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Estimation of Two Sinusoids in a Very Short Signal

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***Abstract*—In the paper, the estimation of the parameters (frequency, amplitude, phase) of two complex-valued sinusoids embedded in a white gaussian circular additive noise is considered. In this context, it is answered what is the minimal necessary number of signal samples needed to reliably estimate all the parameters of both sinusoids. The Cramer-Rao bounds and maximum likelihood estimator are used in the analysis. The answer to the posed question is not straightforward. It is shown that three signal samples are enough only if the difference of phases between both sinusoids meets certain condition, otherwise estimation results are ambiguous. The use of four signal samples has the advantage that reliable estimates can be obtained irrespectively of this phase difference.**

***Keywords*—multiple frequencies estimation, short linear array.**

I. INTRODUCTION

THE problem of estimating multiple complex sinusoids in noise is well known and has many applications, particularly in the areas where signal frequency has some physical interpretation. For example in echolocation systems (radar, sonar, ultrasonography) signal frequency is related to the observed object speed by the Doppler effect. Two frequencies in a received signal occur if two objects simultaneously reflect transmitted wave. Additionally, the transmitted wave is sometimes modulated in such a way that target distance is also encoded in signal frequency (for example – frequency modulated continuous wave radar). Another application is an analysis of vibrations based on recorded acoustic signals (for example generated by rotating machinery parts) or based on directly recorded accelerometer signals. Yet another application is in linear antenna arrays [1] where frequency of a received signal corresponds to the direction of wave arrival. Two complex sinusoidal components occur if two signals arrive to an array. If the two sources of radiation are observed at similar angles then the problem of resolution occurs.

The easiest and the most straightforward way to obtain good frequency resolution is by ensuring long measurement time interval (for time signals) or large aperture (for arrays). Additionally the number of samples must be sufficient to unambiguously measure all the needed signal parameters. In some applications it is not difficult to increase number of samples even much beyond necessary minimum, but there are applications where the number of signal samples should be as small as possible. For example in Doppler radar more samples would mean increased pulse repetition frequency (and this is limited by range ambiguity) or longer time on target (slower space search); additionally in military radar more identical pulses means easier job for electronic counter measures. In array signal processing, samples correspond to array elements, and an increase in number of array elements would increase costs. Longer signals also mean more computationally demanding algorithms, thus more costly hardware and bigger energy consumption.

Hence the question arises: what is the minimal necessary number of signal samples that must be used in order to reliably estimate frequencies (along with other parameters) of two complex sinusoids? Although a large literature corresponds to frequency estimation of multiple sinusoids, this question seems to be not answered. We fill the gap in this paper.

We begin with defining a signal model. Then we analyze Cramer-Rao lower bounds for estimated parameters, and in another section we present maximum likelihood estimates. As a last section we conclude the paper.

A. Signal Model

The analyzed signal model comprises of two complex-valued sinusoids (or cisoids – the short term cisoid is sometimes used in the literature to describe single complex-valued tone) in a complex circular white gaussian noise $\xi(n)$ with variance σ^2. The frequency of the first sinusoid is denoted as f_1 and its complex amplitude as $\tilde{a}_1 = a_1 e^{j\phi_1}$ where a_1 is (real, positive) amplitude and ϕ_1 is a constant phase. Parameters of the second complex sinusoid are denoted as: frequency f_2, amplitude a_2, phase ϕ_2 and complex amplitude $\tilde{a}_2 = a_2 e^{j\phi_2}$. That is, the signal model is:

$$x(n) = \tilde{a}_1 \cdot e^{j2\pi f_1 n} + \tilde{a}_2 \cdot e^{j2\pi f_2 n} + \xi(n), \qquad (1a)$$

or equivalently:

$$x(n) = a_1 \cdot e^{j(2\pi f_1 n + \phi_1)} + a_2 \cdot e^{j(2\pi f_2 n + \phi_2)} + \xi(n). \qquad (1b)$$

