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On the forming of the waves with strong discontinuity in nonlinear hyperbolic thermoelasticity

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Abstract. The most difficult type of waves to analyse in nonlinear thermo-elastic materials are the shock waves. We develop the theory of shock waves in heat conductive elastic materials. We consider the forming of the waves with strong discontinuity (so-called thermal shock waves) in nonlinear thermoelasticity describing the propagation of the heat with finite speed. We proved the mathematical, physical, necessary and sufficient conditions of the forming of the thermal shock wave and mechanical shock wave in the nonlinear thermoelasticity.
Keywords: nonlinear hyperbolic thermoelasticity, thermal shock wave, hyperbolic heat equation, mechanical shock wave

1. Introduction

Before starting the formulation of the problem we recall some facts from the theory of waves.

Wave motion can be treated generally as the propagation of disturbances in a medium.

The wave phenomenon can be described by two features:

— it doesn't affect the mass transportation

— its localization in the space is described by the surface named the front wave.

The local disturbances are the source of the wave motion and are propagated with the finite speed in the space.

If the primary source of the given wave is the harmonic vibration with frequency ω, then such a wave is named the harmonic wave and the equiphase surface is the wave front.

In this case the characteristic value is introduced

$$\lambda = cT$$

where T – is the length period and is given by the formula

$$T = \frac{2\pi}{\omega}.$$

In the investigation of the propagation of the wave we also use the wave vector $\vec{k}$ with the length

$$|\vec{k}| = \frac{2\pi}{\lambda} = k.$$

Vector $\vec{k}$ defines the direction of the wave propagation and is perpendicular to the equiphase plane.
In the description of the nature of the wave phonomena we also use:

— the phase velocity expressed as

$$c_f = \frac{\omega}{k}$$

and
— the group velocity given by the formulae

$$c_g = \frac{dw(k)}{dk}.$$

If we have the equality

$$c_f = c_g$$

the wave is called the non-dispertion wave, but if we have the relation

$$c_f \neq c_g$$

the wave is called the dispersion wave.

With respect to the physical phenomena we can distinguish different kinds of waves:

— acoustic waves
— mechanical waves
— electro-magnetic waves
— thermal waves
— elastic waves
— optical waves

and others.

In the mathematical description, the phenomena of the evolution of the wave in the time space (t, x) where, $t \in \mathbb{R}_+$, and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is determined by a solution of the wave equation, which can be written as follows:

$$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\varphi = f$$

or in the form

$$\Box_c \varphi = f$$

where ∇^2 – is the Laplace's operator while $\Box_c$ – is the d'Alambert's operator.

The geometrical and physical properties of the wave motion depend on the dimension of the space, so we distinguish – one-dimensional wave, two-dimensional wave and three-dimensional wave.

It is worth to mention the so-called Hadamard's principle, which tells us that only in odd dimensional spaces $\mathbb{R}^{2n+1}$ the Huygen's principle is fulfilled and that the propagating sygnal in the space preserves its primary frequency.

Any solution of the wave equation (hyperbolic type) can be represented by the formulae:

$$\varphi(t, x) = H\left(t - \frac{|x|}{c}\right)F(t, x) = \begin{cases} F(t, x) & \text{for } t > \frac{|x|}{c} \\ 0 & \text{for } t < \frac{|x|}{c} \end{cases}$$

where: $H(\cdot)$ – denotes the Heaviside's function.

2. The front of the wave and the surface of the discontinuity

Below, we present a mathematical description and the properties of the front wave.

Let S – be a smooth surface which changes with respect to the times with localization in the Euclidean space E^3.
Let us consider such a moving surface S defined by the equation:

$$S : \psi(t, x) = 0, \quad x \in \mathbb{E}^3, \quad t \in \mathbb{R}_+. \tag{2.1}$$

The unit normal vector to the surface S is expressed by the formula

$$\vec{n} = \frac{\mathrm{grad}\psi}{|\mathrm{grad}\psi|}. \tag{2.2}$$

The velocity with which the surface S is moving is determined by the formula

$$D = -\frac{\partial_t \psi}{|\mathrm{grad}\psi|} \quad \text{on} \quad \vec{D} = D\vec{n}. \tag{2.3}$$

If the surface S is the front of the wave, then $\vec{D}$ denotes the velocity of the propagation of the wave.
In the time-space $I \times \mathbb{E}^3$ (where $I = (0, \infty)$ is the time interval), the wave front is a non-moving hypersurface so-called hyper-cone.
The surface S is the material surface if the velocity $D = v_n$, where v_n – is the velocity of the particle in the normal direction to the given surface.
The wave front is the non-material surface. The wave velocity depends on the physical properties of the medium in which the wave is propagated and on its intensity as well.
From this point of view we distinguish linear and nonlinear waves.
The wave velocity changes its value and the direction of the propagation.

The curve which is tangential to the normal vector is called the rays of the wave. The changes of the direction of the wave velocity lead to the increase in the wave intensity. In this case the rays are intersecting each other and the so-called caustic will be formed.
The change of the value of the wave velocity causes the higher discontinuity of the parameters describing wave motion in the neighbourhood of the wave front i.e. blows up the gradient – the so-called gradient castastrophe.

The waves generating at different moments catch up with one another (are overtaking one another) and affect the wave intensity at one side of the wave front. The next effect is the formation of a strong discontinuity front – a shock – wave front.

These kinds of the waves with strong discontinuity are called shock waves.

The main aim of our paper is the mathematical and physical interpretation of the forming of thermal shock waves.

Let $V(t,x)$ be a scalar or a tensor field defined in the open domain $\Omega \subset \mathbb{E}^3$ in which $S \subset \Omega$. Let us assume that:

- the field V is continuous on both sides of the surface S
- some derivatives of the field V can be discontinuous on the surface S (experience abrupt changes as they pass through the surface S) but are defined on one – side of the surface S with the both sides of the surface S
- the field V is satisfying the equation $LV = 0$ in the domain $\Omega \backslash S$, where L denotes the differential operator defined by a physical process.

