \frac{(t-t_0)^3}{6}\delta_{i,j+3} + \ldots + \frac{(t-t_0)^{2\kappa-1}}{(2\kappa-1)!}\delta_{i,j+2\kappa-1}.$$

Thus any solution of equation (23) with the initial condition $X(t_0) = [C_0,\ldots,C_{2\kappa-1}]^T$, has the form

$$X(t) = \begin{bmatrix} 1 & (t-t_0) & \frac{(t-t_0)^2}{2} & \frac{(t-t_0)^3}{6} & \frac{(t-t_0)^4}{24} & \cdots & \frac{(t-t_0)^{2\kappa-2}}{(2\kappa-2)!} & \frac{(t-t_0)^{2\kappa-1}}{(2\kappa-1)!} \\ 0 & 1 & (t-t_0) & \frac{(t-t_0)^2}{2} & \frac{(t-t_0)^3}{6} & \cdots & \frac{(t-t_0)^{2\kappa-3}}{(2\kappa-3)!} & \frac{(t-t_0)^{2\kappa-2}}{(2\kappa-2)!} \\ 0 & 0 & 1 & (t-t_0) & \frac{(t-t_0)^2}{2} & \cdots & \frac{(t-t_0)^{2\kappa-4}}{(2\kappa-4)!} & \frac{(t-t_0)^{2\kappa-3}}{(2\kappa-3)!} \\ 0 & 0 & 0 & 1 & (t-t_0) & \cdots & \frac{(t-t_0)^{2\kappa-5}}{(2\kappa-5)!} & \frac{(t-t_0)^{2\kappa-4}}{(2\kappa-4)!} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 & (t-t_0) & \frac{(t-t_0)^2}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & (t-t_0) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_{2\kappa-1} \end{bmatrix}.$$

Because of (12)–(13) only initial value conditions $X(0) = [0, C_1, 0, C_3 \ldots, 0, C_{2\kappa-1}]$, $X(1) = [0, C_1, 0, C_3 \ldots, 0, C_{2\kappa-1}]$, are interesting for us. This implies that A is a matrix of the form

$$A(t) = \begin{bmatrix} t & (t-1) & \frac{t^3}{6} & \frac{(t-1)^3}{6} & \frac{t^5}{120} & \ldots & \frac{t^{2\kappa-1}}{(2\kappa-1)!} & \frac{(t-1)^{2\kappa-1}}{(2\kappa-1)!} \\ 1 & 1 & \frac{t^2}{2} & \frac{(t-1)^2}{2} & \frac{t^4}{24} & \ldots & \frac{t^{2\kappa-2}}{(2\kappa-2)!} & \frac{(t-1)^{2\kappa-2}}{(2\kappa-2)!} \\ 0 & 0 & t & (t-1) & \frac{t^3}{6} & \ldots & \frac{t^{2\kappa-3}}{(2\kappa-3)!} & \frac{(t-1)^{2\kappa-3}}{(2\kappa-3)!} \\ 0 & 0 & 1 & 1 & \frac{t^2}{2} & \ldots & \frac{t^{2\kappa-4}}{(2\kappa-4)!} & \frac{(t-1)^{2\kappa-4}}{(2\kappa-4)!} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & \frac{t^2}{2} & \frac{(t-1)^2}{2} \\ 0 & 0 & 0 & 0 & 0 & \ldots & t & (t-1) \\ 0 & 0 & 0 & 0 & 0 & \ldots & 1 & 1cr \end{bmatrix}.$$

In other words $A(t) = [\widehat{\alpha}_{ij}(t)]_{1 \le i,j \le 2\kappa}$, where

$$\widehat{\alpha}_{ij}(t) = \begin{cases} \begin{cases} \frac{1}{(j-i+1)!} t^{j-i+1}, & j \ge i-1 \\ 0, & j < i-1 \end{cases} & \text{for } j = 1, 3, \ldots, 2\kappa - 1 \\ \begin{cases} \frac{1}{(j-i)!} (t-1)^{j-i}, & j \ge i \\ 0, & j < i \end{cases} & \text{for } j = 2, 4, \ldots, 2\kappa \end{cases}$$