We assume that this signal is observed during the discrete-time interval

$$n = -\frac{N-1}{2}, ..., \frac{N-1}{2}, \qquad (2)$$

so that the measurement interval is symmetric with respect to $n = 0$. We note that such a definition implies that for an even number of signal samples, the values of n are not integer, for example for $N = 4$ we have $n \in [-1.5, -0.5, 0.5, 1.5]$. We also note that the typical name "initial phase" for ϕ_1 or ϕ_2 cannot be used, here. To explain this, let us focus on the first complex sinusoid. We see from the equation (1) that the instantaneous phase of this signal is $2\pi f_1 n + \phi_1$. Thus ϕ_1 represents value of this instantaneous phase for $n = 0$. As this is the center of the signal observation interval, the parameter ϕ_1 should rather be named "median phase" or "central phase", but not "initial phase".

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We assume that all the parameters of both complex sinusoids are unknown and are to be estimated. Thus the vector of unknown parameters is

$$\Theta = [f_1, a_1, \phi_1, f_2, a_2, \phi_2]. \quad (3)$$

The SNR will be defined as a ratio of power of the first complex sinusoid to the noise variance, that is:

$$\mathrm{SNR} = \frac{a_1^2}{\sigma^2}. \quad (4)$$

What is the minimal number N of signal samples needed to estimate all the parameters in the vector Θ? It is obviously not enough to have $N = 1$ data point; two signal samples ($N = 2$) could be enough if there was only one sinusoid in the signal. In the case (1) of two sinusoids the smallest feasible number of samples seems to be $N = 3$. Hence we will start analyzes with $N = 3$ and, as it turns out not to be enough in some cases, proceed with $N = 4$.

II. Cramer-Rao Lower Bounds

In this section we present and analyze the Cramer-Rao lower bounds on variance of estimators of unknown parameters stacked in Θ. These bounds hold for all possible unbiased estimators; in other words no unbiased estimator exists that has smaller variance than corresponding Cramer-Rao bound. This means that Cramer-Rao bounds can serve as a benchmark to compare various estimation procedures, but can also be used to assess how much information about unknown parameters is carried in an analyzed signal. For example if the Cramer-Rao bound for a certain parameter is very high then no reliable unbiased estimator exist for this parameter, which means that the analyzed signal is not sufficient and additional information should be provided in order to reliably estimate this unknown parameter.

The Cramer-Rao bounds for parameters in the vector Θ, for the model given by equation (1), can be found in the book by S.M. Kay [2]. For simplicity we present only equations pertaining to the first sinusoidal component, for the second sinusoidal component the equations are analogous. The Cramer-Rao bounds has the following form:

$$\mathrm{CRB}_{f_1} = \beta_f(N, f_1 - f_2, \phi_1 - \phi_2) \cdot \frac{1}{2 \cdot (2\pi)^2 \cdot \mathrm{SNR}} \quad (5)$$

$$\mathrm{CRB}_{a_1} = \beta_a(N, f_1 - f_2, \phi_1 - \phi_2) \cdot \frac{\sigma^2}{2} \quad (6)$$

$$\mathrm{CRB}_{\phi_1} = \beta_\phi(N, f_1 - f_2, \phi_1 - \phi_2) \cdot \frac{1}{2 \cdot \mathrm{SNR}}, \quad (7)$$

where $\beta_f(\cdot)$, $\beta_a(\cdot)$ and $\beta_\phi(\cdot)$ are complicated functions that will be evaluated numerically. We see that each bound depends only on four variables:

- $f_1 - f_2$ – difference of frequencies; it is important how close in frequency the two harmonics are, but the bounds do not depend on f_1 or f_2 independently,
- $\phi_1 - \phi_2$ – only difference of phases matters, there is no dependence on ϕ_1 or ϕ_2 alone,
- N – it can be expected that a longer signal gives better estimates,
- SNR – the higher the SNR the more accurate estimates of frequency and phase,
- σ^2 – only the bound for amplitude depends on noise variance (and not on SNR).

In the following subsection we present and analyze Cramer-Rao bounds for signal comprising of $N = 3$ samples.