The conditions which characterize the properties of the derivatives of the field V and the expresion (LV) by passing though the surface S are called the compatibility conditions.

The surface S is called the surface of discontinuity. In general, we distinguish the compatibility conditions of the first, second and third order (or higher order) with respect to the order of the derivatives of the field V which can get discontinuites and with the order of the operator L.

These conditons are divided into geometrical and kinematical conditions which are connected to the behaviour of the tangential derivatives of the field V on the moving surface S and the substantial derivatives on the moving surface S, respectively.

The different kinds of the conditions are the dynamical compatibility conditions related to the sense in which the operator LV is satisfied on the discontinuous surface S.

They result from the application of the Green-Gauss theorem to the expression LV on the discontinuous surface and are the so-called conservation conditions (or balance conditions) by passing through the surface of discontinuity. These conditons for the first component of the vector field $V = [u, v, w]$ can be written as follows:

$$
\begin{aligned}
&[[\mathrm{grad} u \times \vec{n}]] = 0 && - \text{ geometrical conditon}\\
&\left[\left[\frac{\partial u}{\partial t} + (\mathrm{grad} u) o\vec{n}\right]\right] = 0 && - \text{ kinematical condition}\\
&[[-Du + u(uo\vec{n})]] = 0 && - \text{ dynamical condition}
\end{aligned}
$$

where $[[\cdot]]$ – denotes the jump of the value by passing through the surface of discontinuity.

If the surface S is the wave front i.e. the characteristic surface of the operator L, then in the linear medium the discontinuities of all orders appear only on this surface.
In the nonlinear medium the discontinuities of some order can appear on various surfaces. In the case of the shock waves we will be analyzing the behaviour of the derivative of the first order of the considered field. We can describe in the linear, homogeneous and isotropic medium the unique relation between the direction of the wave propagation and the local direction of the vibration of the field on the wave front. We distinguised two kinds of waves:

— transversal waves
— longitudinal waves

For example: the electromagnetic waves – are the transversal waves, acoustic waves – are the longitudinal waves but the mechanical waves are transversal and longitudinal waves as well (cf. [8], [9], [10], [11], [12]). In the solutions of some boundary value problem there appear the so-called surface waves i.e. Raylaigh's waves, which have a more complicated structure (cf. [8]).

Shock waves in solids

Shock waves develop and propagate in media in which the characteristic of equations describing their motion intersect (cf. [7], [12], [17–23]). Physically speaking this means that disturbances which develop later catch up with the earlier ones and accumulate, and the accumulated dirturbances become strong discontinuity surface – a shock-wave front is formed. The effect occurs when continuous and gradual changes in disturbances are involved.
If the distrubances are generated suddenly for example by a physical, chemical, nuclear explosion or by a collision of bodies at high speeds, then a shock-wave front is formed at once in media with the properties mentioned above.
The forming of the wave with strong discontinuity in the nonlinear thermoelastic medium with propagation of heat with finite speed is considered below.
The main aim of our paper is to designate the necessary and sufficient conditions with which we will describe the forming of thermal shock wave in nonlinear thermoelastic medium with propagation of the heat with finite speeds. Before starting the formulation of our problem we recall some papers devoted to the theory of shock waves published by S. Kaliski (cf. [12, 13]) and E. Włodarczyk (cf. [21–23]) where the problems of shock waves in elastic medium were consider ed. Rakhmatulin (cf. [18]) investigated the

problem of the forming of the wave with strong discontinuity but only in elastic mediums. W. Nowacki was the first in Poland (cf. [13–16]) who developed the classical hyperbolic-parabolic thermoelasticity theory in which the propagation of heat was described by infinite speed (parabolic heat equaitons).

In view of these facts in such thermoelastic medium the "thermal shock wave" does not exsit. In our paper we proved that thermal shock wave appearing in thermoelastic medium which describes the propagation of heat with finite speed. It means that the propagation of heat is presented by hyperbolic heat equation. It is according to physical intuitions. This kind of theory of thermoelasticity is called-hyperbolic thermoelasticity theory (cf. [2], [3], [4], [5], [6], [8], [12]).

The hyperbolic thermoelasticity theory was also investigated by I. Ignaczak, J. Gawinecki (cf. [3], [4], [5], [6], [8]). In order to make our consideration more clear, we consider the initial-boundary value problem for half-infinite bar modelling thermoelastic medium with finite speed propagation of heat.

We consider the system of equations of hyperbolic thermoelasticity theory in the form of the conservation laws:

— the equation of balance of momentum

$$\partial_t(\rho u) - \partial_x \sigma = 0 \tag{3.1}$$

— the equation of balance of energy

$$\partial_t(\rho e) + \partial_x q = 0 \tag{3.2}$$

and the constitutive equation of the form:

— for stress tensor

$$\sigma = \sigma(\varepsilon, \theta) = E(\theta) f(\varepsilon) - \alpha g(\theta) \tag{3.3}$$

— for the internal energy

$$\rho e = \int_{\theta_0}^{\theta} \rho C_v(\eta) d\eta + \gamma(\theta)\varepsilon = e(\theta, \varepsilon) \tag{3.4}$$

— for the heat flux according to the Cattaneo theory (cf. [1])

$$q + \tau_0 \partial_t q + k(\theta) \partial_x \theta = 0 \tag{3.5}$$

with the integral condition of the form:

$$\partial_x u = \partial_t \varepsilon \tag{3.6}$$

where $t \in \mathbb{R}_+$ $x \in (0, \infty)$, with the system of equation (3.1)–(3.6) we associated the initial-boundary condition of the form:

$$\begin{aligned} &v(0,x) = 0, \qquad \varepsilon(0,x) = 0, \qquad \theta(0,x) = 0 \\ &\sigma(t,0) = -p_0(t), \quad q(t,0) = -q_0(t) \ \text{ or } \ \theta(t,0) = \theta_0(t). \end{aligned} \tag{3.7}$$

In the initial-boundary value problem (3.1)–(3.7) we used the notation: ρ – denotes density of the medium, v – the velocity, σ – stress tensor, e – the internal energy, ε – the strain tensor, q – the heat flux, $\theta = T + T_0$ – the relative temperature, T – the absolute temperature, T_0 – the temperature in the reference configuration, c_v – the specific heat, k – the coefficient of heat conductiving, τ_0 – the relaxation time, γ – the coefficient of mechanical – thermal coupling.

