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Determination of the Probability Distribution of the Mean Sound Level

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Assessment of several noise indicators are determined by the logarithmic mean $L_{\text{mean}} = S_n = 10\log\left(\frac{1}{n}\sum_{i=1}^{n} 10^{0.1L_i}\right)$, from the sum of independent random results $L_1, L_2, \ldots, L_n$ of the sound level, being under testing. The estimation of uncertainty of such averaging requires knowledge of probability distribution of the function form of their calculations. The developed solution, leading to the recurrent determination of the probability distribution function for the estimation of the mean value of noise levels and its variance, is shown in this paper.

Keywords: acoustic measurements, statistic analysis of the obtained results, estimation of the distribution, uncertainty.

1. Introduction

According to the requirements of the experimental practice (including standard: PN-EN ISO 10012 (2004)), each investigated vibroacoustic process should have the uncertainty estimation of the analysed variable. Apart from the uncertainty of measuring the equipment calibration, all components of uncertainty, substantial in the given measuring process, should be taken into account in this estimation. In the majority of cases, the problem of uncertainty estimation is reduced to calculation of the standard deviation estimate of the examined variable. It can be related not only to the measurement result but also to the estimate of an arbitrary parameter of the probability distribution of the analysed random variable. Such estimation is referred to as the standard uncertainty of measurement U (*Guide to the Expression...*, 1995). It determines the limit of the uncertainty interval, to which a certain confidence level can be attributed:

$$P = P\{x_0 \in (x - U;\, x + U)\}. \tag{1}$$

It means a probability in which the actual value of the examined quantity can be found.

The acoustic pressure level L_i; $i = 1, 2, \ldots, n$ is the basic measurement variable – in vibroacoustic investigations – determining a noise source emission and the evaluation of the hazards of working environment, or in the surroundings of environment of the tested source. This level determines the analysed acoustic event in the given point of the measurement space for the determined time instant. Its representation is the random variable.

Thus, for the estimation of each class of acoustic event determined by the acoustic pressure levels $L_{A\,i}$, $i = 1, 2, \ldots, n$, it is necessary to calculate their representations, in the logarithmic mean form of the value of the sound level $L_{A\,\mathrm{mean}}$, given by the following relationship:

$$L_{A\,\mathrm{mean}} = 10 \log \left[\frac{1}{n} \sum_{i=1}^{n} 10^{0.1 L_{A\,i}} \right]. \tag{2}$$

Usually, the estimations of $L_{A\,\mathrm{mean}}$ are being done on the basis of limited measurements, resulting from the assumed time schedule of control tests. The problem of assessing their likelihood is related to their estimations. The following question occurs: how to estimate unknown parameters of expected value $E(L_{A\,\mathrm{mean}})$ and variance $\mathrm{Var}(L_{A\,\mathrm{mean}})$ on the basis of random realizations of $L_{A\,i}$; $i = 1, 2, \ldots, n$.

In the estimations of the expected value $E(L_{A\,\mathrm{mean}}) = \mu$ (which is calculated from their random n-independent estimations $L_{A\,\mathrm{mean}\,i}$; $i = 1, \ldots n$, performed at conditions of repeatability of measurements), the estimator of the arithmetic mean is used.

Sufficient application of this estimator is determined by the assumption of the estimator, including the independence and normality of evaluations of $L_{A\,\mathrm{mean}\,i}$; $i = 1, \ldots n$.

The normality condition which occurs in the applied solutions is difficult to accept. Hence, correct solutions of this problem are being looked for. In (Batko, Bal, 2008; 2010) we can find one possible way.

This solution makes the estimating process of expected value for the logarithmic mean of acoustic pressure levels $E(L_{A\,\mathrm{mean}})$ and variance $\mathrm{Var}(L_{A\,\mathrm{mean}})$, independent of the normality of the distribution result of consecutive control evaluations, or from independence of results in the next tests, by accepting some kind of mechanism which describes the changes in consideration of control variables. It makes its identification and approximation by R.G. Brown's Exponential Smoothing adaptation model. The solution (Batko, Bal, 2008) protects the estimation process of searched quantities, in conditioning more easy verification, determining its realization of assumptions (Batko, Bal, 2010).