A. *Cramer-Rao Lower Bounds for Signal Length $N = 3$*

The Cramer-Rao bound for frequency f_1, given that the signal $x(n)$ has only three samples and SNR $\approx$ 30 dB is shown in Fig. 1(a). We see that in certain regions of parameters $(\phi_1 - \phi_2)$ and $(f_1 - f_2)$, the value of the bound is very high which means that, for these regions and for the signal model (1), there is no unbiased estimator for frequency that would give reliable estimates.

This happens if:

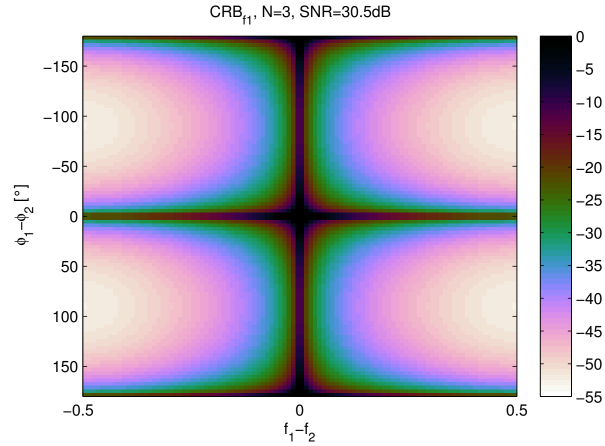
- $f_1 \approx f_2$, that is the two harmonics are very close in frequency. Is it a big region? It is much smaller than the resolution of the periodogram which is approximately $1/N = 0.33$. This suggest that there are ”high-resolution” methods that give a few times better resolution than the standard periodogram. Nevertheless, if the sinusoids are very close in frequency then even “high-resolution” methods should not be expected to give good estimates.
- $\phi_1 - \phi_2 \approx i \cdot 180°$ for $i = -1, 0, 1$. This dependence of frequency estimation accuracy on phase difference is not intuitive. The reason turns out to be ambiguity of signal parameters for $N = 3$. In other words three signal samples are not enough to unambiguously estimate all the unknown parameters. We see that unknown parameter vector Θ comprises of six unknown parameters, whereas $N = 3$ means that we have six real signal samples. This gives us six nonlinear equations with six unknown parameters. For $\phi_1 - \phi_2 \approx i \cdot 180°$ this set of equations cannot be unambiguously solved even if there is no additive noise.
  For example if $\phi_1 - \phi_2 = 180°$ and $a_1 = a_2$ then only $f_1 + f_2$ can be reliably estimated (but not f_1 or f_2 independently) – see example at the end of the paper.

In Figs. 1(b),1(c) and Cramer-Rao bounds for amplitude and phase are presented.

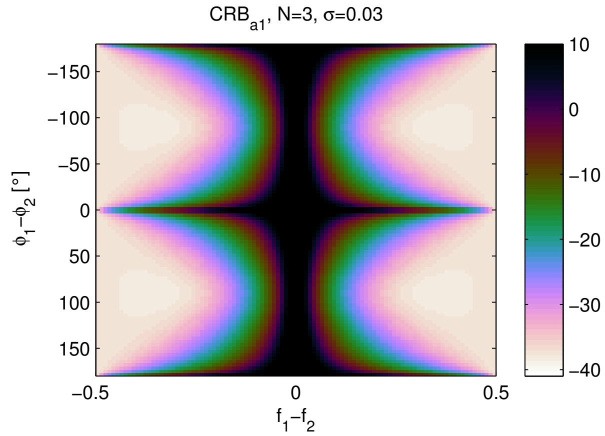
The plot of CRB_{a_1} is similar to that of CRB_{f_1}, hence we have similar problems with reliable estimates for $f_1 \approx f_2$ or $\phi_1 - \phi_2$ being close to zero or $\pm 180°$.

It is interesting that, unlike CRB_{f_1} and CRB_{a_1}, the Cramer-Rao bound for phase CRB_{ϕ_1} do not depend on the phase difference $\phi_1 - \phi_2$. It is very important because it allows for testing if frequency and amplitude estimates are reliable. The algorithm could be as follows: we estimate phases $[\phi_1, \phi_2]$ and compute the difference $\phi_1 - \phi_2$, then we test if the result is close to $i \cdot 180°$ for $i = -1, 0, 1$; if it is not then we can estimate frequency and amplitude, otherwise we know that the estimates would not be reliable.

