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Identification of subspace position in multistage bundle projection in projective space P_n

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Abstract. A multistage projection bundle $\boldsymbol{R}$ is realized in a field of an n-dimensional projection space P_n. An apparatus of this projection is created from:
— the projection plane π, which is a p-dimensional subspace, $1 \le p \le n - 1$,
— a centre of the projection S, which is an s-dimensional subspace $s \ge 0$.
The dimension s of the centre of the projection S decides about the kind of the bundle projection: a single — ($s = 0$) or a multistage ($s > 0$).
In the consecutive steps of a multistage bundle projection, subspaces belonging to the pencil trace system (F) are adopted as projection planes. The pencil trace system (F) is formed by a pencil of the system subspaces $F_1, F_2, \dots, F_k$, $k \ge 2$ and the core F which is a node subspace. The system subspaces $F_1, F_2, \dots F_k$ create a subset of a pencil of the subspaces in the field P_n, i.e., the junction $F_1 F_2 \dots F_k = P_n$, $n \ge 2$.
The relatively easiest solutions can be obtained using double-subspaces pencil trace systems (F_1, F_2) defined in the projective space P_n, $n \ge 2$. This system consists of two different system subspaces F_1, F_2, where $\dim F_1 = \dim F_2 = n - 1$, and the node subspace $F = F_1 \cap F_2$, where $\dim F = n - 2$.
Considering the trace system (F) defined in P_n we can point to two complementary families in the set of all subspaces contained in P_n:
— a family of the trace — determinable subspaces,
— a family of the trace — undeterminable subspaces.
The aim of this article is to determine the conditions which guarantee that a subspace is a trace-determinable one.

Keywords: descriptive geometry, n-dimensional geometry, projective space, bundle projection, multistage projection
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1. Introduction

We define the multistage bundle projection ***R***. The apparatus of this projection is created from:

— the projection plane π which is a p-dimensional subspace, where $1 \leq p \leq n-1$,

— a centre of the projection S which is an s-dimensional subspace, where $s \geq 0$.

If dim $S = s = 0$, then we have a single-stage bundle projection.

If dim $S = s > 0$, then we can realize the projection ***R*** as a superposition of single-stage bundle projections [8].

Their centres S_i and the projection planes π_i, $i = 1, 2, \ldots s, s+1$, satisfy the following conditions [5, 11]:

$$
\begin{aligned}
&— S_1 S_2 \ldots S_{s+1} = S, \quad s \geq 0,\\
&— \pi = \pi_{s+1},\\
&— \pi_i = \pi_{j+1} \ \ S_{j+1}, \quad j = 1, 2, \ldots, s,\\
&— \pi_i S_i = P_n, \quad n \geq 2.
\end{aligned}
\tag{1}
$$

We obtain invertibility of the multistage bundle projection ***R*** in the projection space P_n by using, as we call it, **a pencil trace system** (F)

$$(F) = F_1 F_2 F_3 \ldots F_k, \quad k \geq 2. \tag{2}$$

The system subspaces $F_1, F_2 \ldots, F_k$ are created by a subset of a pencil of subspaces from the field P_n, $P_n = F_1 F_2 \ldots F_k$. The core F of this pencil is a node subspace of a pencil trace system (F) [4].

We can obtain the relatively easiest descriptive solutions using double-subspaces pencil trace systems (F_1, F_2) defined in the projective space P_n, $n \geq 2$. This system is created by two different system subspaces F_1, F_2 where dim F_1 = dim $F_2 = n - 1$, and the node subspace $F = F_1 \cap F_2$, where dim $F = n - 2$ [6, 11].

The identification of a position of any subspace $A \subset P_n$ in relation to the trace system $(F) = \{F_1, F_2, \ldots F_k\}$, $k \geq 2$ discriminated in P_n, is carried out thanks to products (traces) of the subspace A with system planes:

$$A \cap F_1 = A_1, \quad A \cap F_2 = A_2, \ldots, \quad A \cap F_k = A_k, \tag{3}$$

and the product of the subspace A with the node subspace F of this system (F):

$$A \cap F = A_F. \tag{4}$$

The subspace A_F is called a **node of the subspace *A*.**

Taking the trace system (F), defined in P_n, into account, we can distinguish two complementary families in the set of all subspaces contained in P_n:

— a family of trace-determinable subspaces,

— a family of trace-undeterminable subspaces.

Definition 1

The subspace $A \subset P_n$ is a member of a family included in the family of trace-determinable subspaces in the given trace system $(F) = \{F_1, F_2, ... F_k\}$ when the junction of traces

$$A_1 = A \cap F_1, \quad A_2 = A \cap F_2, \ldots \quad A_k = A \cap F_k$$

is identical with A,

$$A = A_1 A_2 ... A_k. \tag{5}$$

Definition 2

The subspace $A \subset P_n$ belongs to the family of trace-undeterminable subspaces in the given trace system $(F) \subset P_n$ when the junction of the traces $A_1 A_2 ... A_k$ is different than A,

$$A_1 A_2 ... A_k \neq A. \tag{6}$$

We assume that double subspaces trace system $(F) = \{F_1 F_2\}$ is determined in the projective space P_n, $n \geq 2$. We will try to formulate the conditions which the subspace $A \subset P_n$ should satisfy to be included in the family of trace-determinable subspaces in relation to the system (F).

