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Dynamic response of a spherical ballistic casing loaded explosively to current movement of boundary conditions limiting surfaces

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Abstract. Dynamic fields of: displacements, strains, and stresses in a spherical thick-walled ballistic casing loaded internally by the pressure of detonation products were studied. The casing material was assumed to be homogenous, isotropic, and elastically incompressible. It turns out that this kind of casing loaded as mentioned above oscillates radially with specific angular frequency, alike the mechanical system with one degree of freedom. Two mathematical models of the studied problem were considered: the linear model, in which boundary conditions were applied to the initial position of limiting surfaces of the casing (Lagrangian coordinates), and the non-linear model, taking into account the movement of casing limiting surfaces in boundary conditions (Eulerian coordinates). For the linear model, the analytic closed form solution to the problem was obtained. In case of very small elastic strains, less than 1%, the results obtained for this model are convergent to the non-linear solution. Only in this range of strains , it can be used in engineering calculations. For larger strains, the errors resulting from the linearization of the problem are of the order of dozen and more per cent. The linearization of the problem distorts quantitative and qualitative view of casing dynamic parameters.
Keywords: dynamics, ballistic casing, explosive load, incompressible material

1. Introduction

Spatially one-dimensional initial-boundary value problems connected with an explosive loading of various media and constructions of cylindrical and spherical symmetry have been studied by many researchers [1-10]. The dynamic loading of thick-walled cylindrical pipes and spherical reservoirs (ballistic casings) by the internal pressure belongs, amongst other things, to this group of issues. The autofrettage is used in the process of production of gun barrels and high-pressure reservoirs. It is a self-strengthening of objects mentioned above with the use of high pressure (gaseous mixture explosion) causing the plastic strain in their inner layers. After such an overstraining, the nominal pressure applied inside these objects causes only elastic strains.

From selected technical issues quoted above it results the conclusion that the dynamics of ballistic casings loaded explosively is an important problem deserving detailed theoretical studies from different points of view. In the general approach, they are mathematically complicated initial-boundary value problems. To overcome mathematical difficulties, the reasonable compromise with the physics of phenomenon is practiced. Therefore theoretical predictions should be taken with reserve and experimentally validated.

In order to obtain an analytic solution to the specific initial-boundary value problems, the model simplifications, based on the physical premises of this issue, are assumed. For example, basing on linear elasticity theory of small displacements and strains, the movement of limiting surfaces in boundary conditions during the process of deformation of given system is neglected [1, 10]. The boundary conditions are formulated for the initial position of limiting surfaces (boundaries). This assumption is made arbitrarily without limitation of its application. The available literature lacks any quantitative estimation of the influence of this simplification on the dynamic characteristics of the studied objects.

This paper is an attempt of quantitative determination of error caused by this simplification. The problem was considered with the use of example of the dynamics of a thick-walled spherical ballistic casing loaded by the internal pressure of detonation products of high explosive.

2. Formulation of the problem

Dynamic states of mechanical characteristics in a metal thick-walled spherical casing loaded internally by the pressure of detonation products of gaseous explosive mixture will be determined.

The material of casing is assumed to be homogenous, isotropic, and elastically incompressible. The casing movement is characterized by a spherical symmetry and it is determined by Hook's law. Let a and b denote initial radii of the casing, internal and external, respectively. A spherical system of the coordinates r, φ, θ is used. Therefore the states of stress and strain in the casing material are represented by the following principal components of stress and strain tensors:

σ_r — radial stress,

$\sigma_\varphi = \sigma_\theta$ — tangential stresses,

ε_r — radial strain,

$\varepsilon_\varphi = \varepsilon_\theta$ — tangential strains.

From the spherical symmetry of the problem, it results that it is spatially one-dimensional. Therefore the parameters characterizing dynamic state of the casing depend on one spatial coordinate, *r* and the time, *t*. The theory of small strains is used, therefore the arguments *r*, *t* can be treated as Lagrangian or Eulerian coordinates. In the following considerations in Eulerian system, for the distinguishing purpose, the spatial coordinate is denoted with the letter *R*.

The problem is solved on the basis of the linear elasticity theory, according to which we can write the following relations [3, 9]:

$$\varepsilon_r(r,t) = \frac{\partial u(r,t)}{\partial r}, \qquad \varepsilon_\varphi(r,t) = \varepsilon_\theta(r,t) = \frac{u(r,t)}{r}, \tag{2.1}$$

$$\sigma_\varphi(r,t) - \sigma_r(r,t) = 2\mu\left[\varepsilon_\varphi(r,t) - \varepsilon_r(r,t)\right], \tag{2.2}$$

where u denotes the radial displacement of casing infinitesimal element and μ is the shear modulus:

$$\mu = \frac{E}{2(1+\nu)}. \tag{2.3}$$

In turn, the symbols E and ν denote Young's modulus and Poisson's ratio, respectively.

From the mass conservation law, written for the casing infinitesimal element in the Lagrangian coordinates *r*, *t* in spherical symmetry we have:

$$(r+u)^2\left(1+\frac{\partial u}{\partial r}\right) = \frac{\rho_0}{\rho} r^2, \tag{2.4}$$

where the symbols ρ_0 and ρ denote the initial and current density of the casing material.

For metals, at pressure values of the order of a few thousands MPa it can be assumed that $\rho \approx \rho_0 = \text{const}$. An error caused by this simplification is of the order of a per cent fraction [10]. Taking this assumption into account, for small strains ($\varepsilon_\varphi \varepsilon_r = \frac{u}{r}\frac{\partial u}{\partial r} \approx 0$, $\varepsilon_\varphi^2 = \left(\frac{u}{r}\right)^2 \approx 0$ and $\varepsilon_\varphi^2 \varepsilon_r = \left(\frac{u}{r}\right)^2 \frac{\partial u}{\partial r} \approx 0$), Eq. (2.4) can be reduced to the following form:

$$\frac{\partial u}{\partial r} + 2\frac{u}{r} = 0. \tag{2.5}$$

The equation of motion of the casing infinitesimal element written by means of the Lagrangian coordinates r, t has the form:

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = \left(1+\frac{u}{r}\right)^2 \frac{\partial \sigma_r}{\partial r} + 2\left(1+\frac{u}{r}\right)\left(1+\frac{\partial u}{\partial r}\right)\frac{\sigma_r - \sigma_\varphi}{r}. \tag{2.6}$$

For small displacements and strains, determined by formula (2.1), taking into account that $\varepsilon_\varphi = (u/r) << 1$ and $\varepsilon_r = (\partial u/\partial r) << 1$, Eq. (2.6) can be simplified to the following form:

$$\frac{\partial \sigma_r}{\partial r} + 2\frac{\sigma_r - \sigma_\varphi}{r} = \rho_0 \frac{\partial^2 u}{\partial t^2}. \tag{2.7}$$

The identical form of the equation is obtained on the basis of the linear elasticity theory in the Eulerian coordinates, R, t, namely:

$$\frac{\partial \sigma_r}{\partial R} + 2\frac{\sigma_r - \sigma_\varphi}{R} = \rho_0 \frac{\partial^2 u}{\partial t^2}. \tag{2.8}$$

Now, we will deal with the modelling of motion of gaseous detonation products of explosive mixture contained in the casing.