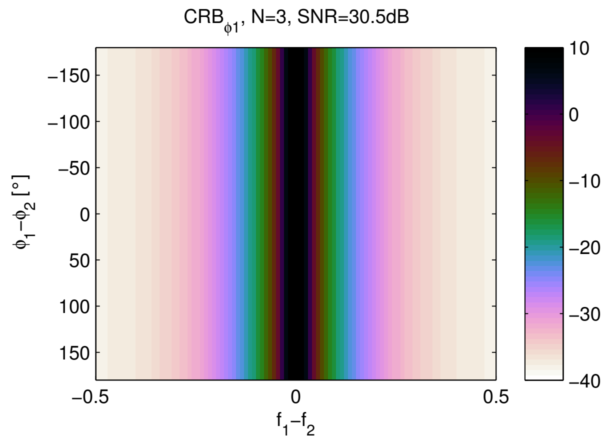
Summing up, we can reliably estimate frequencies of two sinusoids analyzing just three samples of signal, but only if the phase difference $\phi_1 - \phi_2$ is not too close to zero or $\pm 180°$.



(a) bound for frequency f_1



(b) bound for amplitude a_1



(c) bound for phase ϕ_1

Fig. 1. Cramer-Rao bounds for frequency f_1, amplitude a_1 and phase ϕ_1; signal length is $N = 3$.

Now we calculate Cramer-Rao bounds for a signal of length $N = 4$ and see advantages of increasing signal length by 1 sample.

B. Cramer-Rao Lower Bounds for Signal Length $N = 4$

The Cramer-Rao bound for frequency in the case of a four-sample signal is depicted in Fig. 2(a). We again expect unreliable estimates for $f_1 \approx f_2$, but this time this region of bad estimates is narrower (harmonics can be closer in frequency) than it was for $N = 3$. It is natural as a longer signal gives a greater normed frequency resolution.

Dependence on the phase difference $\phi_1 - \phi_2$ is still visible but it is much smaller than it was for $N = 3$. This means that one additional signal sample allows us to unambiguously estimate unknown parameters for all values of $\phi_1 - \phi_2$. It is worth noting, that dependence of frequency estimation accuracy on phase difference exists for even longer signals – in the paper by Trunk et al. [3] it was observed for $N = 21$.

In Figs. 2(b) and 2(c) the bounds for amplitude and phase are presented. On both plots it is again visible that for very close frequencies, obtaining reliable estimate of any signal parameter is impossible. On the amplitude plot the dependence on phase difference $\phi_1 - \phi_2$ is much smaller than it was for $N = 3$. Phase estimates can paradoxically be expected to be even more accurate for $\phi_1 - \phi_2 \approx i \cdot 180°$ than for other values of the phase difference.

III. Maximum Likelihood Estimator

Maximum likelihood (ML) estimator of parameters (3) can be found for example in [4] or [2]. The ML estimator of frequencies of the two sinusoids is:

$$[\hat{f}_1, \hat{f}_2] = arg \max_{f_1, f_2} L(f_1, f_2) \tag{8}$$

that is we choose such parameters f_1, f_2 that $L(f_1, f_2)$ is the largest. The maximized two-dimensional function is defined as

$$L(f_1, f_2) = \frac{N[|X(f_1)|^2 + |X(f_2)|^2] - 2\mathrm{Re}[X^*(f_1)X(f_2)\gamma(f_1, f_2)]}{N^2 - |\gamma(f_1, f_2)|^2}, \tag{9}$$

where

$$\gamma(f_1, f_2) = \sum_{n=-(N-1)/2}^{(N-1)/2} e^{-j2\pi f_1 n} e^{j2\pi f_2 n} \tag{10}$$

is the dot product of the functions $e^{j2\pi f_2 n}$ and $e^{j2\pi f_1 n}$, while