2. Trace-determinable subspaces

2.1. Conditions for trace-determinable subspaces

2.1.1. The general condition

Lemma 1

If the double-subspaces trace system $(F) = \{F_1 F_2\}$ is given in the projective space (subspace) P_n with a dimension $n \geq 2$ then, the subspace $A \subset P_n$ is trace determined in this system if and only if

$$(FA_1) \cap (F_2 A) = (F_1 \cap F_2) A. \tag{7}$$

Proof

Suppose that the subspace $A \subset P_n$ satisfies conditions (1). Because dimensions of identical subspaces are equal to each other, we have:

$$\dim\left[\left(F_1 A\right) \cap \left(F_2 A\right)\right] = \dim\left[\left(F_1 \cap F_2\right) A\right]. \tag{8}$$

Making equivalent transformation of Eq. (8) we gain:

$$\begin{aligned}
&\dim\left[\left(F_1 \cap F_2\right) A\right] = \dim\left(F_1 A\right) + \dim\left(F_2 A\right) - \dim\left(F_1 A F_2 A\right) = \\
&= \dim\ \left(F_1 \cap F_2\right) + \dim A - \dim\left(F_1 \cap F_2 \cap A\right) \quad \Leftrightarrow \\
&\Leftrightarrow \quad \dim F_1 + \dim A - \dim\left(F_1 \cap A\right) + \dim F_2 + \dim A - \dim\left(F_2 \cap A\right) - n = \\
&= \dim F_1 + \dim F_2 - n + \dim A - \dim\left(F_1 \cap F_2 \cap A\right) \quad \Leftrightarrow \\
&\Leftrightarrow \quad -\dim\left(F_1 \cap A\right) + \dim A - \dim\left(F_2 \cap A\right) = -\dim\left(F_1 \cap F_2 \cap A\right) \quad \Leftrightarrow \\
&\Leftrightarrow \quad \dim A = \dim\left(F_1 \cap A\right) + \dim\left(F_2 \cap A\right) - \dim\left(F_1 \cap F_2 \cap A\right) \quad \Leftrightarrow \\
&\Leftrightarrow \quad \dim A = \dim\left[\left(F_1 \cap A\right)\left(F_2 \cap A\right)\right].
\end{aligned} \tag{9}$$

On the other hand

$$\left[\left(F_1 \cap A \subset A\right) \wedge \left(F_2 \cap A\right) \subset A\right] \Rightarrow \left[\left(F_1 \cap A\right)\left(F_2 \cap A\right) \subset A\right]. \tag{10}$$

In comparison with Eq. (9), it leads to the conclusion that

$$A = \left(F_1 \cap A\right)\left(F_2 \cap A\right). \tag{11}$$

According to the definition given above it means that the subspace A is a trace-determinable subspace in the system (F).

We shall still assume that the subspace $A \subset P_n$ is a trace-determinable subspace in the given double-subspaces trace system$(F) = \{F_1 F_2\}$, so

$$A = \left(F_1 \cap A\right)\left(F_2 \cap A\right). \tag{12}$$

Writing Eq. (12) using arithmetic of dimensions and making equivalent transformation in the same way as we have presented in the first part of the proof, but in the opposite sequence, we receive Eq. (8).

Equation (8) together with the obvious correlation

$$A\left(F_1 \cap F_2\right) \subset \left(F_1 A\right) \cap \left(F_2 A\right), \tag{13}$$

let us deduce that subspaces $A(F_1 \cap F_2)$ and $(F_1A) \cap (F_2A)$ are identical

$$A(F_1 \cap F_2) = (F_1A) \cap (F_2A). \tag{14}$$

The last statement together with the earlier arguments completes the proof of lemma 1.

2.1.2. Other conditions

Analysing the properties of the family of the trace-determinable subspaces in the double trace system $(F) = \{F_1, F_2\} \subset P_n$ described in definition 1 and lemma 1, we can prove that this family can be divided into two subsets:

1. A subset of trace subspaces.
   This means a subset of subspaces which are contained in minimum one of the system subspaces of the trace system (F).
   For example, if $A \subset F_1$, then $A_1 = A \cap F_1 = A$ and $A_2 = A \cap F_2$, and these lead to $A_1A_2 = A(A \cap F_2) = A$;
2. A subset of the unrestricted subspaces in relation to the node subspace F, which satisfy the conditions of lemma 2.

Lemma 2

If the double-subspaces trace system $(F) = \{F_1, F_2\}$ is given in the projective space (subspace) P_n, $n \geq 2$ and the node A_F of the subspace $A \subset P_n$ ($\dim A = a \geq 1$) is an (a-2)-dimensional subspace, then A is the trace-determinable subspace in the system (F).

