Transformation of the Network Model with Complete Information of the Network Structure into a Network Model with Incomplete Information of the Network Structure – A Game Theory Approach

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In this paper we consider the formation of directed networks, i.e. networks represented by directed graphs. By *information* (a rather common use of this term) we mean good information that flows in the network. Each member of the network is endowed with some amount of resources and has also a payoff function, which depends positively on the amount of information he has access to. Knowing the network structure players can gain access to the information possessed by others by creating links. The problem is to specify which network structures can be a strategic equilibrium and whether they are optimal (effective) [3], [2].
Further on we introduced a model in which players do not have a complete knowledge of the network structure, but only a part of it. Decisions they make base on incomplete information. The problem is to define the equilibrium and to find out what strategies can lead to the equilibrium.

Keywords: network formation, Nash equilibrium, incomplete information of network structure.

1. Network model with complete information of the network structure

Let $N = \{1,\ldots,n\}$ be a finite set of agents (for example web pages, members of a social network etc.), who we will refer to as players. Every player is endowed with a finite amount of resource $X \in (0,1)$, that can be used to create links. Every player yields a benefit from accessing other network members, directly or indirectly.
Before we begin with formulating the game theory elements we will start with a basic definition.

Definition 1.1. A digraph (directed graph) $g =< N, E >$, consists of vertices N (which we will identify as players) and directed edges E ,

$$E \subset \{(i, j) \mid i, j \in N, i \neq j\}$$

We write (i, j) which means an edge starting from *i* and pointing at j .

Remark 1.1. In the further discussion we will assign a positive value to every edge, it is then justified to use a word network instead of a graph. By g_N we denote a family of networks with a set of vertices N .

Only connected graphs will be considered.

Pure strategies

A pure strategy is an n-element vector, which informs us with whom and with what value a player wishes to establish a link. When the j-th coordinate of the strategy vector is positive, i wishes to create a link with j . Formally, we introduced a definition:

Definition 1.2. A pure strategy of player $i \in N$ is a vector $x(i) = (x_j(i))_{j \in N}$ with n coordinates, so that

$$x_j(i) \in [0, X] \text{ for all } j \in N$$

and

$$x_i(i) = 0$$

and

$$\sum_{j=1}^{n} x_j(i) = X .$$

By $X(i)$ we denote a set of all pure strategies of player $i \in N$.

When $x_j(i) > 0$ it means that player $i \in N$ creates a link with $j \in N$ consuming a positive part of his resource **X** to establish that link. A value $x_j(i)$ is assigned to (i, j) edge.

A strategy profile $x=(x(1),x(2),\ldots,x(n))$, where $x(1)\in X(1)$, $x(2)\in X(2)$ and so on, can be identified as a directed graph g, with positive values assigned to every edge, fulfilling conditions from definition 1.2.

When $(i,j)\in g$ it means that $x_j(i)>0$, whereas $(i,j)\notin g$ means that $x_j(i)=0$.

By $X=X(1)\times X(2)\times\ldots X(n)$ we denote a set of all pure strategies. To conclude all the above statements:

Preposition 1.1. There exists a one-to-one map between directed, valued graphs among n nodes and strategy profiles in $X=X(1)\times X(2)\times\ldots X(n)$.

Proof. Given any weighted, directed graph g, for all $i\in N$ we define $x(i)$ in the following way: for all $j\neq i$ put $x_j(i)=0$ if there is no edge from i to j in g, otherwise to $x_j(i)$ put the value of the weight assigned to (i,j) edge. Therefore, we defined a strategy which generates graph g.

Given any strategy profile $x\in X$, we construct a graph g in the following way: if $x_j(i)=0$ then there is no edge from i to j in g; if $x_j(i)>0$ we construct an edge from i to j in g with a weight $x_j(i)$. We defined a weighted, directed graph generated by a strategy profile $x\in X$.

Walks, paths and circles in a graph

Definition 1.3. A sequence of edges $(v_0,v_1),(v_1,v_2),\ldots,(v_{m-1},v_m)$ in graph g is a *walk*, when every next edge is incident. We call v_0 a start vertice and v_m a final vertice of a walk. We can say then, that there is a walk *from* v_0 *to* v_m. Number of edges in that walk is a walk length.

Definition 1.4. A walk is a *path* when every edge in a walk are different. A walk or a path is closed when $v_0=v_m$. A closed path with at lest one edge is a *cycle*.

Definition 1.5. A *value of path* $(v_0,v_1),(v_1,v_2),\ldots,(v_{m-1},v_m)$ is a product of values assigned to every edge in a path, that is:

$$\prod_{k=0}^{m} x_{v_{k+1}}(v_k).$$

By $P_{(i,j)}(g)$ we denote a set of all paths from i to j in graph g. Let $p\in P_{(i,j)}(g)$ be any path from i to j; by $b(p)$ we denote a value of this path.

