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On some elliptic operator of order one

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Abstract. The purpose of this paper is to determine explicitly well-posed transmission problems for some elliptic system. It is well known that for this system there is no well-posed boundary value problem in any bounded domain in R^4.

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Consider the following operator in R^4:

$$A(\frac{\partial}{\partial x}) = \begin{pmatrix} \partial_1 & -\partial_2 & -\partial_3 & -\partial_4 \\ \partial_2 & \partial_1 & -\partial_4 & \partial_3 \\ \partial_3 & \partial_4 & \partial_1 & -\partial_2 \\ \partial_4 & -\partial_3 & \partial_2 & \partial_1 \end{pmatrix},$$

where $\partial_k = \frac{\partial}{\partial x_k}$.

Note that A is the elliptic operator. $\det A(\xi) = |\xi|^4, \ |\xi|^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2$. Let $u =^t [u_1, u_2, u_3, u_4]$, $f =^t [f_1, f_2, f_3, f_4]$. If we write the vectors u and f as the quaternions $u = u_1 + iu_2 + ju_3 + ku_4$, $f = f_1 + if_2 + jf_3 + kf_4$ then the equation $Au = f$ has the form

$$\partial_x u = f,$$

where $\partial_x = \partial_1 + i\partial_2 + j\partial_3 + k\partial_4$, $i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j$.

Let Ω be the open bounded subset of R^4 with the smooth (C^∞) boundary $\partial\Omega$. M. Solomyak proved in [1] that every boundary problem

$$Au = f \qquad \text{in} \quad \Omega,$$

$$Bu_{|\partial\Omega} = \varphi,$$

is not well posed. Here B is a 2×4-matrix of pseudodifferential operators and ${}^t\varphi = [\varphi_1, \varphi_2]$ is a given vector-function on $\partial\Omega$.

Let Ω^+ be the bounded domain in R^4, $\Omega_- = R^4 \setminus \Omega^+$, and $\partial\Omega^+ = \Gamma$ be the smooth manifold. In this paper we study the following transmission boundary value problems:

$$\begin{cases} \partial_x u^+ = 0 \quad in \quad \Omega^+ \\ \partial_x u^- = 0 \quad in \quad \Omega_- \\ u+ = Gu^- + g \quad on \quad \Gamma \\ u^-(\infty) = 0 \end{cases}, \tag{1}$$

$$\begin{cases} \partial_x u^+ = 0 \quad in \quad \Omega^+ \\ \partial_x u^- = 0 \quad in \quad \Omega_- \\ u+ = u^- G + g \quad on \quad \Gamma \\ u^-(\infty) = 0 \end{cases}, \tag{2}$$

where $G = G_1 + +iG_2 + jG_3 + kG_4$, $g = g_1 + ig_2 + jg_3 + kg_4$ are Γ smooth enough quaternions on Γ. We give also an example of well-posed boundary value problem for the operator

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

in $R^4_+ = \{x : x_1 > 0\}$. We treat Eqs. (1) and (2) as the systems of equations for the functions u_l^+, u_l^- ($l = 1, 2, 3, 4$) in Sobolev spaces. If u satisfies the equation $\partial_z u = 0$ in Ω^+ and is continuous in $\bar{\Omega}^+$ then (see [2])

$$u(z) = \frac{1}{4\pi^2} \int_\Gamma [\partial_z \frac{1}{r^2}] n u(x) d\sigma_x,$$

where $r = |z - x|$ and n is the exterior unit normal to Γ in the point x. We look for the solution of Eq. (1) of the form

$$u(z) = \frac{1}{4\pi^2} \int\limits_{\Gamma} [\partial_z \frac{1}{r^2}] n\varphi(x) d\sigma_x.$$

According to formulae

$$u^+(y) = \lim_{\Omega^+ \ni z \to y \in \Gamma} u(z) = \frac{\varphi(y)}{2} + \frac{1}{4\pi^2} PV \int\limits_{\Gamma} [\partial_y \frac{1}{r^2(x,y)}] n\varphi(x) d\sigma_x,$$

$$u^-(y) = \lim_{\Omega^- \ni z \to y \in \Gamma} u(z) = \frac{-\varphi(y)}{2} + \frac{1}{4\pi^2} PV \int\limits_{\Gamma} [\partial_y \frac{1}{r^2(x,y)}] n\varphi(x) d\sigma_x,$$

one can transform problem (1), to the following equivalent system of integral equations:

$$\varphi + C\varphi = -G\varphi + GC\varphi + g, \tag{3}$$

where

$$C\varphi(y) = \frac{1}{2\pi^2} PV \int\limits_{\Gamma} [\partial_y \frac{1}{r^2(x,y)}] n\varphi(x) d\sigma_x.$$

Now, consider the operator S acting on φ given by Eq. (3).

Theorem. *The operator $S : H^s(\Gamma) \to H^s(\Gamma)$ is elliptic (problem (1) is well posed) iff $G_1(x) \neq 0$.*

Proof. It is enough to calculate the symbol $\sigma(\xi')$ of the operator S ($\xi' = (\xi_1, \xi_2, \xi_3)$) on $|\xi'| = 1$. Let

$$\begin{aligned} a &= 1 + G_1 - G_2 i\xi_3 + G_3 i\xi_2 + G_4 i\xi_1 \\ b &= G_2 - i\xi_3 + G_1\xi_3 + G_4 i\xi_2 - G_3 i\xi_1 \\ c &= G_3 + i\xi_2 + G_4 i\xi_3 - G_1 i\xi_2 + G_2 i\xi_1 \\ d &= G_4 + i\xi_1 - G_3 i\xi_3 - G_2 i\xi_2 - G_1 i\xi_1 \end{aligned},$$

then

$$\sigma(\xi')_{|\xi'|=1} = \det \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}$$

$$= (a^2 + b^2 + c^2 + d^2)^2 = (4G_1 + 4i(G_4\xi_1 + G_3\xi_2 - G_2\xi_3))^2,$$

which completes the proof. □

Similarly one can prove that the problem (2) is well posed iff

$$2G_1(G_1^2 + G_2^2 + G_3^2 + G_4^2)+ \\ + 2G_1 - 4(G_1^2 + G_2^2 + G_3^2 + G_4^2) - (G_1 + G_2 + G_3 + G_4) \neq 0.$$

Now let $u =^t [u_1, u_2, ..., u_8]$. It is easy to show that there exists the constant d such that the boundary value problem

$$Pu = F \quad \text{in} \quad R^4_+,$$

$$Bu = \varphi \quad \text{on} \quad \partial R^4_+,$$

with

$$P = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -d & 1 & 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

is well posed in Sobolev spaces.

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O pewnym operatorze eliptycznym pierwszego rzędu

Streszczenie. Celem tej pracy jest wydzielenie pewnej klasy dobrze postawionych zagadnień transmisji dla operatora eliptycznego, o którym wiadomo, że nie istnieje dla niego dobrze postawione zagadnienie brzegowe i to na dowolnym obszarze.
Słowa kluczowe: zagadnienia poprawnie postawione, zagadnienie transmisji
Symbole UKD: 51