The most universal solution of the problem and calculations of expected value $E(L_{A\,\mathrm{mean}})$ and variance $\mathrm{Var}(L_{A\,\mathrm{mean}})$, is determining of the form of probability distribution for the function which describes the mean sound level. This problem has not been solved yet in literature of the subject. The aim of this work is to solve the above problem. The algorithm determining the analytic form of this function is given in this article.

2. Procedure of determining the probability distribution for the logarithmic mean of the sound level measurement results

Determination of distribution of random variable

$$L_{\mathrm{mean}} = S_n = 10\log\left(\frac{1}{n}\sum_{i=1}^{n} 10^{0.1L_i}\right),$$

defined by values of L_1, L_2, ..., L_n of tested sound levels, which are independent random variables with equal distribution $\rho(x)$ and distribution function $\Psi(x)$, $x \in \langle 0, +\infty)$, can be defined by the following relations.

We need to determine the recurrent formula for probability distribution for the logarithmic mean of the sound level. We have to divide our reasoning into a few parts. First let L_1, L_2, ..., L_n be independent random results of the tested sound level with equal distribution $\rho(x)$ and distribution function $\Psi(x)$, $x \in \langle 0, +\infty)$.

Related to them exposures $X_i = 10^{0.1L_i}$, $i = 1, \ldots, n$ are also independent random variables X_i, $i = 1, \ldots n$ due to the independence of variables L_i. Let the distribution of random variable X_i be determined by $\rho_{X_i}(\cdot)$ and a distribution function by $\Psi_{X_i}(\cdot)$.

$$\begin{aligned}\Psi_{X_i}(y) = P(X_i < y) = P(10^{0.1L_i} < y) &= P(0.1L_i < \log y)\\ &= P(L_i < 10\log y) = \Psi(10\log y);\end{aligned}$$

as a deduction

$$\Psi_{X_i}(y) = \Psi(10\log y), \qquad y \in \langle 1, +\infty),$$

therefore

$$\rho_{X_i}(y) = \rho(x) = \frac{10}{\ln 10}\frac{\rho(10\log y)}{y}, \qquad y \in \langle 1, +\infty). \tag{3}$$

Let us assume

$$\begin{aligned}&Y_1 = X_1,\\ &Y_2 = X_1 + X_2,\\ &Y_3 = X_1 + X_2 + X_3,\\ &\ldots\\ &Y_n = X_1 + X_2 + \ldots + X_n.\end{aligned}$$

Distribution of the random variable Y_i is determined by $\rho_{Y_i}(z)$ (Billingsley, 1986; Gersternkorn, Śródka, 1983) from (3)

$$\rho_{Y_1}(z) = \rho_{X_1}(z) = \frac{10}{\ln 10}\frac{\rho(10\log z)}{z},$$

for $z \in \langle 1, +\infty)$, therefore

$$\begin{aligned}\rho_{Y_2}(z) &= \int\limits_1^{z-1} \rho_{Y_1}(x)\rho_{X_2}(z-x)\,\mathrm{d}x \\ &= \Big(\frac{10}{\ln 10}\Big)^2 \int\limits_1^{z-1} \frac{\rho(10\log x)}{x}\frac{\rho(10\log(z-x))}{z-x}\,\mathrm{d}x,\end{aligned} \tag{4}$$

for $z \in \langle 2, +\infty)$,

$$\begin{aligned}\rho_{Y_3}(z) &= \int\limits_2^{z-1} \rho_{Y_2}(x_1)\rho_{X_3}(z-x_1)\,\mathrm{d}x_1 \\ &= \Big(\frac{10}{\ln 10}\Big)^3 \int\limits_2^{z-1}\Bigg(\int\limits_1^{x_1-1} \frac{\rho(10\log x)}{x}\frac{\rho(10\log(x_1-x))}{x_1-x}\,\mathrm{d}x\Bigg)\frac{\rho(10\log(z-x_1))}{z-x_1}\,\mathrm{d}x_1,\end{aligned} \tag{5}$$