Payoffs

Definition 1.6. A benefit $b(i,j)$ that player i receives from accessing j is the maximum value of all paths from i to j in g,

$$b(i,j)=\max_{p\in P_{i,j}(g)} b(p).$$

Definition 1.7. A *payoff* of player $i\in N$ is a sum of every benefit the player receives from accessing every other player in g, i.e.

$$\varphi_i(g)=\sum_{j\in N} b(i,j).$$

The above construction of the payoff function reflects the intuition that the benefit a player receives from accessing other players, decreases with the distance between the players.

Example 1.1. Let $g=<N,E>$ with the set of players $N=\{1,2,3\}$ and the set of edges $E=\{(1,2),(2,3)\}$. Then, according to the model, a value **X** will be assigned to every edge. A benefit that player 1 receives from accessing player 2 is $b(1,2)=X$, whereas a benefit from accessing player 3 is $b(1,2)=X\cdot X=X^2$ (we omit *max* since there is only one path from 1 to 3).

Network efficiency. Nash equilibrium

The welfare value of the network

Definition 1.8. The *welfare value* of the network is a sum of all payoffs, i.e. $W(g)=\sum_{i\in N}\varphi_i(g)$.

We say that a network is effective when there is no other network which welfare value is greater.

Nash equilibrium

Since there is a one-to-one mapping between a network g and a strategy profile $x\in X$, we can write $g=g(x(1),x(2),\ldots,x(n))$.

Definition 1.9. A strategy profile $x^*=(x^*(1),\ldots,x^*(n))$ is a Nash equilibrium, if

$$\varphi_i(g(x^*(1),\ldots,x^*(n)))\geq$$
$$\geq\varphi_i(g(x^*(1),\ldots,x(i),\ldots,x^*(n)))$$

for all $i\in N$ and for all $x(i)\in X(i)$.

2. The Network Model with Complete Information of the Network Structure

In this section we will define a network construction game with incomplete information of the network structure. We considered a graph $g =< N,E >$, with a set of players N and a set of links E.

Player's knowledge of the network structure

Definition 2.1. A player's $i\in N$ knowledge of the network structure is a subgraph $w_i =< N_i,E_i >$ of graph g. We require that w_i includes every direct neighbours of i and every edge incident with i.

When $w_i = g$ one can say that i's knowledge of the network structure is complete. Otherwise its knowledge of the network structure is indeed incomplete.

Remark 2.1. Sum of knowledge of all players covers $g =< N,E >$.

Further on we will assume that a set N_i can only expand by adding new vertices. A vertice can not be deleted from N_i once it is in it. On the other hand, edges can be added to or deleted from E_i, since the set E_i reflects the network structure.

Pure strategies

In the network model with complete information a player has a knowledge of the network structure and makes his decisions on its basis. The only constraint is the resource **X**, which can be consumed to create links. Whereas in the following model with incomplete information, a player can add or delete links with players who belong to knowledge $w_i =< N_i,E_i >$.

Let $N_i=\{k_1,k_2,\ldots,k_{n_i}\}$ and n_i be the number of elements the set, i.e. $n_i=|N_i|$. Remind that $i\in N_i$.

Definition 2.2. A pure strategy of player $i\in N$ is a vector $x(i)=(x_j(i))_{j\in N_i}$ with n_i coordinates enumerated with elements of N_i, so that $x_j(i)\in[0,X]$ for all $j\in N_i$ and

$$x_i(i)=0$$

and

$$\sum_{j\in N_i} x_j(i)=X .$$

The set of all pure strategies of $i\in N$ will be denoted by $X(i)$.

Payoffs

We introduce the following definition of the payoff function, analogous to the definition in the model with complete information.

Definition 2.3. A sequence of edges $(v_0,v_1),(v_1,v_2),\ldots,(v_{m-1},v_m)$ in graph w_i is a *walk*, when every next edge is incident.

Definition 2.4. A walk is a *path* when all edges in a walk are different.

By $P_{(i,j)}(w_i)$ we denote a set of all paths from i to j in graph w_i.

Definition 2.5. A *value of path* $(v_0,v_1),(v_1,v_2),\ldots,(v_{m-1},v_m)$ in graph w_i is a product of values assigned to every edge in a path, that is:

$$\prod_{k=0}^{m} x_{v_{k+1}}(v_k).$$

By $P_{(i,j)}(w_i)$ we denote a set of all paths from i to j in graph w_i. Let $p\in P_{(i,j)}(w_i)$ be any path from i to j in w_i; by $b(p)$ we denote a value of this path.

Preposition 2.1. $P_{(i,j)}(w_i)\subseteq P_{(i,j)}(g)$.

Definition 2.6. A benefit $b(i,j)$ that player i receives from accessing j is the maximum value of all paths from i to j in w_i,

$$b(i,j)=\max_{p\in P_{(i,j)}(w_i)} b(p).$$

Definition 2.7. A *payoff* of player $i\in N$ is a sum of every benefit the player receives from accessing every other player in w_i, i.e.

$$\varphi_i(g)=\sum_{j\in N_i} b(i,j) .$$

Remark 2.2. Basing on Preposition 2.1 one can make a hypothesis that a player's $i \in N$ payoff in the complete information model is no less than his/her payoff in the model with incomplete information. The intuition is that the w_i subgraph can include at most all paths that g includes. Therefore, the *max* in Definition 2.5 is taken from a smaller set of paths.