We assume that the initiation of explosive mixture detonation starts in the centre of the casing. After the end of detonation process, the explosion products do not diffuse into a metal and, like during underwater explosions [2], they create a gas cavity of the initial radius $R_a(0) = a$. At the moment of hitting of the detonation wave on the inner surface of casing ($r = a$) it undergoes refraction and two divergent shock waves are created. The first wave moves towards the outside across the casing wall, while the other one concentrically propagates in the detonation products towards the centre of the gas cavity. In the following part of this process, the successive shock waves' reflections from the centre of the gas cavity and from the casing surfaces take place. Because of high velocities of shock waves propagation (of the order of a few thousands m/s) and finite casing dimensions, the nonstationary process of waves refraction is of short duration. In this period, the considerable amount of energy is irreversibly transferred from the cavity to the casing wall. For example, during underwater explosions about 2/3 of detonation products energy is irretrievably transferred to the surroundings [1, 2]. This energy is transformed into heat and mechanical work. As a result of this phenomenon, the pressure in the gas cavity decreases intensively, its oscillations decay, and the whole detonation products expand approximately adiabatically. The nonstationary shock wave refraction process of short duration (of the order of a few μs) in the cavity and in the casing wall is neglected. During this process, the casing material is strengthened (autofrettage).

After decaying of oscillations of the pressure in the cavity, the thermodynamic characteristics of detonation products can be approximated by the polytropic gas model [1, 8]. After these simplifications, from the mass conservation law for the gas in the cavity and Poisson's adiabate it results that the pressure acting on the inner casing surface after the end of explosive mixture detonation is determined by the following formula [6, 8]:

$$p\left[R_a(t)\right] = p_0\left[\frac{a}{R_a(t)}\right]^{3k}, \quad p_0 = np_J, \quad n < 1, \tag{2.9}$$

where $R_a(t)$ is the current inner radius of casing (Eulerian coordinate), i.e.:

$$R_a(t) = a + u(a,t), \tag{2.10}$$

and the n coefficient characterizes the amount of internal energy transferred from the gas cavity to the casing wall during shock waves refraction. The symbols k and p_J denote the Poisson's polytropic curve exponent and the pressure of detonation products in Chapman-Jouget's point, respectively.

To close the equation system of the problem, the initial and the boundary conditions for the considered system should be determined.

Linearized system of Eqs. (2.5) and (2.8) was solved with the following boundary conditions:

$$\sigma_r\left[R_a(t),t\right] = -p_0\left[\frac{a}{R_a(t)}\right]^{3k} \quad \text{for} \quad R = R_a(t), \tag{2.11}$$

$$\sigma_r\left[R_b(t),t\right] \equiv 0 \quad \text{for} \quad R = R_b(t), \tag{2.12}$$

and homogeneous initial conditions:

$$u(r,0) \equiv 0, \qquad v(r,0) = \left.\frac{\partial u}{\partial t}\right|_{t=0} \equiv 0, \tag{2.13}$$

where

$$R_b(t) = b + u(b,t). \tag{2.14}$$

The solution to the problem formulated above is presented in the next section.

3. Solution to the initial-boundary value problem

The general integral of Eq. (2.5) has the following form:

$$u(r,t)=\frac{C(t)}{r^2}, \tag{3.1}$$

where $C(t)$ denotes the arbitrary, continuous and twice differentiable function of the time t.

Displacements of the limiting surfaces of the casing can be expressed using their Eulerian coordinates $R_a(t)$ and $R_b(t)$, namely:

$$u(a,t)=R_a(t)-a, \qquad u(b,t)=R_b(t)-b. \tag{3.2}$$

Next, from Eqs. (3.1) and (3.2) it follows that $C(t)$ function is also uniquely determined by the coordinate $R_a(t)$ or $R_b(t)$, i.e.:

$$C(t)=a^2\left[R_a(t)-a\right] \quad \text{or} \quad C(t)=b^2\left[R_b(t)-b\right]. \tag{3.3}$$

This expressions yield:

$$R_b(t)=b+\frac{a^2\left[R_a(t)-a\right]}{b^2}. \tag{3.4}$$

Finally, from Eqs. (3.1) and (3.2) we obtain:

$$u(r,t)=\frac{a^2\left[R_a(t)-a\right]}{r^2}\approx\frac{a^2\left[R_a(t)-a\right]}{R^2}, \tag{3.5}$$

$$\frac{\partial^2 u(r,t)}{\partial t^2}=\frac{a^2}{r^2}\ddot{R}_a(t)\approx\frac{a^2}{R^2}\ddot{R}_a(t); \qquad \ddot{R}_a(t)=\frac{\mathrm{d}^2 R_a(t)}{\mathrm{d}t^2}. \tag{3.6}$$

After using relation (3.5), expressions (2.1) and (2.2) can be written as follows:

$$\begin{aligned} &\varepsilon_r(r,t)=-2\frac{a^2\left[R_a(t)-a\right]}{r^3}, \\ &\varepsilon_\varphi(r,t)=\varepsilon_\theta(r,t)=\frac{a^2\left[R_a(t)-a\right]}{r^3}=-\frac{1}{2}\varepsilon_r(r,t), \end{aligned} \tag{3.7}$$

$$\sigma_\varphi(r,t)-\sigma_r(r,t)=6\mu\frac{a^2\left[R_a(t)-a\right]}{r^3}\approx 6\mu\frac{a^2\left[R_a(t)-a\right]}{R^3}\approx 6\mu\varepsilon_\varphi(r,t). \tag{3.8}$$

After substitution of relations (3.6) and (3.8) into Eq. (2.8) we have:

$$\frac{\partial \sigma_r}{\partial R} = 12\mu \frac{a^2\left[R_a(t)-a\right]}{R^4} + \rho_0 \frac{a^2}{R^2}\ddot{R}_a(t),$$

and after integration in respect to R we obtain:

$$\sigma_r(R,t) = -4\mu \frac{a^2\left[R_a(t)-a\right]}{R^3} - \rho_0 \frac{a^2}{R}\ddot{R}_a(t) + A(t), \tag{3.9}$$

where $A(t)$ is the arbitrary time-dependent function.