for $z \in \langle 3, +\infty)$,

$$\dots$$

$$\begin{aligned}\rho_{Y_n}(z) &= \int\limits_{n-1}^{z-1} \rho_{Y_{n-1}}(x_{n-2})\rho_{X_n}(z-x_{n-2})\,\mathrm{d}x_{n-2} \\ &= \Big(\frac{10}{\ln 10}\Big)^n \int\limits_{n-1}^{z-1}\int\limits_{n-2}^{x_{n-2}-1}\dots\int\limits_2^{x_2-1}\Bigg[\int\limits_1^{x_1-1}\Big(\frac{\rho(10\log x)}{x}\frac{\rho(10\log(x_1-x))}{x_1-x}\Big)\mathrm{d}x \\ &\qquad \cdot \frac{\rho(10\log(x_2-x_1))}{x_2-x_1}\Bigg]\,\mathrm{d}x_1 \dots \frac{\rho(10\log(x_{n-1}-x_{n-2}))}{x_{n-1}-x_{n-2}}\,\mathrm{d}x_{n-2},\end{aligned} \tag{6}$$

for $z \in \langle n, +\infty)$.

Now from Eqs. (4), (5) and (6) we will receive explicit equations for probability distribution S_n.

Determining:

$$S_1 = 10\log(Y_1),$$
$$S_2 = 10\log\left(\frac{1}{2}Y_2\right),$$
$$\ldots$$
$$Sn = 10\log\left(\frac{1}{n}Y_n\right),$$

for which S_i distribution is defined $\rho_{S_i}(s)$

$$S_1 = 10\log Y_1 = 10\log X_1 = L_1,$$

$$\rho_{S_1}(s) = \rho_{L_1}(s) = \rho(s), \tag{7}$$

$$S_2 = 10\log\frac{1}{2}Y_2 \iff Y_2 = 2\cdot 10^{0.1S_2}. \tag{8}$$

From (4) and (8), the variable S_2 has a distribution which can be presented as follows:

$$\begin{aligned}\rho_{S_2}(s) &= 2\cdot 0.1\cdot \ln 10\cdot 10^{0.1s}\rho_{Y_2}(2\cdot 10^{0.1s})\\ &= 2\cdot\frac{10}{\ln 10}\cdot 10^{0.1s}\int\limits_{1}^{2\cdot 10^{0.1s}-1}\frac{\rho(10\log x)}{x}\frac{\rho(10\log(2\cdot 10^{0.1s}-x))}{2\cdot 10^{0.1s}-x}\,\mathrm{d}x.\end{aligned} \tag{9}$$

Assuming:

$$S_3 = 10\log\frac{1}{3}Y_3 \iff Y_3 = 3\cdot 10^{0.1S_3}. \tag{10}$$

from Eq. (5) and (10), the probability distribution for his variable S_3 is defined as follows:

$$\begin{aligned}\rho_{S_3}(s) &= 3\cdot 0.1\ln 10\cdot 10^{0.1s}\rho_{Y_3}(3\cdot 10^{0.1s})\\ &= 3\cdot\left(\frac{10}{\ln 10}\right)^2\cdot 10^{0.1s}\int\limits_{2}^{3\cdot 10^{0.1s}-1}\int\limits_{1}^{x_1-1}\frac{\rho(10\log x)}{x}\frac{\rho(10\log(x_1-x))}{x_1-x}\,\mathrm{d}x\\ &\qquad\cdot\frac{\rho(10\log(3\cdot 10^{0.1s}-x_1))}{3\cdot 10^{0.1s}-x_1}\,\mathrm{d}x_1,\end{aligned} \tag{11}$$

where $s\in\langle 0,+\infty)$

$$\ldots$$

$$S_n = 10\log\frac{1}{n}Y_n \iff Y_n = n\cdot 10^{0.1S_n}. \tag{12}$$

Using Eqs. (4) and (10), we receive the probability distribution of sum of the independent random variable. S_n is defined by the following relation:

$$\rho_{S_n}(s) = n \cdot 0.1 \ln 10 \cdot 10^{0.1s} \rho_{Y_n}(n \cdot 10^{0.1s}) = n \cdot \left(\frac{10}{\ln 10}\right)^{n-1} \cdot 10^{0.1s}$$

$$\cdot \int\limits_{n-1}^{n \cdot 10^{0.1s}-1} \int\limits_{n-2}^{x_{n-2}-1} \ldots \left(\int\limits_{1}^{x_1 - 1} \frac{\rho(10\log x)}{x} \frac{\rho(10\log(x_1 - x))}{x_1 - x} \, \mathrm{d}x \right)$$

$$\cdot \frac{\rho(10\log(x_2 - x_1))}{x_2 - x_1} \, \mathrm{d}x_1 \ldots \frac{\rho(10\log(n \cdot 10^{0.1s} - x_{n-3})}{n \cdot 10^{0.1s} - x_{n-3}} \, \mathrm{d}x_{n-2}, \tag{13}$$

where $s \in \langle 0, +\infty)$.