The Equation (3.5) is the Cattaneo equation for the heat flux which leads us to the description of the the propagation of heat with finite speed. If $\tau_0 \to 0$ then from equation (3.5) we get the classical Fourier law for the heat flux.

The system of equations is the nonlinear system of hyperbolic thermoelasticity theory. Below, we will lead out the necessary and sufficient conditions for the forming of thermal shock waves in the initial-boundary value problem (3.1)–(3.7) describing the half-infinite bar being modelled as the nonlinear thermoelastic medium. It is worth to emphasize that it will be done for the first time. So, the results are also oryginal and innovatory ones. Up till now, the shock waves have been investigated in an elastic medium (cf. [12], [13–21]) but not in the thermoelastic medium.

In order to prove our results, we will act as follows.

At first, after some transformations we can write the system of equations (3.1)–(3.6) as follows:

$$\begin{aligned} &\partial_t(\rho v) - \frac{\partial \sigma}{\partial \varepsilon}\partial_x \varepsilon + \frac{\partial \sigma}{\partial \theta}\partial_x \theta = 0 \\ &\frac{\partial e}{\partial \theta}\partial_t \theta + \frac{\partial e}{\partial \varepsilon}\partial_t \varepsilon + \partial_x q = 0 \\ &\tau_0 \partial_t q + q + k(\theta)\partial_x \theta = 0 \\ &\partial_x u = \partial_t \varepsilon. \end{aligned} \tag{3.8}$$

Introducing the notation:

$$\frac{\partial \sigma}{\partial \varepsilon} = \rho a_s^2, \quad \frac{\partial e}{\partial \theta} = \rho c_v, \quad \frac{\partial \sigma}{\partial \theta} = -\beta, \quad \frac{\partial e}{\partial \varepsilon} = \beta \theta \tag{3.9}$$

where: a_s – is the local velocity of the propagation of mechanical perturbation in the medium, β – is the thermo-mechanical coupling coefficient. We can write the system of equations (3.8) as follows:

$$\begin{aligned} &\partial_t(\rho v) - \rho a_s^2 \partial_x \varepsilon + \beta \partial_x \theta = 0 \\ &\rho c_v \partial_t \theta + \beta \theta \partial_t \varepsilon + \partial_x q = 0 \\ &\tau_0 \partial_t q + q + k \partial_x \theta = 0 \\ &\partial_x v - \partial_t \varepsilon = 0. \end{aligned} \tag{3.10}$$

So, we obtain the system of equations for the vector $V = [v, \varepsilon, \theta, q]$ with the parameters $(a_s, \beta, c_v, k, \tau_0)$.
Now, we consider the following problem at the phase plane (t, x). Let the smooth normal curve be given on the phase plane (t, x)

$$C : x = \varphi(t).$$

The values of the derivatives of the components of the vector V in the tangential direction to the curve are as follows:

$$\begin{aligned} \frac{dv}{dt} &= \partial_t v + \partial_x v \frac{dx}{dt} \\ \frac{d\varepsilon}{dt} &= \partial_t \varepsilon + \partial_x \varepsilon \frac{dx}{dt} \\ \frac{d\theta}{dt} &= \partial_t \theta + \partial_x \theta \frac{dx}{dt} \\ \frac{dq}{dt} &= \partial_t q + \partial_x q \frac{dx}{dt}. \end{aligned} \tag{3.11}$$

We can write the system of equations (3.10) and (3.11) in the matrix form

$$MX = F \tag{3.12}$$

where:

$$X = \begin{bmatrix} \partial_t v \\ \partial_x v \\ \partial_t \varepsilon \\ \partial_x \varepsilon \\ \partial_t \theta \\ \partial_x \theta \\ \partial_t q \\ \partial_x q \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -q \\ \frac{dv}{dt} \\ \frac{d\varepsilon}{dt} \\ \frac{d\theta}{dt} \\ \frac{dq}{dt} \end{bmatrix} \tag{3.13}$$

$$M=\begin{bmatrix} \rho & 0 & 0 & -\rho a^2 & 0 & \beta & 0 & 0\\ 0 & 0 & \beta\theta & 0 & \rho c_v & 0 & 0 & 1\\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & k & \tau_0 & 0\\ 1 & \frac{dx}{dt} & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & \frac{dx}{dt} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & \frac{dx}{dt} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{dx}{dt} \end{bmatrix}. \tag{3.14}$$

The curve C: $x=\varphi(t)$ on which the system of equations (3.12) doesn't have the unique solution (cf. [7], [13]) is named the characteristic curve. The directions $\frac{dx}{dt}$ tangential to the curve C are-called the characteristic directions.