From boundary condition (2.12) and expression (3.9) it follows that:

$$A(t) = 4\mu \frac{a^2\left[R_a(t)-a\right]}{R_b^3(t)} + \rho_0 \frac{a^2}{R_b(t)}\ddot{R}_a(t),$$

and taking relation (3.4) into account we have:

$$A(t) = 4\mu \frac{a^2\left[R_a(t)-a\right]}{\left\{b + \frac{a^2\left[R_a(t)-a\right]}{b^2}\right\}^3} + \frac{\rho_0 a^2}{b + \frac{a^2\left[R_a(t)-a\right]}{b^2}}\ddot{R}_a(t). \tag{3.10}$$

From relations (3.9) and (3.10) it results that the stress can be determined by the following formula:

$$\begin{aligned}\sigma_r(R,t) = &-\frac{\rho_0 a^2}{R}\frac{b^3 + a^2\left[R_a(t)-a\right] - b^2 R}{b^3 + a^2\left[R_a(t)-a\right]}\ddot{R}_a(t) + \\ &-4\mu \frac{a^2\left[R_a(t)-a\right]}{R^3}\frac{\left\{b^3 + a^2\left[R_a(t)-a\right]\right\}^3 - b^6 R^3}{\left\{b^3 + a^2\left[R_a(t)-a\right]\right\}^3}.\end{aligned} \tag{3.11}$$

In turn, through the substitution of expression (3.11) into boundary condition (2.11) and transformations we obtain the non-linear differential equation of the second order which the function $R_a(t)$ should satisfy, namely:

$$\ddot{R}_a(t) = \frac{R_a(t)}{\rho_0 a^2} \frac{b^3 + a^2\left[R_a(t) - a\right]}{b^3 + a^2\left[R_a(t) - a\right] - b^2 R_a(t)} \left\{ p_0 \left[\frac{a}{R_a(t)}\right]^{3k} + \right.$$
$$\left. -4\mu \frac{a^2\left[R_a(t) - a\right]}{R_a^3(t)} \frac{\left\{b^3 + a^2\left[R_a(t) - a\right]\right\}^3 - b^6 R_a^3(t)}{\left\{b^3 + a^2\left[R_a(t) - a\right]\right\}^3} \right\}. \tag{3.12}$$

It seems that for small displacements and strains of the casing elements this equation can be linearized. In fact, taking into account that $(1 + u/a) \approx 1$, because $(u/a) << 1$, after this simplification from Eq. (3.12) we obtain:

$$\ddot{R}_a(t) + 4\frac{\mu}{\rho_0}\frac{b^2 + ab + a^2}{a^2 b^2} R_a(t) - 4\frac{\mu}{\rho_0}\frac{b^2 + ab + a^2}{ab^2} - \frac{p_0}{\rho_0}\frac{b}{a\,(b-a)} \approx 0. \tag{3.13}$$

The function $R_a(t)$ characterizing the oscillations of the inner surface of the casing, according to Eq. (2.14) should satisfy the following initial conditions:

$$R_a(0) = a\,, \quad \dot{R}_0(0) = 0. \tag{3.14}$$

In order to simplify the further analysis of the problem, the following dimensionless quantities were introduced:

$$\left.\begin{aligned} &\xi = \frac{r}{a}, \qquad \eta = \frac{t}{T_0}, \qquad \xi_a = \frac{R_a}{a}, \qquad \hat{a} = \frac{b}{a}, \qquad U = \frac{u}{a}, \\ &\xi_R = \frac{R}{a} = \frac{r}{a} + \frac{u}{a} = \xi + U, \qquad \bar{\omega}_0 = \frac{\omega_0}{(c/a)} = \frac{2\sqrt{\beta^2 + \beta + 1}}{\beta}, \\ &S_r = \frac{\sigma_r}{p_0}, \qquad S_\varphi = \frac{\sigma_\varphi}{p_0}, \qquad S_z = S_\varphi - S_r, \qquad P = \frac{p_0}{\mu}, \end{aligned}\right\} \tag{3.15}$$

as well as additional parameters:

$$\begin{aligned} &c = \sqrt{\frac{\mu}{\rho_0}}, \qquad \omega_0^2 = 4\frac{\mu}{\rho_0}\frac{b^2 + ab + a^2}{a^2 b^2} = 4\left(\frac{c}{a}\right)^2 \frac{\beta^2 + \beta + 1}{\beta^2}, \\ &\omega_0 = \frac{2\sqrt{\beta^2 + \beta + 1}}{\beta}\frac{c}{a} = \frac{2\pi}{T_0}, \end{aligned} \tag{3.16}$$

where the symbols c, ω_0, and T_0 denote the propagation velocity of shear wave in the casing material, the angular frequency and period of the casing free vibrations, respectively.