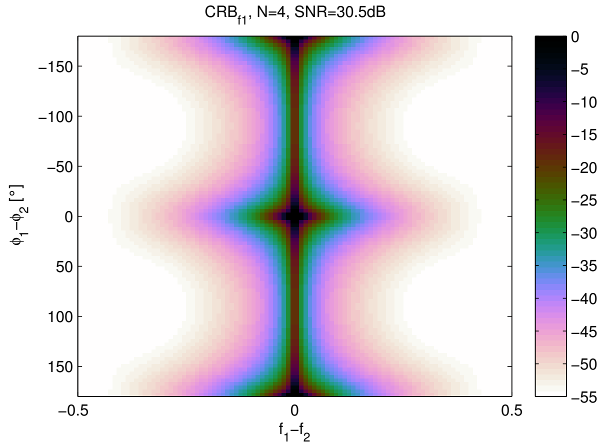
$$X(f) = \sum_{n=-(N-1)/2}^{(N-1)/2} x(n) e^{-j2\pi f n} \tag{11}$$

is the discrete Fourier transform of the analyzed signal $x(n)$ at frequency f.

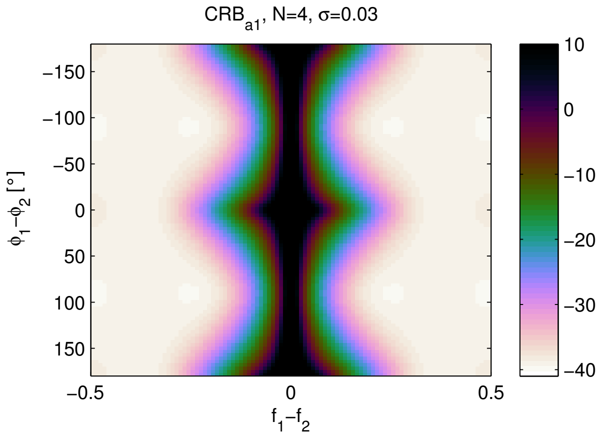
The complex amplitude is estimated as

$$\hat{\mathbf{a}} = (\mathbf{E}^H \mathbf{E})^{-1} \mathbf{E}^H \mathbf{x} \tag{12}$$

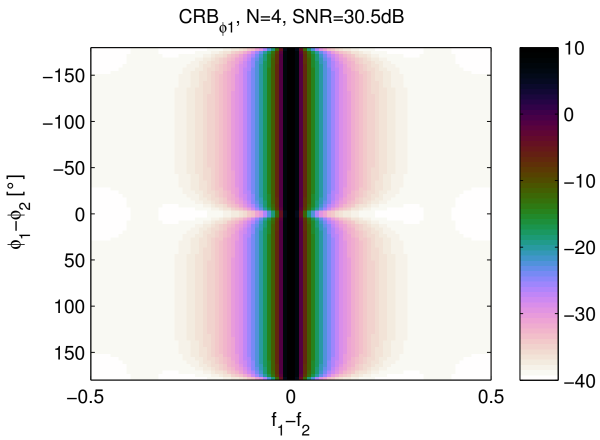
where $\mathbf{E} = [\mathbf{e}_1, \mathbf{e}_2]$, $\mathbf{e}_i = [e^{-j\pi f_i (N-1)/2}, \cdots, e^{j\pi f_i (N-1)/2}]^T$ and $\mathbf{x} = [x(-\frac{N-1}{2}), ..., x(\frac{N-1}{2})]^T$.