Proof

The dimension of the subspace A equals a,

$$\dim A = a. \tag{15}$$

The dimension of the node subspace $F = F_1 \cap F_2$ from the trace system $(F) = \{F_1, F_2\}$ equals $n - 2$;

$$\dim F = n - 2. \tag{16}$$

The intersection of these subspaces $A \cap F$ is the node A_F and its dimension equals $a - 2$;

$$\dim A_F = a - 2. \tag{17}$$

However, the junction of the subspaces A and F is the space P_n ($AF = P_n$) so their dimensions are as follows:

$$\dim (AF) = a + (n - 2) - (a - 2) = n. \tag{18}$$

It follows from this that each junction of the subspace A with the system subspaces F_i, ($i = 1, 2$) which contains the node subspace F is also identical with P_n,

$$AF_1,= AF_2 = P_n. \tag{19}$$

Furthermore,

$$(AF_1)\cap(AF_2) = P_n = AF = A(F_1\cap F_2). \tag{20}$$

According to lemma 1, it means that A is the trace-determinable subspace in the given trace system (F).

3. Conclusion

In the consecutive steps of a multistage bundle projection, subspaces belonging to the pencil trace system (F) are adopted as projection planes. The pencil trace system (F) is formed by the pencil of the system subspaces $F_1, F_2, \ldots, F_k$, $k \geq 2$ and the core F which is a node subspace. The system subspaces $F_1, F_2, \ldots, F_k$ create a subset of a pencil of the subspaces in the field P_n, i.e., the junction $F_1 F_2 \ldots F_k = P_n$, $n \geq 2$.

Considering the trace system (F) defined in P_n we can point to two complementary families in the set of all subspaces contained in P_n:

— a family of the trace-determinable subspaces,

— a family of the trace-undeterminable subspaces.

In this article we determined the conditions which guarantee that a subspace is trace-determinable.

Trace systems together with bundle projections let us create invertible projections which are realized on planes contained in the projection space P_n, ($n \geq 2$). The set of bundle projections and trace systems creates an apparatus of invertible projection in the projection space P_n.

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E. ZARZEKA-RACZKOWSKA

Identyfikacja położenia podprzestrzeni w złożeniowym rzutowaniu wiązkowym *n*-wymiarowej przestrzeni rzutowej P_n

Streszczenie. W przestrzeni rzutowej *n*-wymiarowej P_n ($n \geq 2$) zostało zdefiniowane złożeniowe rzutowanie wiązkowe ***R***. Aparat tego odwzorowania tworzą:

— rzutnia π, podprzestrzeń *p*-wymiarowa, $1 \leq p \leq n - 1$,

— środek rzutowania *S*, podprzestrzeń o wymiarze *s*, $s \geq 0$.

Wymiar *s* środka rzutowania *S* decyduje o tym, czy mamy do czynienia z rzutowaniem wiązkowym prostym ($s = 0$) czy też z rzutowaniem wiązkowym złożeniowym ($s > 0$).

W poszczególnych etapach rzutowania wiązkowego złożeniowego na rzutnie obierane są podprzestrzenie wchodzące w skład tzw. pękowego układu śladowego (*F*). Pękowy układ śladowy (*F*) tworzy pęk podprzestrzeni układowych $F_1, F_2, \ldots F_k$, $k \geq 2$ o rdzeniu *F*, będącym podprzestrzenią węzłową. Podprzestrzenie układowe $F_1, F_2, \ldots F_k$ stanowią podzbiór pęku podprzestrzeni o polu P_n, tzn. złącz $F_1 F_2 \ldots F_k = P_n$, $n \geq 2$.

Stosunkowo najprostsze rozwiązania uzyskuje się przy wykorzystaniu dwupodprzestrzeniowych pękowych układów śladowych (F_1, F_2) określonych w przestrzeni rzutowej P_n, $n \geq 2$. Układ ten składa się z dwóch różnych podprzestrzeni układowych F_1, F_2, przy czym $\dim F_1 = \dim F_2 = n - 1$ oraz podprzestrzeni węzłowej *F*, $F = F_1 \cap F_2$, gdzie $\dim F = n - 2$.

Z uwagi na wyróżniony w P_n układ śladowy (*F*) w zbiorze wszystkich podprzestrzeni zawartych w P_n wyróżniamy dwie uzupełniające się rodziny:

— rodzinę podprzestrzeni śladowo-wyznaczalnych,

— rodzinę podprzestrzeni śladowo-niewyznaczalnych.

W artykule przedstawiono ponadto warunki, jakie musi spełniać dana podprzestrzeń, aby była ona śladowo-wyznaczalna.

Słowa kluczowe: geometria wykreślna, geometria *n*-wymiarowa, przestrzeń rzutowa, rzutowanie wiązkowe, rzutowanie wieloetapowe
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