Equations (3.12) and (3.13) as well as initial conditions (3.14) expressed correspondingly by dimensionless quantities (3.15) and after using Eq. (3.16) have the following form:

$$\ddot{\xi}_a(\eta) = \pi^2 \frac{\beta^2}{\beta^2+\beta+1} \frac{\xi_a\left(\beta^3-1+\xi_a\right)}{\beta^3-1-\left(\beta^2-1\right)\xi_a} \left\{P\xi_a^{-3k} + \right. \\ \left. -4\frac{\xi_a-1}{\xi_a^3} \frac{\left\{\beta^3-1+\xi_a\right\}^3-\beta^6\xi_a^3}{\left\{\beta^3-1+\xi_a\right\}^3}\right\}, \tag{3.17}$$

$$\ddot{\xi}_a(\eta) = (2\pi)^2\left(-\xi_a + \frac{\beta^3}{4\left(\beta^3-1\right)}P + 1\right) = f\left(\xi_a\right), \tag{3.18}$$

$$\xi_a(0) = 1, \qquad \dot{\xi}_a(0) = 0, \tag{3.19}$$

where:

$$f\left(\xi_a\right) = (2\pi)^2\left(1 + \frac{\beta^3}{4\left(\beta^3-1\right)}P - \xi_a\right). \tag{3.20}$$

The analysis of the relations derived so far leads to the conclusion that all mechanical characteristics of the casing after explosive loading are determined by analytic formulae (3.5), (3.7), (3.8), (3.11), and (3.12), where Eulerian coordinate of the inner surface of the casing, $R_a(t)$, is the unknown function. It is uniquely derived from Eq. (3.12) and, in linearized form, from relation (3.13) and initial conditions (3.14). Equation (3.12) is non-linear and in a general case it is integrated numerically, e.g. with the use of Runge-Kutta method whereas, Eq. (3.13) is solved analytically in closed form in the next section.

4. Analytic solution to the problem for the linearized equation

From the analysis of Eq. (3.13) and (3.18) as well as boundary conditions (3.14) and (3.19) it results that they constitute the mathematical model for the casing loaded by the internal surge pressure p_0 which is constant during all motion of the casing. The problem simplified in this way can be solved analytically. In order to do this we transform Eq. (3.18) into the form:

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left[\dot{\xi}_a^2(\eta)\right]=2\dot{\xi}_a(\eta)f(\xi_a),$$

which after integration and considering the initial conditions yields:

$$\dot{\xi}_a^2(\eta)=2\int_1^{\xi_a} f(x)\mathrm{d}x=$$

$$=2(2\pi)^2\int_1^{\xi_a}\left[-x+\frac{\beta^3}{4(\beta^3-1)}P+1\right]\mathrm{d}x=(2\pi)^2\left\{-x^2+2\left[\frac{\beta^3}{4(\beta^3-1)}P+1\right]x\right\}_1^{\xi_a}$$

or

$$\dot{\xi}_a(\eta)=\frac{\mathrm{d}\xi_a}{\mathrm{d}\eta}=2\pi\left\{-\xi_a^2+2\left[\frac{\beta^3}{4(\beta^3-1)}P+1\right]\xi_a-\left[\frac{\beta^3}{2(\beta^3-1)}P+1\right]\right\}^{\frac{1}{2}}.$$

After separation of variables in the above-mentioned expression and integration we have:

$$2\pi\eta=\text{arc cos}\frac{-2\xi_a+2\left[\frac{\beta^3}{4(\beta^3-1)}P+1\right]}{2\left[\frac{\beta^3}{4(\beta^3-1)}P\right]}$$

or

$$\xi_a(\eta)=1+\frac{\beta^3}{4(\beta^3-1)}P(1-\cos 2\pi\eta). \tag{4.1}$$

The analytic shape of $\xi_a(\eta)$ function allows us to determine, in the closed analytic form, all mechanical characteristics of the casing loaded by the internal surge pressure p_0, namely:

$$U(\xi,\eta)=\frac{\xi_a-1}{\xi^2}=\frac{\beta^3}{4(\beta^3-1)}P\frac{1-\cos 2\pi\eta}{\xi^2}, \tag{4.2}$$

$$\varepsilon_\varphi(\xi,\eta)=-\frac{1}{2}\varepsilon_r(\xi,\eta)=\frac{\beta^3}{4(\beta^3-1)}P\frac{1-\cos 2\pi\eta}{\xi^3}, \tag{4.3}$$

$$S_r(\xi,\eta) = -\frac{1}{\beta^3-1}\left[\left(\frac{\beta}{\xi}\right)^3 - 1\right] + A_r(\xi)\cos 2\pi\eta,$$
$$A_r(\xi) = \frac{1}{\beta^3-1}\left[\left(\frac{\beta}{\xi}\right)^3 - \left(\beta^2+\beta+1\right)\left(\frac{\beta}{\xi}-1\right) - 1\right], \tag{4.4}$$

$$S_\varphi(\xi,\eta) = \frac{1}{2\left(\beta^3-1\right)}\left[2 + \left(\frac{\beta}{\xi}\right)^3\right] - A_\varphi(\xi)\cos 2\pi\eta, \tag{4.5}$$
$$A_\varphi(\xi) = \frac{1}{2\left(\beta^3-1\right)}\left[\left(\frac{\beta}{\xi}\right)^3 + 2\left(\beta^2+\beta+1\right)\left(\frac{\beta}{\xi}-1\right) + 2\right],$$

$$S_z(\xi,\eta) = S_\varphi(\xi,\eta) - S_r(\xi,\eta) = \frac{3}{2}\frac{\beta^3}{\beta^3-1}\frac{1-\cos 2\pi\eta}{\xi^3} = \frac{6}{P}\frac{U(\xi,\eta)}{\xi} \tag{4.6}$$

In the case when the pressure p_0 acts statically on the casing inner surface, we obtain:

$$U_s(\xi) = \frac{\beta^3}{4\left(\beta^3-1\right)}P\frac{1}{\xi^2}, \quad \varepsilon_{\varphi s}(\xi) = -\frac{1}{2}\varepsilon_{rs}(\xi) = \frac{U_s(\xi)}{\xi}, \tag{4.7}$$

$$S_{rs}(\xi) = -\frac{1}{\beta^3-1}\left[\left(\frac{\beta}{\xi}\right)^3 - 1\right], \quad S_{\varphi s}(\xi) = \frac{1}{2\left(\beta^3-1\right)}\left[2 + \left(\frac{\beta}{\xi}\right)^3\right], \tag{4.8}$$

$$S_{zs}(\xi) = S_{\varphi s}(\xi) - S_{rs}(\xi) = \frac{3}{2}\frac{\beta^3}{\beta^3-1}\frac{1}{\xi^3}. \tag{4.9}$$

From the introductory analysis of the formulae quoted above, the general conclusion follows that the studied casing loaded by the internal surge pressure p_0 = const responses like the mechanical system with one degree of freedom which oscillates radially with the angular frequency ω_0 determined by formula (3.16). As it can be seen, this frequency varies in direct proportion to the shear wave velocity in the casing material and in inverse proportion to the inner radius of the casing. Moreover, the quantity ω_0 decreases with the increase in the casing wall thickness (β parameter — Fig. 1), what is an obvious fact, taking into account an increase in casing mass.