The characteristic directions can be determined by the characteristic equation

$$\det M = 0 \tag{3.14a}$$

where the determined M, is given by the formula

$$\det M=\tau_0\rho^2 c_v\left(\frac{dx}{dt}\right)^4-\left(\tau_0\rho^2 c_v a_s^2+\tau_0\beta^2\theta+k_\rho\right)\left(\frac{dx}{dt}\right)^2+k_\rho a_s^2 \tag{3.15}$$

or after some calculation

$$\det M=\left[\left(\frac{dx}{dt}\right)^4-\left(a_s^2+\frac{b}{\tau_0\rho c_v}+\frac{\beta^2\theta}{\rho^2 c_v}\right)\left(\frac{dx}{dt}\right)^2+\frac{k}{\tau_0\rho c_v}a_s^2\right]\frac{1}{\tau_0\rho c_v} \tag{3.15a}$$

or finally

$$\det M=\left[\left(\frac{dx}{dt}\right)^4-\left(\frac{dx}{dt}\right)^2(a_s^2+a_T^2\delta^2)+a_s^2a_T^2\right]\frac{1}{\tau_0\rho^2 c_v}$$

where:

$$a_T^2=\frac{k}{\tau_0\rho c_v},\qquad \delta^2=\frac{\beta^2\theta}{\rho^2 c_v}. \tag{3.15b}$$

After some calculations, the characteristic equation has the form:

$$\tau_0\rho^2 c_v\left(\frac{dx}{dt}\right)^4-\left(\tau_0\rho^2 c_v a_s^2+\tau_0\beta^2\theta+k\rho\right)\left(\frac{dx}{dt}\right)^2+k\rho a_s^2=0 \tag{3.16}$$

or

$$\left(\left(\frac{dx}{dt}\right)^2-a_s^2\right)\left(\left(\frac{dx}{dt}\right)^2-a_T^2\right)-\left(\frac{dx}{dt}\right)^2\delta^2=0 \tag{3.17}$$

where: $a_T^2 = \frac{k}{\tau_0 \rho c_v}$ – is the velocity of thermal wave; $\delta^2 = \frac{\beta^2 \theta}{\rho^2 c_v}$ – is the thermo-mechanical coupling coefficient

Remark 3.1.

— If $\tau_0 \to 0$ (cf. (3.15) we obtain the classical theory of thermoelasticity in (3.16)) so we have:

$$-k_\rho \left(\left(\frac{dx}{dt} \right)^2 - a_s^2 \right) = 0 \tag{3.18}$$

it means that we have only one characteristic direction $\left(\frac{dx}{dt}\right)^2 = a_s^2$ (describing the mechanical wave).

— If $\delta \to 0$ (cf. (3.17)) (i.e. thermo-mechanical coupling does not exist) from (3.17) we have

$$\left(\left(\frac{dx}{dt} \right)^2 - a_s^2 \right) \left(\left(\frac{dx}{dt} \right)^2 - a_T^2 \right) = 0$$

i.e. we obtain two characteristic directions

$$\left(\frac{dx}{dt} \right)^2 = a_s^2 \quad \text{or} \quad \left(\frac{dx}{dt} \right)^2 = a_T^2$$

where a_T – determines the thermal wave for wave equation without thermo-mechanical coupling.

Generally, equation (3.16) has two solutions:

$$\begin{aligned} \left(\frac{dx}{dt} \right)_1^2 &= \frac{1}{2}((a_s^2 + a_T^2 + \delta^2) + \sqrt{\Delta}) = c_1^2 > 0 \\ \left(\frac{dx}{dt} \right)_2^2 &= \frac{1}{2}((a_s^2 + a_T^2 + \delta^2) - \sqrt{\Delta}) = c_2^2 > 0 \end{aligned} \tag{3.18a}$$

where:

$$\Delta = (a_s^2 - a_T^2)^2 + 2\delta^2(a_s^2 + a_T^2) + \delta^4 > 0. \tag{3.19}$$

The values c_1^2 and c_2^2 denote the coupled velocity of the **mechanical--thermal wave** and the **thermo-mechanical wave**, respectively.

From formula (3.18a) it follows that

$$c_1^2 > c_2^2 \tag{3.20}$$

Moreover, in general case, c_1^2 and c_2^2 are the functions of the strain tensor ε and the temperature θ.
So, we appointed two characteristic directions given by the formulae

$$\begin{aligned} \left(\frac{dx}{dt}\right)^2 &= c_1^2(\varepsilon,\theta) \\ \left(\frac{dx}{dt}\right)^2 &= c_2^2(\varepsilon,\theta). \end{aligned} \tag{3.21}$$

Analyzing the properties of the system of equations (3.12) we get the solution in the form of the Riemann's wave for which the solution of equation (3.20) for the initial data equal to zero has the form:

$$\begin{gathered} x = c_1(\tau)(t-\tau) \\ \text{and} \\ x = c_2(\tau)(t-\tau) \end{gathered} \tag{3.22}$$

where τ – denotes the initial parameter of the starting of the line i.e. $t=\tau$, $x=0$ (cf. Fig. 3.1).

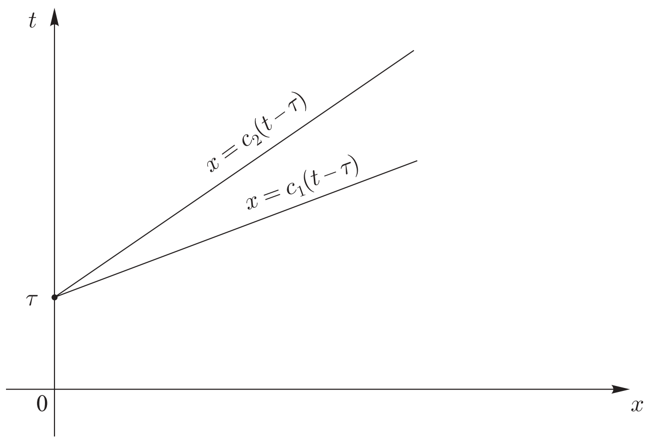

Fig. 3.1. The characteristic direction on the phase plane (t,x), $(t \geqslant 0)$, $(x \geqslant 0)$

The Riemann's wave can propagate up till the moment of the intersection of the two sucessive characteristic directions (generated in the chronological successions).