According to established notation (see (14)) $\widehat{\alpha}_{ij}(.) = \alpha_j^{(i-1)}(.)$, where for $j = 2l - 1$ and $j = 2l$ we have respectively $\alpha_{2l-1}(t) = \frac{t^{2l-1}}{(2l-1)!}$, $\alpha_{2l}(t) = \frac{(t-1)^{2l-1}}{(2l-1)!}$, $l = 1, 2, \ldots, \kappa$. The last expressions are solutions that are the basis for $D_0(0, \ldots, 0)$ and $D_1(0, \ldots, 0)$, respectively. Thus for $p = 0, 1, \ldots, 2\kappa - 1$, we have

$$\alpha_{2l-1}^{(p)}(t) = \begin{cases} \frac{t^{2l-p-1}}{(2l-p-1)!} & \text{for } l \ge \frac{p+1}{2} \\ 0 & \text{for } l < \frac{p+1}{2} \end{cases}$$
$$\alpha_{2l}^{(p)}(t) = \begin{cases} \frac{(t-1)^{2l-p-1}}{(2l-p-1)!} & \text{for } l \ge \frac{p+1}{2} \\ 0 & \text{for } l < \frac{p+1}{2} \end{cases},$$

for $l = 1, 2, \ldots, \kappa$. Notice that if we multiply the $2s$-th columns of A by -1 next add to $(2s-1)$-th columns, $s = 1, \ldots, \kappa$, then we obtain a triangular matrix that contains ones on the main diagonal. Therefore

$$W(t) = \det A(t) = 1, \quad \text{for } t \in [0, 1].$$

If we denote $\mathcal{G}^{(p),\kappa} := \mathcal{H}^{(p),\kappa}_{0\ldots 0}$, for $p = 0, 1, \ldots, 2\kappa - 1$. Then by (21) we get

$$\mathcal{G}^{(p),\kappa}(t,s) := \begin{cases} \sum\limits_{l=\lceil \frac{p+1}{2} \rceil}^{\kappa} \dfrac{t^{2l-p-1}}{(2l-p-1)!} \det A_{2l-1}(s) & \text{for } 0 \le t \le s \le 1 \\ \sum\limits_{l=\lceil \frac{p+1}{2} \rceil}^{\kappa} \dfrac{(t-1)^{2l-p-1}}{(2l-p-1)!} \det A_{2l}(s) & \text{for } 0 \le s \le t \le 1 \end{cases},$$

where $\lceil . \rceil \colon \mathbb{R} \to \mathbb{Z}$, is given by the formula $\lceil x \rceil := \begin{cases} [x]+1 & \text{for } x \notin \mathbb{Z} \\ x & \text{for } x \in \mathbb{Z} \end{cases}$. Furthermore corollary 2.2 implies that p-th derivative, $p = 0, \ldots, 2\kappa - 1$, of any solution to (22) has the form

$$x^{(p)}(t) = \int_0^1 \mathcal{G}^{(p),\kappa}(t,s)\, \widehat{f}(s) ds. \tag{25}$$

The Lidstone BVP is mathematical description of many physical and mechanical phenomena, thus the knowledge about the form of the exact solutions is very important from practical point of view. We have explained that to start seeking the solution to (2) it is necessary to find the form of the corresponding Green's function first. We have given recipes how to do it, therefore the results can be interesting for specialists from different part of science.

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Pewne aspekty dotyczące jawnych rozwiązań zagadnienia brzegowego Lidstone'a

Streszczenie. Celem pracy jest rozważenie pewnych aspektw egzystencjalnych dotyczących zagadnienia brzegowego Lidstone'a, postaci

$$\begin{cases} x^{(2k)}(t) - \sum_{i=1}^{k} \lambda^i x^{(2k-2i)}(t) = f\left(t, x(t), x''(t), \ldots, x^{(2k-2)}(t)\right) \\ x^{(2s)}(0) = x^{(2s)}(1) = 0, \quad s = 0, \ldots, k-1 \end{cases}.$$

Jest widoczne, że problem istnienia rozwiązań zależy od części liniowej powyższego problemu. Dokładniej dla pewnego zestawu parametrw λ_i operator rżniczkowy odpowiadający lewej stronie przedstawionego problemu może być odwracalny lub nie. W związku z tym praca została podzielona na dwie części, w pierwszej skupiamy się na wyznaczeniu k-wymiarowego widma oraz opisaniu jego własności. Druga część jest poświęcona podaniu metod wyznaczania jawnej postaci funkcji Greena dla powyższego problemu w zbiorze rezolwenty wspomnianego operatora rżniczkowego.
Słowa kluczowe: problem brzegowy Lidstone'a, wartość własna