(a) bound for frequency f_1



(b) bound for amplitude a_1

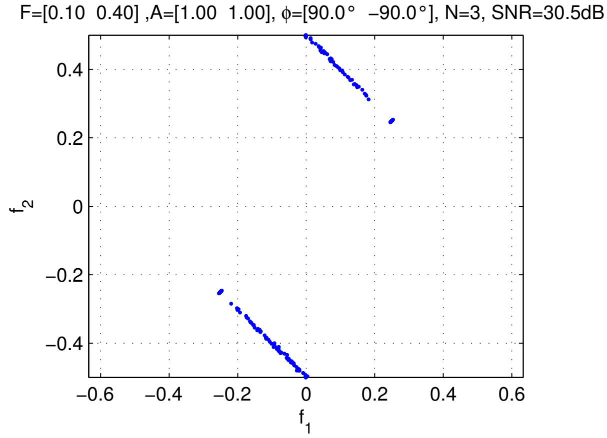


(c) bound for phase ϕ_1

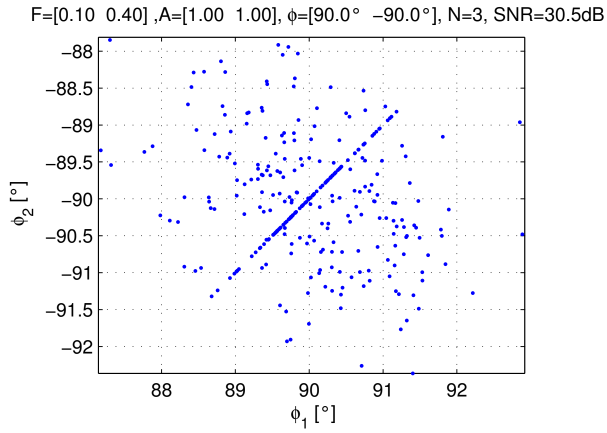
Fig. 2. Cramer-Rao bounds for frequency f_1, amplitude a_1 and phase ϕ_1; signal length is $N = 4$.

A. Example 1

As an example, let the parameters of the signal model (1) be: $a_1 = 1$, $f_1 = 0.1$, $\phi_1 = 90°$; $a_2 = a_1$, $f_2 = 0.4$, $\phi_2 = -90°$, $\sigma = 0.03$. This is "a difficult set of parameters", because $\phi_1 - \phi_2 = 180°$. In simulations the signal $x(n)$ was generated and parameters estimated by the use of ML estimator for 300 times. The results are presented in Fig. 3 for $N = 3$ and Fig. 4 for $N = 4$.



(a) estimates of frequencies



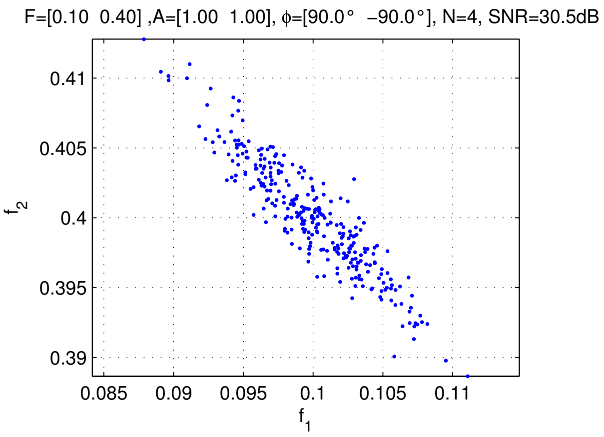
(b) estimates of phases

Fig. 3. Estimates of frequencies $[f_1, f_2]$ and phases $[\phi_1, \phi_2]$ for signal length $N = 3$ and signal parameters from example 1.

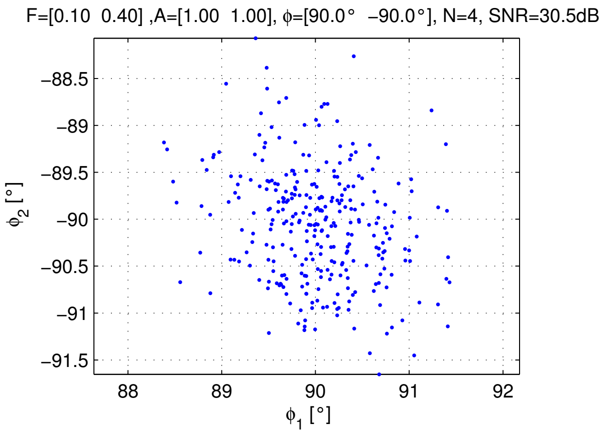
As we see on Fig. 3(a), for $N = 3$ the estimates of f_1 and f_2 are incorrect, but $\hat{f}_1 + \hat{f}_2$ is always approximately equal to 0.5 (or 0.5+1 due to periodicity) which is a good result. This is in accord with theoretical analysis from the previous section – error in this case is not caused by additive noise, but mainly by ambiguity between parameters caused by insufficient number of signal samples.