The dynamic state of mechanical parameters of the casing varies in course of time periodically around their static values. Analogously to the mechanical system with one degree of freedom, the dynamic coefficient of inner surge load of the casing, according to Eqs. (4.2) and $(4.7)_1$, is determined by the expression:

$$\Psi(\eta) = \frac{U(\xi,\eta)}{U_s(\xi)} = 1 - \cos 2\pi\eta.$$

As it can be seen, the maximum value of Ψ coefficient is 2.

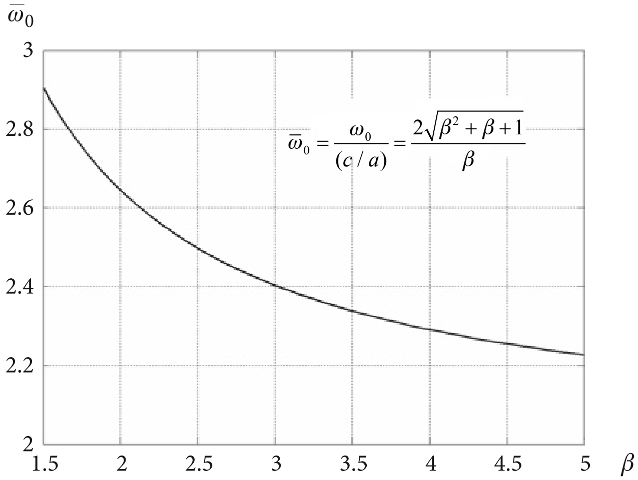

Fig. 1. Variation of relative angular frequency of the casing free vibrations versus β parameter

It follows directly from expressions (4.2)-(4.6) that the maximum absolute values of dynamic characteristics of the casing occur in the middle of the vibration period ($\eta = 0.5$) on the casing inner surface, i.e., for $\xi = 1$ and with the increase of ξ coordinate they intensively decrease. The displacement decreases in inverse proportion to ξ^2 and the remaining quantities — approximately to ξ^3. The reason of these variations is a phenomenon of spatial divergence of studied quantities.

The spatial graphs of the functions $U(\xi,\eta)/P$, $S_r(\xi,\eta)$, $S_\varphi(\xi,\eta)$ and $S_z(\xi,\eta)$ are depicted in Figs. 2-5. Please note some characteristic features of these quantities.

From the analysis of formulae (4.4) and graphs shown in Fig. 3 it results the conclusion that the function $S_r(\xi, \eta)$ can change its sign from negative to positive (radial tension of casing parts). This phenomenon occurs if β parameter characterizing the wall thickness of the casing satisfies the inequality:

$$\beta^4 + 2\beta^3 - 5\beta^2 - 6\beta + 5 > 0. \tag{4.11}$$

The minimum β value satisfying this inequality is 1.79. Therefore in the casing for which $\beta = (b/a) < 1.79$, the radial tensile stress will not appear.

The maximum value of the function $S_r(\xi, 0.5)$ is reached in the section:

$$\xi = \xi_e(\beta) = \sqrt{\frac{6\beta^2}{\beta^2 + \beta + 1}}, \qquad \beta > 1.79 \tag{4.12}$$

and amounts:

$$S_{r\max} = S_r(\xi_e; 0.5) = \frac{1}{\beta - 1}\left(\frac{\beta}{\xi_e} - 1\right) - \frac{2}{\beta^3 - 1}\left[\left(\frac{\beta}{\xi_e}\right)^3 - 1\right]. \tag{4.13}$$

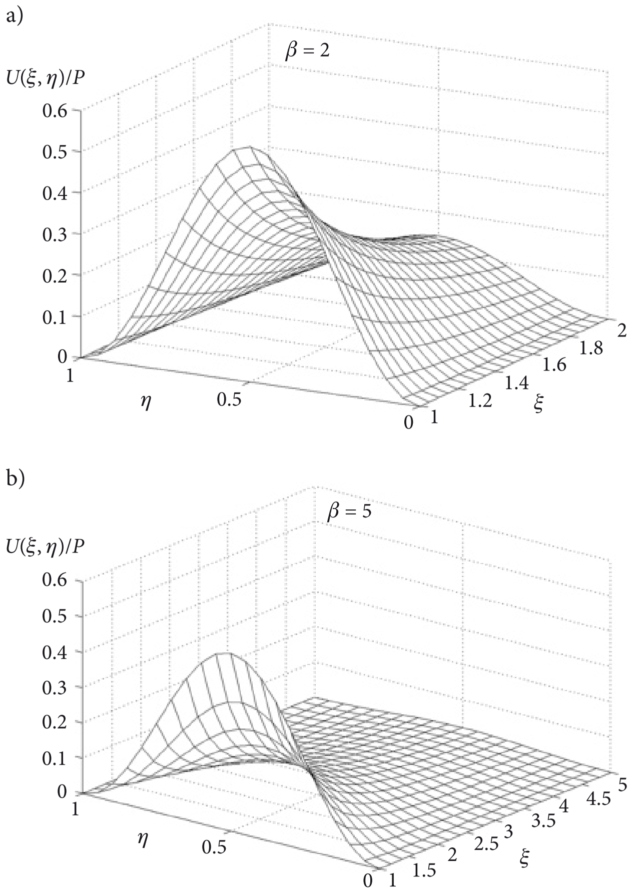

Fig. 2. Spatial graphs of the function $U(\xi, \eta)/P$ for $\beta = 2$ and $\beta = 5$