Now, we consider the situation described above on the phase plane (cf. Fig. 3.2).

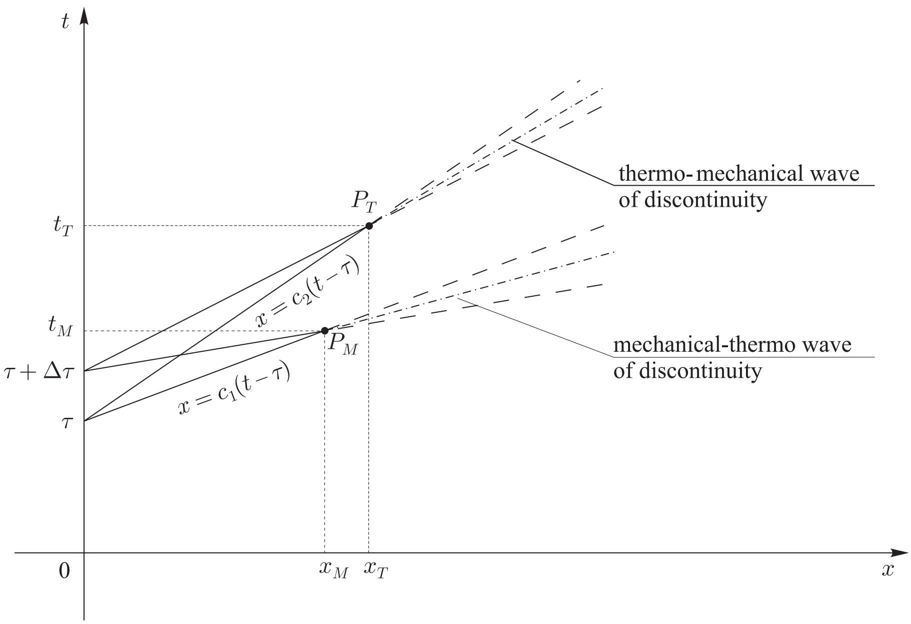

Fig. 3.2. Forming of the wave of discontinuity

Now, we investigate the existence of the **thermal shock wave** in the thermoelastic medium. At first, we lead out the beginning of the propagation of the shock wave in the thermoelastic medium.

We take into account the case of the wave formulation by the Riemann's wave propagating with the velocity c_1 the so-called mechanical-thermal wave (i.e. a wave without the thermo-mechanical coupling i.e. if $\delta \to 0$ becomes a mechanical wave).

In order to prove it, we write the equation of the two characteristics following each other

$$x = c_1(\tau)(t-\tau) \quad \text{and} \quad x = c_1(\tau + \Delta\tau)((t-\tau) - \Delta\tau). \tag{3.22a}$$

Using the extension into Taylor series, we get:

$$x = c_1(\tau)(t-\tau).$$

and

$$x = c_1(\tau)(t-\tau) + \frac{dc_1(\tau)}{d\tau}(t-\tau)\Delta\tau + \frac{1}{2}\frac{d^2c(\tau)}{d\tau^2}(t-\tau)(\Delta\tau)^2 + \cdots - c_1(\tau)\Delta\tau - \frac{dc_1(\tau)}{d\tau}(\Delta\tau)^2 + \cdots . \tag{3.23}$$

In the point of intersection we have*

$$t-\tau = \frac{c_1(\tau) + \frac{c_1(\tau)}{d\tau}(\Delta\tau) + o(\Delta\tau)^2}{\frac{dc_1(\tau)}{d\tau} + \frac{1}{2}\frac{d^2c_1(\tau)}{d\tau}\Delta\tau + o(\Delta\tau)^2)}. \tag{3.24}$$

From (3.24) we get:

$$\begin{aligned} t_{P_1} &= \tau + \lim_{\Delta\tau\to 0}\frac{c_1(\tau) + \frac{dc_1(\tau)}{d\tau}\Delta\tau + o(\Delta\tau)^2}{\frac{dc_1(\tau)}{d\tau} + \frac{1}{2}\frac{d^2c_1(\tau)}{d\tau^2}\Delta\tau + o(\Delta\tau)^2} \\ t_{P_1} &= \tau + \frac{c_1(\tau)}{\frac{dc_1(\tau)}{d\tau}} = t_M \end{aligned} \tag{3.25}$$

and

$$x_{P_1} = \frac{c_1^2(\tau)}{\frac{dc_1(\tau)}{d\tau}} = x_M .$$

From formulae (3.25) it follows that the necessary conditions of the forming of **the mechanical-thermal shock wave** is as follows

$$\frac{dc_1(\tau)}{d\tau} > 0. \tag{3.26}$$

Analysing in a similar way the thermo-mechanical wave (i.e. a wave which under the condition $\delta \to 0$ becomes a thermal wave with velocity c_2), we obtain

$$t_T = t_{P_2} = \tau + \frac{c_2(\tau)}{\frac{dc_2(\tau)}{d\tau}}, \qquad x_{P_2} = \frac{c_2^2(\tau)}{\frac{dc_2(\tau)}{d\tau}} = x_T. \tag{3.27}$$

From (3.27) it follows that the necessary condition to form a **thermo-mechanical shock wave** is as follows

$$\frac{dc_2(\tau)}{dt} > 0. \tag{3.28}$$

* $o(\Delta\tau)$ – denotes the Landau symbol.