In Fig. 3(b) corresponding estimates of phase $[\phi_1, \phi_2]$ are presented. It is interesting that although frequency estimates were incorrect, phase is estimated quite accurately. This is in accord with theoretical Cramer-Rao bounds presented in the previous section. The importance of having available good phase estimate is that we know that the phase difference $\phi_1 - \phi_2$ is close to 180° and hence we know that frequency estimates would be poor. Thanks to that we can set the flag "unreliable frequency estimate" so that our system does not treat incorrect frequency estimates as correct. Without this knowledge, we would not be able to do much use of the three-sample signal.

The case $N = 4$ is presented in Fig. 4. This time we obtain satisfactory result – all the estimates of phases and frequencies are correct. The lack of circular symmetry in Fig. 4(a) means



(a) estimates of frequencies



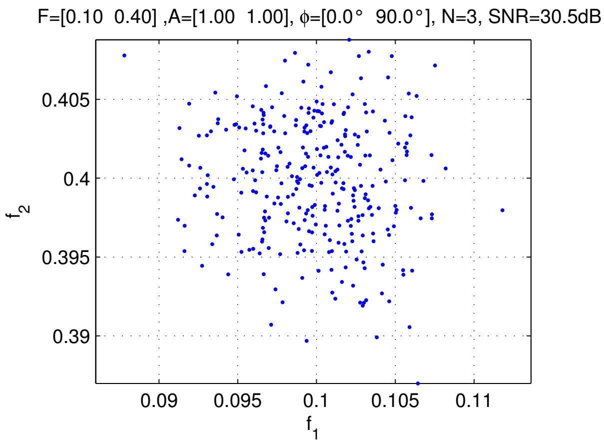
(b) estimates of phases

Fig. 4. Estimates of frequencies $[f_1, f_2]$ and phases $[\phi_1, \phi_2]$ for signal length $N = 4$ and signal parameters from example 1.

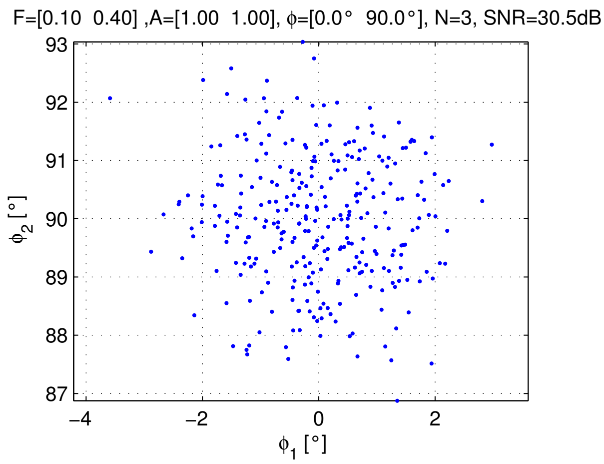
that there is nonzero correlation between $\hat{f}_1$ and $\hat{f}_2$. We see that for $N = 4$, estimation errors are caused by an additive noise only and not by ambiguities as it was for $N = 3$.

B. Example 2

In the first example we saw that for $N = 3$ and "difficult" phase difference of 180° the estimation results were not reliable. In this second example we again analyze short signal of $N = 3$ samples, but with a different phase difference value. Let the parameters of the signal model (1) be: $a_1 = 1$, $f_1 = 0.1$, $\phi_1 = 0°$; $a_2 = a_1$, $f_2 = 0.4$, $\phi_2 = 90°$, $\sigma = 0.03$. We see that this time, it is "an easy set of parameters", because the phase difference is equal to exactly $\phi_1 - \phi_2 = 90°$. Just like in the first example, we used 300 repetitions of signal computer generation and parameters ML estimation. We note that the only difference in parameters between examples 1 and 2 is that in the first example phase difference was 180° (which was a difficult case according to the results of our earlier analysis of Cramer-Rao bounds) and in the second example it is 90° (which is an easy case). The results are presented in Fig. 5. We see in the figure that estimation results are correct, which is in contrast with example 1 where for $N = 3$ estimates were generally unreliable. This shows how important the phase



(a) estimates of frequencies



(b) estimates of phases

Fig. 5. Estimates of frequencies $[f_1, f_2]$ and phases $[\phi_1, \phi_2]$ for signal length $N = 3$ and signal parameters from example 2.

difference is in case of an extremely short signal of $N = 3$ samples.