To get the recurrent formula for distribution of S_n, we will use partial calculation and explicit formula for the probability, described by Eqs. (7), (9), (11) and (13).

Putting to (9) correlations (3) and (7), we obtain:

$$\rho_{S_2}(s) = 2 \cdot \left(\frac{10}{\ln 10}\right) \cdot 10^{0.1s} \int\limits_{1}^{2 \cdot 10^{0.1s}-1} \rho_{S_1}(10\log x) \frac{\rho(10\log(2 \cdot 10^{0.1s} - x))}{x(2 \cdot 10^{0.1s} - x)} \, \mathrm{d}x. \tag{14}$$

Distribution of random variable S_2 is defined by (9), substituting

$$s = 10\log \frac{1}{2} x_1$$

we obtain

$$\rho_{Y_2}(x_1) = \frac{\rho_{S_2}\left(10\log \frac{1}{2} x_1\right)}{0.1 \cdot \ln 10 \cdot x_1}. \tag{15}$$

Therefore to Eq. (11) we put (3) and (15). We receive distribution of S_3:

$$\rho_{S_3}(s) = 3 \cdot \left(\frac{10}{\ln 10}\right) \cdot 10^{0.1s} \int\limits_{2}^{3 \cdot 10^{0.1s}} \rho_{S_2}\left(10\log \tfrac{1}{2} x_1\right) \frac{\rho(10\log(3 \cdot 10^{0.1s} - x_1))}{x_1(3 \cdot 10^{0.1s} - x_1)} \, \mathrm{d}x_1. \tag{16}$$

For the distribution of the random variable S_{n-1} we have (13). Substituting

$$s = 10\log \frac{1}{n-1} x_{n-1}$$

we receive:

$$\rho_{Y_{n-1}}(x_{n-2}) = \frac{\rho_{S_{n-1}}\left(10\log \frac{1}{n-1} x_{n-2}\right)}{0.1 \cdot \ln 10 \cdot x_{n-2}}; \tag{17}$$

putting to (13) Eqs. (3) and (17) we obtain:

$$\rho_{S_n}(s) = n \cdot \left(\frac{10}{\ln 10}\right) \cdot 10^{0.1s} \int\limits_{n-1}^{n10^{0.1s}} \rho_{S_{n-1}}\left(10 \log \frac{1}{n-1} x_{n-2}\right) \cdot \frac{\rho(n \cdot 10^{0.1s} - x_{n-2})}{x_{n-2}(n \cdot 10^{0.1s} - x_{n-2})} \, \mathrm{d}x_{n-2}. \tag{18}$$

Equations (7) and (18), giving the recurrent formula for distribution of S_n, were the expected value and variance is given by (19), (20) and (21)

$$ES_n = \int\limits_0^{+\infty} s\rho_{S_n}(s) \, \mathrm{d}s, \tag{19}$$

$$ES_n^2 = \int\limits_0^{+\infty} s^2\rho_{S_n}(s) \, \mathrm{d}s, \tag{20}$$

$$\mathrm{Var}S_n = ES_n^2 - (ES_n)^2. \tag{21}$$

3. Conclusion

Determination the probability distribution for the basic vibroacoustic operation, which is the calculation of the logarithmic mean of the sound level measurement results – presented in this paper, constitutes broadening of the actually used assessment methods of uncertainty of the acoustic measurements results. It is free from the arbitral and difficult to be accepted assumptions (Kirpluk, 2008).

The knowledge of the form of probability distribution of the controlled measurement results was assumed at defining the realisation of the derivation of the solution proposed in the paper. Broadening of the presented solution by means of their detailed specifications will enable the transformation of the general formulas – given in the paper – into the tabular forms of the probability distribution of the mean sound level values. Thereby, their wider practical application in procedures of uncertainty assessments of mean value calculations will become possible. This problem will constitute the next stage of investigations and further developments of the topic.

The derivation given in the present paper seems to be a good illustration and contains important directions for searching for estimation formulas of uncertainty of other noise indicators.

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