Remark 3.2. It is worth to emphasize that the intersection of the mechanical-thermal wave and the thermo-mechanical wave appears only if $t_{P_3} = \tau$ and $x_{P_3} = 0$ i.e. then the shock wave doesn't form.
From the necessary conditions

$$\frac{dc_1(\tau)}{d\tau} > 0 \quad \text{or} \quad \frac{dc_2(\tau)}{d\tau} > 0 \tag{3.29}$$

which express the existance of the positive acceleration of the velocity of the propagation of thermal disturbance in the thermoelastic medium, we get the physical interpretation of the conditions of forming the wave with strong discontinuity in the nonlinear thermoelasticity theory.
In our consideration, below we base on the relation:

$$\frac{dc_i(\varepsilon(\tau), \theta(\tau))}{d\tau} = \frac{\partial c_i}{\partial \varepsilon}\frac{d\varepsilon}{d\tau} + \frac{\partial c_i}{\partial \theta}\frac{d\theta}{d\tau} > 0 \tag{3.30}$$

where $i = 1, 2$.
We start with the following lemma:

Lemma 3.1. *If the thermo-mechanical coupling parameter δ (cf. (3.15b)) formula satisfies the inequality*

$$\delta << 1 \tag{3.31}$$

then the velocities given by formula (3.18a) satisfy the relation

$$c_1^2 \approx a_s^2(1 + \delta_1^2) \quad \text{and} \quad c_2^2 \approx a_T^2(1 - \delta_1^2) \tag{3.32}$$

where:

$$\delta_1^2 = \frac{\delta^2}{a_s^2 - a_T^2}$$

and

$$a_s^2 = \frac{1}{\rho}\frac{d\sigma}{d\varepsilon}, \quad a_T^2 = \frac{1}{\tau}\kappa^2$$

$\kappa^2 = \frac{k}{\rho c_v}$ *– is the coefficient of the termal diffusion.*

Sketch of proof
From formulae (3.18a) it follows that:

$$c_i^2 = \frac{1}{2}((a_s^2 + a_T^2 + \delta^2) \pm \sqrt{(a_T^2 + a_s^2 + \delta^2)^2 - 4a_T^2 a_s^2}) \tag{3.33}$$

for $i = 1$ we have for "+" and for $i = 2$ we have "−" in the bracket.
We can represent Δ as follows

$$\begin{aligned}\Delta &= (a_T^2 + a_s^2 + \delta^2)^2 - 4a_T^2 a_s^2 \\ &= ((a_s - a_T)^2 + \delta^2)((a_s + a_T)^2 + \delta^2) \\ &= (a_s^2 - a_T)^2 (a_s + a_T)^2 + \delta^2[(a_s - a_T)^2 + (a_s + a_T)^2] \\ &+ \delta^4 = (a_s - a_T)^2 (a_s + a_T)^2 \left[1 + \frac{\delta^2 (a_s^2 + a_T^2)}{(a_s - a_T)^2 (a_s + a_T)^2}\right. \\ &\left. + \frac{\delta^4}{(a_s - a_T)^2 (a_s + a_T)^2}\right] \\ &= (a_s^2 - a_T^2)^2 \left[1 + \frac{\delta^2 (a_s^2 + a_T^2)}{(a_s^2 - a_T^2)^2} + \frac{\delta^4}{(a_s^2 - a_T^2)^2}\right].\end{aligned} \tag{3.34}$$

Putting the formulae (3.34) into (3.33) we get:

$$\begin{aligned}c_1^2 &\cong \frac{1}{2}\left[a_s^2 + a_T^2 + \delta^2 + a_s^2 - a_T^2 + \frac{\delta^2}{2} \frac{(a_s^2 + a_T^2)}{a_s^2 - a_T^2}\right. \\ &\left. + \frac{\delta^4}{(a_s^2 - a_T^2)}\right] = a_s^2 \left(1 + \delta^2 \frac{1}{a_s^2 - a_T^2} + \frac{\delta^4}{4a_s^2 (a_s^2 - a_T^2)}\right) \\ c_1^2 &\approx a_s^2 \left(1 + \delta^2 \frac{1}{a_s^2 - a_T^2}\right)\end{aligned}$$

and acting in a similar way, we obtain

$$c_2^2 \approx a_T^2 \left(1 - \delta^2 \frac{1}{a_s^2 - a_T^2}\right).$$

This ends the proof of theorem 3.1.
Now, we lead out the physical conditions which are necessary for the formation of a thermal shock wave

Theorem 3.1. *Let us assume that*

$$\frac{\partial c_1}{\partial \theta} \approx 0 \tag{3.35}$$

i.e. if we deal with a discontinuous wave with mechanical predomination then the necessary conditions for forming the discontinuous wave is as follows:

$$\gamma^2 \frac{d^2\sigma}{d\varepsilon^2} \frac{d\sigma}{dt} > 0 \tag{3.36}$$

where γ^2 is some coefficient.

Sketch of proof
In view of (3.32) we have

$$c_1^2 = \frac{1}{\rho}\frac{\partial\sigma}{\partial\varepsilon}(1+\delta_1^2) \tag{3.37}$$

Differentiating (3.37) with respect to τ we get:

$$\begin{aligned}\frac{d(c_1^2)}{d\tau} &= 2c_1\frac{dc_1}{d\tau} = \frac{1}{\rho}\left\{\frac{d}{d\tau}\left[\frac{\partial\sigma}{\partial\varepsilon}(1+\delta_1^2)\right]\right\}\\ &= \frac{1}{\rho}\left(\frac{\partial^2\sigma}{\partial\varepsilon^2}\frac{d\varepsilon}{d\tau}(1+\delta_1^2) + \frac{\partial^2\sigma}{\partial\theta\partial\varepsilon}\frac{d\theta}{d\tau}(1+\delta_1^2)\right)\\ &\quad + \frac{1}{\rho}\frac{\partial\sigma}{\partial\varepsilon}\left(\frac{\partial(\delta_1^2)}{\partial\varepsilon}\frac{d\varepsilon}{d\tau} + \frac{\partial(\delta_1^2)}{\partial\theta}\frac{d\theta}{d\tau}\right).\end{aligned} \tag{3.38}$$