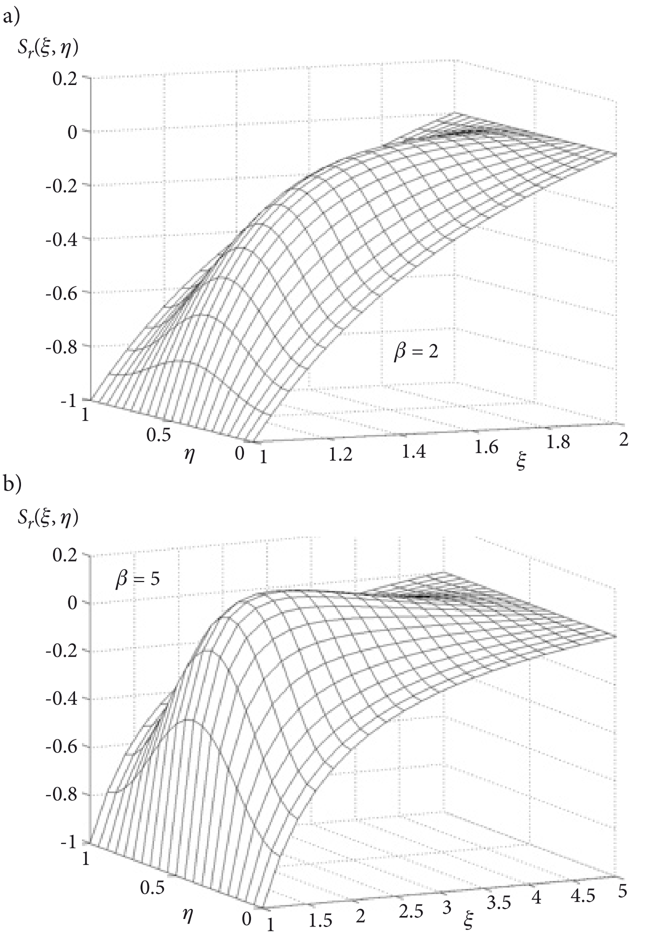

Fig. 3. Spatial graphs of the function $S_r(\xi, \eta)$ for $\beta = 2$ and $\beta = 5$

The change of sign of the function $S_r(\xi, \eta)$ is the result of interaction of the inertial forces on the free surface of the casing. In case of brittle materials, the tensile radial stresses can cause a local fracture in the casing.

It results directly from graphs depicted in Fig. 4 and the analysis of formulae (4.5) that the function $S_\varphi(\xi, \eta)$ changes its sign twice in the time interval $0 \le t \le T_0 (0 \le \eta \le 1)$. This phenomenon occurs irrespective of the parameter β value in every section of casing $\xi < \beta$. Let us notice here that the casing inner radius during vibrations does not decrease below the initial value, i.e., $R_a(\eta) > a$ (Fig. 2).

At the initial instant function, $S_\varphi(\xi, \eta)$ satisfies the inequality:

$$S_\varphi(\xi, \eta) = S_r(\xi, \eta) = -\frac{\beta^2 + \beta + 1}{\beta^3 - 1}\left(\frac{\beta}{\xi} - 1\right) \le 0 \tag{4.14}$$

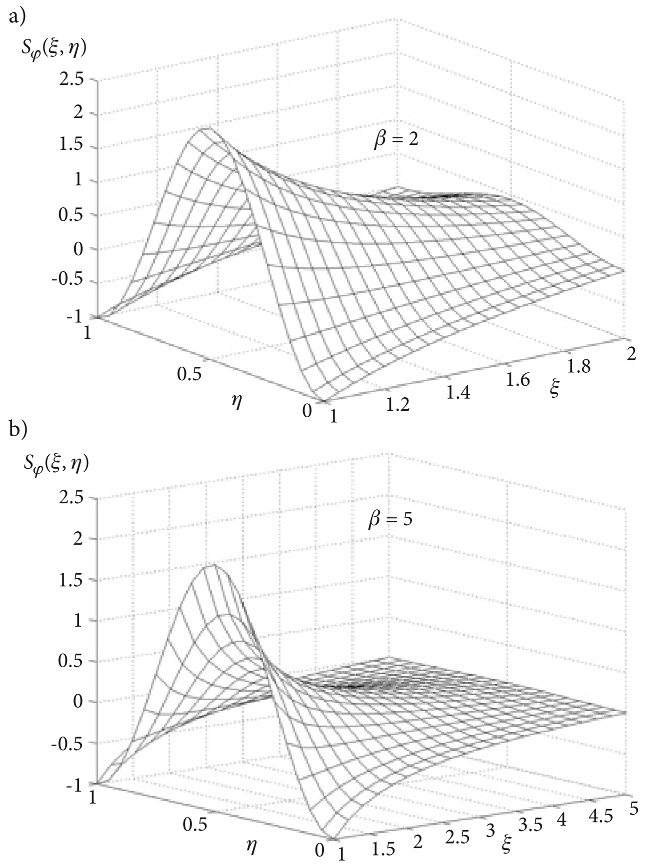

Fig. 4. Spatial graphs of the function $S_\varphi(\xi, \eta)$ for $\beta = 2$ and $\beta = 5$

Then, during the movement of the casing wall, the function $S_\varphi(\xi, \eta)$ increases and at a determined value of η changes its sign to a positive one. The function $S_\varphi(\xi, \eta)$ reaches its maximum positive value in the middle of the vibration period ($\eta = 0.5$) for $\xi = 1$ which amounts:

$$S_{\varphi \max} = S_\varphi\left(1\,;0.5\right) = \frac{2\beta^3 + 1}{\beta^3 - 1}. \tag{4.15}$$

The relative reduced stress $S_z(\xi, \eta)$ according to Eq. (4.6), like the displacement, increases monotonically from zero to maximum value which is reached also after half of the vibration period ($\eta = 0.5$ — Fig. 5). As it can be seen, the inertial forces cause some "inertia" in the increasing reduced stress to its maximum value in comparison with the sudden increase in the pressure p_0. The maximum value of the function $S_z(\xi, \eta)$ is determined by the formula:

$$S_{z\max} = S_z(1\,;0.5) = 3\frac{\beta^3}{\beta^3 - 1}. \tag{4.16}$$

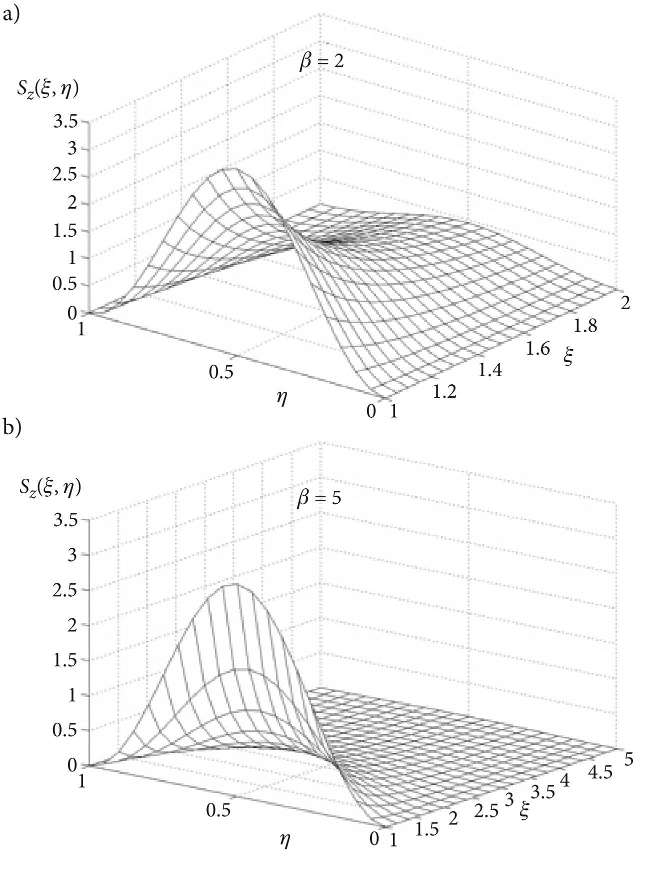

Fig. 5. Spatial graphs of the function $S_z(\xi, \eta)$ for $\beta = 2$ and $\beta = 5$

It follows from the formulae presented above and the diagrams shown in Figs. 2-5 that the maximum values of dynamic characteristics of the casing are determined by β parameter (the casing wall thickness). The main changes of characteristics' courses take place in the range $1 < \beta \leq 5$. Whereas, for $\beta > 5$, the influence of the casing wall thickness for changes of characteristics courses vanishes. In this case, the results obtained from formulae (4.2)-(4.6) are comparable with these for a spherical cavity in unbounded medium for $\nu = 0.5$ [10]. The differences between them do not exceed a per cent fraction.

5. Comparison of solutions for non-linear and linearized equations

The non-linear equation (3.17) describing the problem was solved numerically with the use of Runge-Kutta method. The results of calculations are presented as graphs depicted in Figs. 6-9. The courses obtained from the non-linear equation are drawn with solid lines and the parameters of linearized model are presented as series of circles. The graphs show courses for the casing inner surface $\xi = 1$, because the absolute values of casing dynamic characteristics reach their maxima there. In further spherical sections of casing, the variations of characteristics versus η variable are analogous. They are only respectively smaller because of the phenomenon of spatial divergence.

The variation of quantity $U(1, \eta)/P$ versus η for a few values of the parameters P, k, and β is depicted in Fig. 6. The quantities P and k were correlated according to the thermodynamic properties of explosive mixtures [6, 8]. The function $U(1, \eta)$ represents the tangential strain of the casing inner surface. The comparison of graphs shows that the results obtained from both solutions are convergent for $P = 0.01$, this means for very small strains of the order of a per cent fraction. For strains of the order of a few per cent, the differences between results exceed 10%. Whereas, for $P = 0.5$ ($\varepsilon_\varphi \approx 0.2$) the errors are greater than 50%. As it can be seen, the use of the linearized solution is strongly limited.

The graphs representing the functions $S_r(1,\eta)$, $S_\varphi(1,\eta)$ and $S_z(1,\eta)$ are depicted in successive Figs. 7, 8, and 9, respectively, in similar way as in Fig. 6. Likewise for the function $U(1, \eta)$, both solutions give the comparable results only for very small strains, i.e., for $P < 0.1$. Whereas, for $P \geq 0.1$ the results are divergent. The differences are of a dozen and more per cent. Particularly large quantitative and qualitative differences occur for the relative reduced stress $S_r(1, \eta)$. It results from the fact that in the non-linear model, the pressure inside the casing varies exponentially during vibrations, and on the contrary in the linear system p_0 = const.

The dynamic yield point in metals is always finite. It is greater than the static one. The obtained solution of the problem is valid only in the elastic range. From this fact it follows the limitation of the maximum value of pressure created in the casing, i.e., $p_0 \leq p_{max}$. The value of p_{max} can be significantly increased through the initial plastic strains caused by the strong shock wave generated in the casing by the refraction of detonation wave.

As it is known, plastic strains in metals are caused by the components of stress deviator. It can be assumed on this basis that the condition of beginning of material plastic flow depends only on the difference of the stresses $\sigma_\varphi - \sigma_r$. In point of fact, the expression $(\sigma_\varphi - \sigma_r)/2$ determines the maximum value of shear stress. Therefore according to Tresca condition of plasticity, and in case of spherical symmetry — also to Huber-Misses-Hencky, we have:

$$\sigma_z = \sigma_\varphi - \sigma_r = \sigma_0, \tag{5.1}$$

where σ_0 is the value of dynamic yield point obtained in the tensile test of given material.

It follows from graphs presented in Figs. 6 and 9 and from relation (4.6) that the quantity $S_z(1,\eta)$ varies similarly to the function $U(1,\eta)$. It reaches its maximum in the middle of the period ($\eta = 0.5$) in the linearized system. The non-linearity of system suppresses the reduced stress. With the increase in the casing wall thickness, its inertial forces significantly decrease $S_{z\,\max}(1,\eta)$ in both systems.

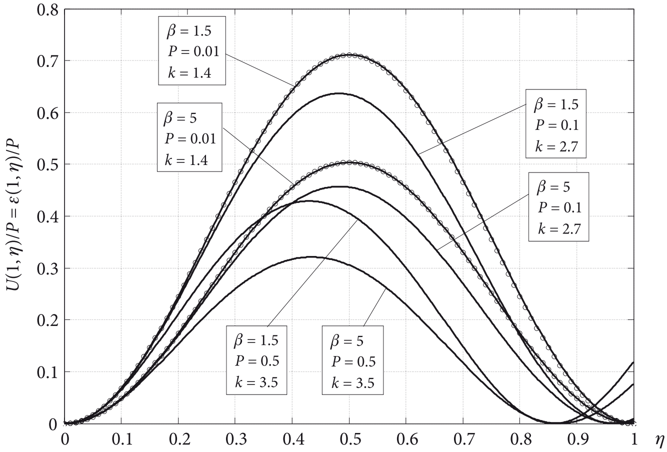

Fig. 6. Comparison of relative displacement of the casing inner surface (ξ = 1) obtained from the linearized (∘∘∘) and non-linear (—) solution