C. Example 3

It was noticed in previous examples that for a short signal of $N = 3$ samples the results are reliable if the phase difference $\phi_1 - \phi_2$ is equal to 90° but unreliable if it is equal to 180°. Hence, in this example we analyze this dependence more thoroughly. In the Fig. 6 the dependence of estimation accuracy on phase difference is depicted. All signal parameters, apart from phase, had the same values as in the previous two examples. We see that the simulation results are in accord with Cramer-Rao bounds. If the phase difference is close to 0° or ±180° then estimation results are not reliable. On the other hand, the best accuracy is obtained for phase difference close to ±90°. For phase differences equal to exactly 0° or ±180° the Cramer-Rao bounds are not shown because the Fisher information matrix is singular, which we interpret that there is no sufficient information on estimated parameter and it is not possible to obtain reliable estimates.

The same dependence for $N = 4$ is presented in Fig. 7. Again, the simulation results are in accord with theoretical

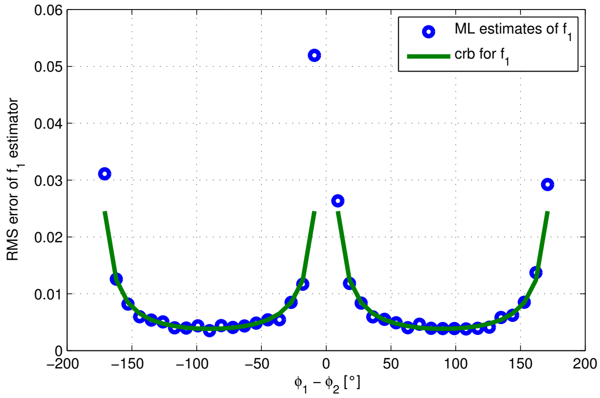


Fig. 6. Dependence of frequency estimation rms error on phase difference; signal length $N = 3$.

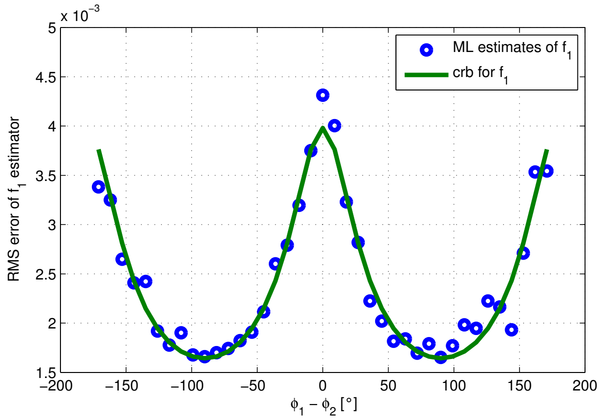


Fig. 7. Dependence of frequency estimation rms error on phase difference; signal length $N = 4$.

Cramer-Rao bounds. We still have worse estimates for phase difference close to $0°$ or $\pm 180°$, but the results are much more accurate then for $N = 3$.

IV. Conclusions

The problem of estimating two complex-valued sinusoids involves estimation of six unknown parameters – namely frequency, amplitude and phase for each of the two. We analyzed possibility of estimating all those unknown parameters based only on $N = 3$ signal samples and showed that it is possible only if difference of phases is not near zero or $\pm 180°$ – otherwise there is an ambiguity between estimates of certain parameters. This ambiguity vanishes for $N = 4$ thanks to information given by one additional signal sample.

Summing up, if an analyzed signal comprises of two sinusoids resolvable in frequency, then it is possible to estimate those sinusoids: (a) for $N = 3$ if $\phi_1 - \phi_2$ is not close to $i \cdot 180°$, $i = -1, 0, 1$, and (b) for $N = 4$ always, but still if $\phi_1 - \phi_2$ is close to the multiplicity of $180°$ the frequency estimates will be less accurate.

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