In view of assumption (3.38) we have:

$$\frac{d(c_1^2)}{d\tau} \approx \frac{1}{\rho}\left(\frac{\partial^2\sigma}{d\varepsilon^2}\frac{d\varepsilon}{d\tau} + \frac{\partial^2\sigma}{\partial\theta\partial\varepsilon}\frac{d\theta}{d\tau}\right). \tag{3.39}$$

Taking into account that

$$\frac{d\sigma}{d\tau}(\varepsilon,\theta) \approx \frac{\partial\sigma}{\partial\varepsilon}\frac{d\varepsilon}{d\tau} = \rho a_s^2\frac{d\varepsilon}{d\tau} \tag{3.40}$$

and basing on assumption (3.35) finally we obtain

$$\frac{d\varepsilon}{d\tau} \approx \gamma^2\frac{d^2\sigma}{d\varepsilon^2}\frac{d\sigma}{d\tau}$$

Taking into account (3.29) we have

$$\frac{d^2\sigma}{d\varepsilon^2}\frac{d\sigma}{d\tau} > 0 \tag{3.41}$$

This ends the proof of Theorem 3.1.

Theorem 3.2. *Let us assume that*

$$\frac{\partial c_2}{\partial\varepsilon} \approx 0 \tag{3.42}$$

i.e. if we deal with the discontinuous wave with thermal predomination, then the necessary conditions for forming the thermal shock wave are as follows

$$\alpha^2 \frac{d\kappa}{d\theta}\frac{d\theta}{d\varepsilon} > 0 \tag{4.43}$$

where α^2 – is some parameter.

Sketch of proof
We have (cf. (3.32))

$$c_2^2 = \frac{1}{\tau_0}\kappa^2(1-\delta_1^2). \tag{3.44}$$

Differentiating (3.44) with respect to τ we obtain:

$$\begin{aligned}\frac{d(c_2^2)}{d\tau} &= 2c_2\frac{dc_2}{d\tau} = \frac{1}{\tau_0}\frac{d}{d\tau}[\kappa^2(1-\delta_1^2)]\\ &= \frac{1}{\tau_0}\left(\frac{\partial\kappa^2}{\partial\varepsilon}\frac{d\varepsilon}{d\tau}(1-\delta_1^2) + \kappa^2\left(\frac{\partial(-\delta_1^2)}{\partial\varepsilon}\right)\frac{\partial\varepsilon}{d\tau}\right)\\ &\quad + \frac{1}{\tau_0}\left(\frac{\partial\kappa^2}{\partial\theta}\frac{d\theta}{d\tau}(1-\delta_1^2)\right) + \kappa^2\frac{d}{\partial\theta}\left(\delta_1^2\right)\frac{d\theta}{d\tau}\Bigg).\end{aligned} \tag{3.45}$$

Taking into account (3.43) from it (3.45) (3.29) we get:

$$\frac{d(c_2^2)}{d\tau} \simeq c_2\frac{\partial\varepsilon}{d\tau} = \frac{1}{\tau_0}\frac{\partial\kappa^2}{\partial\theta}\frac{d\theta}{d\tau} = \frac{1}{\tau_0}\kappa\frac{\partial\kappa}{\partial\theta}\frac{d\theta}{d\tau} \tag{3.46}$$

so, in view of the above we obtain:

$$\alpha^2\frac{d\kappa}{\partial\theta}\frac{d\theta}{d\tau} > 0 \tag{3.47}$$

where $\alpha^2 = \frac{1}{\tau_0}\kappa$. This ends the proof of Theorem 3.2.

Remark 3.3. From theorem 3.2 (cf. formula (3.43)) it follows, that the thermal wave with discontinuous – so-called thermal shock wave appears, if

$$\frac{d\kappa}{d\theta}\frac{d\theta}{d\tau} > 0. \tag{3.48}$$

Since the diffusion coefficient $\kappa(\theta)$ increases with temperature, so we have $\frac{dk(\theta)}{d\theta} > 0$ and also $\frac{d\theta}{d\tau} > 0$.
It means the system is submitted to the increasing influence of temperature.

According to Einstein's theorem (cf. [1]), we have that

$$c_v = c_{v_0} \left(\frac{T_0}{\theta}\right)^2 \frac{e^{\frac{T_0}{\theta}}}{\left(e^{\frac{T_0}{\theta}} - 1\right)^2},$$

which supports our statements above.

Remark 3.4. From theorem (3.1) cf. formulae (3.30) it follows that in the case of mechanical wave with discontinuity – so-called mechanical shock wave the condition of their forming is as follows

$$\frac{d^2\sigma}{d\varepsilon^2}\frac{d\sigma}{d\tau} > 0. \tag{3.49}$$

Since there are materials with $\frac{d^2\sigma}{d\varepsilon^2} > 0$ and with $\frac{d^2\sigma}{d\varepsilon^2} < 0$ (cf. Fig. 3.3) we have two cases:

— $\frac{d^2\sigma}{d\varepsilon^2} > 0$ and $\frac{d\sigma}{dt} > 0$ – loading process
— $\frac{d^2\sigma}{d\varepsilon^2} < 0$ and $\frac{d\sigma}{dt} < 0$ unloading the process of stress increase

in which a loading shock wave and an unloading shock wave occur (cf. Fig.3.3).