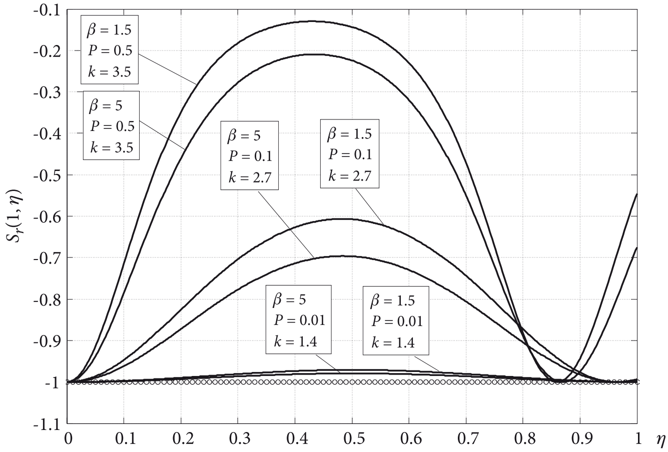

Fig. 7. Comparison of relative radial stress on the casing inner surface (ξ = 1) obtained from the linearized (∘∘∘) and non-linear (—) solution

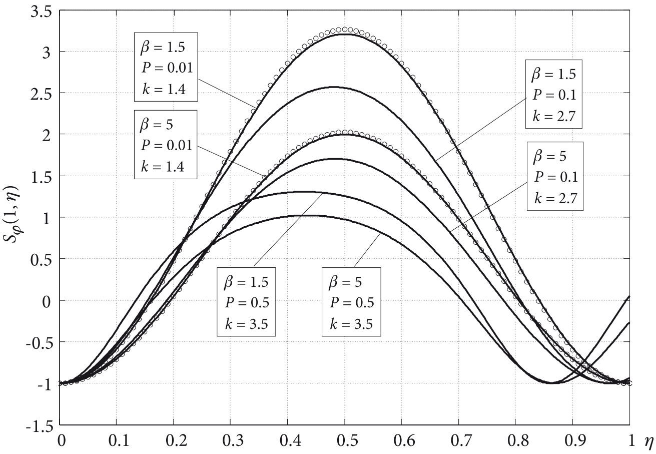

Fig. 8. Comparison of relative tangential stress on the casing inner surface ($\xi = 1$) obtained from the linearized (∘∘∘) and non-linear (—) solution

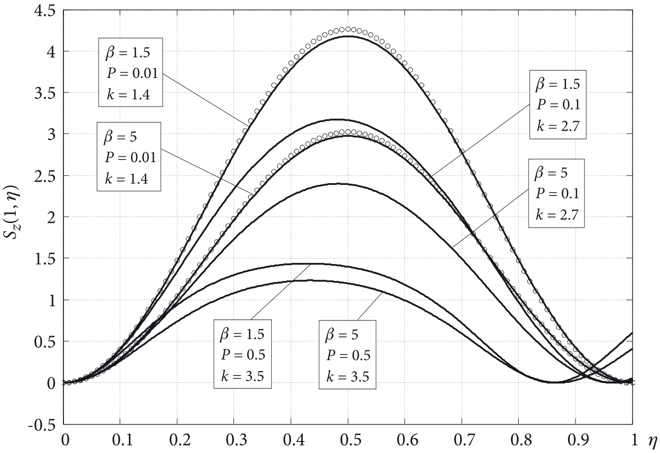

Fig. 9. Comparison of relative reduced stress on the casing inner surface ($\xi = 1$) obtained from the linearized (∘∘∘) and non-linear (—) solution

6. Final conclusions

The following conclusions result from the analysis of the studied problem:

1. A thick-walled spherical casing, made of incompressible isotropic elastic material, loaded internally by the pressure of detonation products of high explosive, after the initial period of detonation and shock waves refraction on the casing surfaces, vibrates with determined angular frequency and responses like the mechanical system with one degree of freedom.

2. The dynamics of a spherical ballistic casing was the example to present the effective mathematical non-linear model of solving one-dimensional dynamic initial-boundary value problems in the incompressible elastic media loaded explosively. This model takes into account the movement of limiting surfaces in boundary conditions and represents well the real technical issues.
3. Neglecting the movement of limiting surfaces in boundary conditions, often used in literature [1], distorts the qualitative and quantitative view of dynamic field of: displacements, strains, and stresses in the studied objects. The errors exceed a dozen, and even several dozen per cent.
4. The linear model that does not take into account the movement of limiting surfaces in boundary conditions yields comparable results with the non-linear model only for very small strains not exceeding one per cent.
5. The solution presented above can be applied to estimate the strength of spherical ballistic casings used at explosive driving of thin-walled rings in the research of dynamic properties of metals. Besides, the presented results of studies are the modest contribution of knowledge to the vibration theory of continuous technical systems.

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Dynamiczna reakcja kulistej osłony balistycznej obciążonej wybuchowo na bieżący ruch powierzchni granicznych warunków brzegowych

Streszczenie. Zbadano dynamiczne pola: przemieszczeń, odkształceń i naprężeń w kulistej grubościennej osłonie balistycznej, obciążonej wewnętrznie ciśnieniem produktów detonacji materiału wybuchowego. Założono, że materiał osłony jest jednorodny izotropowy i sprężyście nieściśliwy. Okazuje się, że taka osłona pod wymienionym obciążeniem drga radialnie z określoną częstością kołową, podobnie jak układ mechaniczny o jednym stopniu swobody. Rozpatrzono dwa modele matematyczne badanego zagadnienia: liniowy, w którym warunki brzegowe lokalizowano na początkowym położeniu powierzchni granicznych osłony (współrzędne Lagrange'a) i nieliniowy, uwzględniający ruch granicznych powierzchni osłony w warunkach brzegowych (współrzędne Eulera). Dla modelu liniowego uzyskano analityczne zamknięte rozwiązanie problemu. W przypadku bardzo małych odkształceń sprężystych, mniejszych od 1% otrzymuje się z niego wyniki zbieżne z rozwiązaniem nieliniowym. Tylko w tym przedziale odkształceń można go stosować w inżynierskich obliczeniach. Dla większych odkształceń błędy wynikające z linearyzacji problemu są rzędu kilkunastu i więcej procent. Linearyzacja zagadnienia zniekształca ilościowy i jakościowy obraz dynamicznych parametrów osłony.

Słowa kluczowe: dynamika, osłona balistyczna, obciążenie wybuchowe, materiał osłony sprężyście nieściśliwy