Equilibrium

We define an equilibrium as an analogue to the equilibrium in the complete information model taking into account that the player's knowledge of network structure is limited to the subgraph w_i.

Let $w_i = < N_i, E_i >$ and $n_i = |N_i|$. We can enumerate the elements of N_i in the following way: $N_i = \{k_1, k_2, \dots, k_{n_i}\}$ (a reminder that there exists such an index j that $k_j = i$. We can write:

$$w_i = w_i(x(k_1), x(k_2), \dots, x(k_{n_i}))$$

which means that w_i depends only on the elements of N_i.

Definition 2.8. A strategy profile $x^* = (x^*(k_1), x^*(k_2), \dots, x^*(k_{n_1}))$ is an equilibrium according to knowledge w_i, when

$$\varphi_i(w_i(x^*(k_1), x^*(k_2), \dots, x^*(k_{n_1}))) \geq$$
$$\geq \varphi_i(w_i(x^*(k_1), x^*(k_2), \dots, x(k_j), \dots, x^*(k_{n_1})))$$

for all $k_i \in N_i$ and for all $x(k_i) \in X(k_i)$.

3. A Game with Complete Information of Network Structure as a Special Case of a Game with Incomplete Information

A game model with complete information of the network structure consists of:

- graph $g = < N, E >$
- strategy sets $x(i)$ defined for all $i \in N$
- payoff functions $\varphi_i(g)$ defined for all $i \in N$.

Whereas a game model with incomplete information consists of:

- graph $g = < N, E >$
- a family $\{w_i\}_{i \in N}$ of subgraphs of g
- strategy sets $x(i)$ defined for all $i \in N$ and for all w_i
- payoff functions $\varphi_i(w_i)$ defined for all $i \in N$ and for all w_i.

One can notice that the model with incomplete information is indeed an extension of the model with complete information. In a special case, when $w_i = g$ for all $i \in N$ (i.e. the state of knowledge of every player is equal to g) then:

1. Since $N = N_i$ for all $i \in N$, then $x(i) = (x_j(i))_{j \in N_i} = (x_j(i))_{j \in N}$ which means that the definition of strategy in the model with incomplete information is identical with the definition of strategy in the complete information model. This is because in both cases the strategies are an n-element vectors, which fulfill the same conditions.
2. If $w_i = g$, then $N_i = N$ for all $i \in N$. Then the payoff function meets the condition:
$$\varphi_i(w_i) = \sum_{j \in N_i} b(i, j) = \sum_{j \in N} b(i, j) = \varphi_i(g)$$
for all $i \in N$.
3. A reminder that the knowledge of every player in the complete information model is equal to g. Then
$$g = g(x^*(i), \dots, x^*(n)) =$$
$$= w_i(x^*(k_1), \dots, x^*(k_{n_i})) = w_i$$
for all $i \in N$.

   The condition:
$$\varphi_i(w_i(x^*(k_1), x^*(k_2), \dots, x^*(k_{n_i}))) \geq$$
$$\geq \varphi_i(w_i(x^*(k_1), x^*(k_2), \dots, x(k_j), \dots, x^*(k_{n_i})))$$
for all $i \in N$ and for all $x(i) \in X(i)$, can be written in the following way:
$$\varphi_i(g(x^*(1), \dots, x^*(n))) \geq$$
$$\geq \varphi_i(g(x^*(1), \dots, x(i), \dots, x^*(n)))$$
for all $i \in N$ and for all $x(i) \in X(i)$, which is the definition of the equilibrium in the complete information model.

4. Bibliography

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Transformacja modelu z pełną informacją o sieci użytkowników do modelu z niekompletną informacją Podejście wykorzystujące narzędzia teorii gier

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W tym artykule zajmiemy się modelowaniem sieci skierowanych, to znaczy przedstawionych za pomocą grafów skierowanych. Przez „informację” (w raczej ogólnym użyciu tego słowa) będziemy rozumieć dobro, którego przepływ następuje w sieci. Każdy uczestnik jest obdarzony pewnym zasobem, ale posiada również funkcję wypłaty, która wprost zależy od ilości informacji, do których dany uczestnik ma dostęp. Znając strukturę sieci gracze przez ustanowienie połączeń do innych uczestników uzyskują dostęp do posiadanej przez nich informacji. Problem polega na określeniu, jakie konfiguracje połączeń mogą prowadzić do równowagi oraz czy takie konfiguracje są optymalne (efektywne) [3], [2].
W dalszej części wprowadzamy model, w którym gracze nie posiadają wiedzy na temat struktury całej sieci a jedynie pewnego fragmentu. Decyzje podejmowane są na podstawie cząstkowej (niekompletnej) informacji. Podstawowym problemem jest zdefiniowanie równowagi w takim modelu a następnie zbadanie, jakie postępowanie prowadzi do równowagi.

Słowa kluczowe: formowanie sieci, równowaga Nash'a, niekompletna informacja o strukturze sieci.