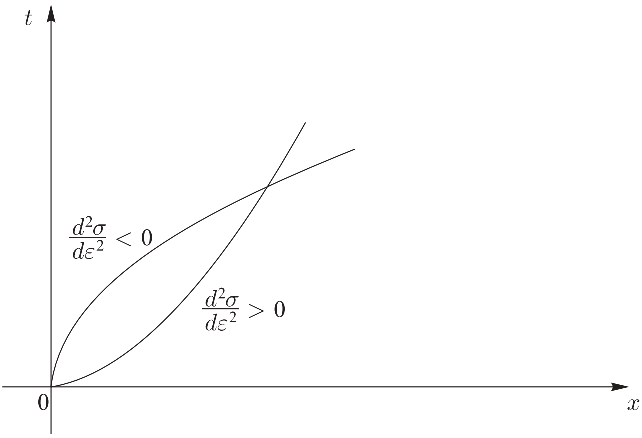

Fig. 3.3. Convex and noconvex depedning of the σ

Remark 3.5. In the linear case $\frac{d^2\sigma}{d\varepsilon^2} = 0$, all directions (for arbitrary τ) are parallel. In this case shock waves do not occur.
Putting $\tau = 0$ into (3.23) we get the domain on the phase plane in which Riemann's waves occur. In this case we can determine the solution moving the stress values along the characteristic (cf. Fig. 3.4).

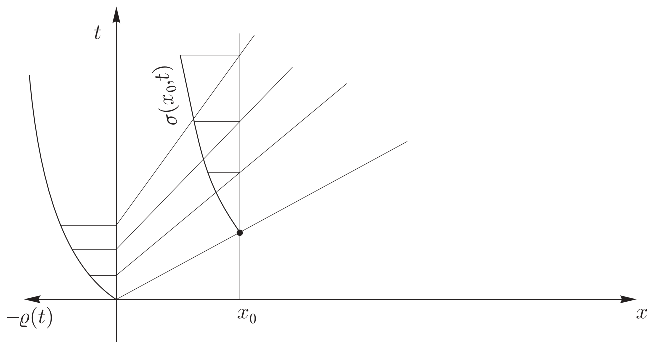

Fig. 3.4. Graphical solution for the Riemann's wave

Remark 3.6. If $\frac{d\sigma}{dt} \to +\infty$ (i.e. $\sigma(t)$ has the jump at $\tau = \tau_0 = 0$), then the beginning of the forming of the discontinuity is located in the point $x = 0$ (there is no domain of the Riemann's waves). This occurs in the case of the unloading process.

Remark 3.7. (Riemann's paradox) Riemann assumed that the shape of the shock wave is the envelope of the Riemann's waves (cf. Fig. 3.5a). But this is not true because in this case the equation of the balance of energy is not satisfied (cf. Fig. 3.5a and Fig. 3.5b).

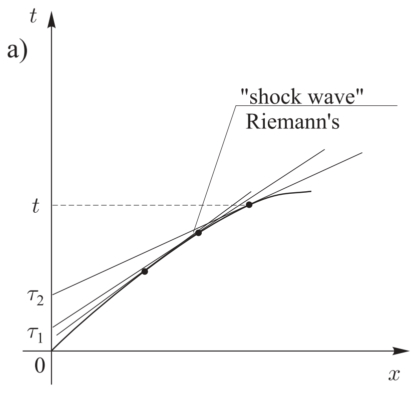

Fig. 3.5. a) $x = \phi_R(t)$ velocity of the wave $D_R = \frac{d\phi_R(t)}{dt} = c(\tau)$

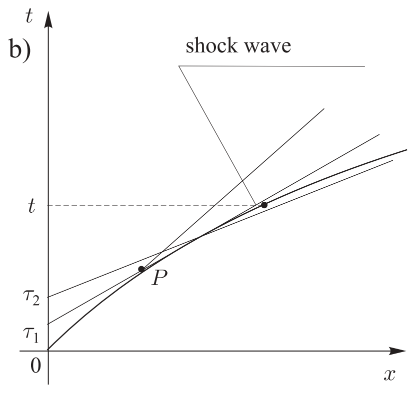

Fig. 3.5. b) $x = \phi(t)$ velocity of the wave $D = \frac{d\phi(t)}{dt}$, $c(\tau_1) < D < c(\tau_2)$

From our consideration if follows that the forming of the strong discontinuity:

— depends on the physical properties of the nonlinear thermoelastic medium expressed for example by the dependence of the stress tensor σ

on ε and conductivity coefficient k on i.e. $\sigma = \sigma(\varepsilon)$, $k = k(\theta)$ and other parameters (cf. (3.3), (3.9))
— depends on the smoothness of the functions describing the external influence on to thermoelastic medium given for example by $p_0(t)$, $\theta_0(t)$ for $x = 0$ (cf. (3.7))

4. Concluding remarks

In our paper we proved that in the nonlinear hyperbolic thermoelasticity medium two waves with strong discontinuity exist:

— the thermal shock wave
— the mechanical shock wave

We formulated and proved the necessary and sufficient mathematical and physical conditions for the forming of the thermal and mechanical shock waves. Additionally, we show that the intersection of the mechanical-thermal and the thermo-mechanical waves is impossible (since $c_1^2 > c_2^2$) so the shock wave cannot be formed by the influence of the thermal and mechanical wave (cf. Fig. 3.2).

Using the method presented in our paper, we can extend our consideration to the proof of the existence of thermal shock waves in nonlinear hyperbolic thermoelasticity theory with two relaxation times (for the equations cf. [3], [4], [5], [8], [12]). It will be done in the future paper.

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Powstawanie fal silnych nieciągłości w nieliniowej hiperbolicznej termosprężystości

Streszczenie. W pracy przedstawiono teorię fal uderzeniowych w materiałach sprężystych przewodzących ciepło. Rozważano formowanie się fal silnych nieciągłości tzw. termicznych fal uderzeniowych w nieliniowej hiperbolicznej termosprężystości, opisujących propagację ciepła ze skończoną prędkością. W pracy wyprowadzono i udowodniono matematyczne i fizyczne warunki konieczne i wystarczające do powstania termicznych fal uderzeniowych.

Słowa kluczowe: nieliniowa hiperboliczna termosprężystość, termiczne fale uderzeniowe, hiperboliczne równanie ciepła, mechaniczne fale uderzeniowe