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Unconventional procedure for inversion of polarimetric data: numerical calculation for a simple model of toroidal plasma

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Abstract. A gradient method is proposed for inversion of polarimetric data for tokamak plasma. It is assumed that the electron density and the normalized magnetic profile along the ray are known from experiments (scattering and magnetic measurements) in the spirit of the EFIT approach. Under these assumptions the maximum value of the electron density and maximum value of the magnetic field may be estimated by the gradient method. It is shown that with this method it is possible to achieve accuracy of about 0.1% already after 3–4 iterations.

Key words: Cotton-Mouton effect • Faraday effect • gradient method

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Introduction

A recently proposed approach to the inversion of polarimetry data [4, 5] is based on the angular variable technique (AVT) [2]. In this paper we outline an alternative method for this task, based on the gradient approach.

Basic equations for the inversion are presented in section 'The algorithm for inversion'. A numerical procedure for finding solutions of these equations by means of the gradient method is described in section 'Gradient method for inversion of the polarimetry data'. Section 'Numerical example' contains a numerical example illustrating implementation of the new inversion procedure. In 'Discussion' we discuss some aspects of the new approach.

The algorithm for inversion

The algorithm for inversion proposed in [4, 5] is based on the angular variables technique (AVT), which involves angular variables ψ (the azimuthal angle) and χ (the ellipticity angle). The evolution equations for the angular parameters of the polarization ellipse in a magnetized plasma have the form [2]

$$(1)\quad \begin{cases} \dot{\psi} = (1/2)\left[\Omega_3 - (\Omega_1 \cos 2\psi + \Omega_2 \sin 2\psi)\ \ \mathrm{tg}2\chi\right] \\ \dot{\chi} = (1/2)(\Omega_1 \sin 2\psi - \Omega_2 \cos 2\psi) \end{cases}$$

The derivatives of ψ and χ are taken over the arc length along the ray, denoted σ:

$$\dot{\psi} = \frac{d\psi}{d\sigma}, \quad \dot{\chi} = \frac{d\phi}{d\sigma}$$

The constans Ω_1 and Ω_2 are plasma parameters that characterize the Cotton-Mouton effect (see [10])

$$\Omega_1 = \Omega_\perp \cos 2\alpha_\perp \quad (2)$$
$$\Omega_2 = \Omega_\perp \sin 2\alpha_\perp$$
$$\Omega_\perp = k_0 X Y^2 \sin^2 \alpha_{||}$$

and Ω_3 characterizes the Faraday effect [10]:

$$\Omega_3 = ½\, k_0 XY \cos \alpha_{||} \quad (3)$$

The parameters $\Omega_{1,2,3}$ depend on the wave number $k_0 = \omega/c$ of the sounding beam, the longitudinal ($\alpha_{||}$) and transverse ($\alpha_\perp$) angles characterizing orientation of the beam relative to the static magnetic field B_0 [2], and dimensionless plasma parameters

$$X = \frac{\omega^2_{pl}}{\omega^2}, \quad Y = \frac{\omega_{ce}}{\omega} \quad (4)$$

Because of dependence on the squared plasma frequency ω^2_{pl}, the parameter X is in effect proportional to the electron density N_e, and because of the dependence on the electron cyclotron frequency the parameter Y is proportional to the magnetic field B, i.e.

$$X \propto N_e, \quad Y \propto B \quad (5)$$

Following the knowledge-based approach [1, 2] we write the parameters X and Y in the form

$$X(\sigma) = \bar{X} \cdot x(\sigma), \quad Y(\sigma) = \bar{Y} \cdot y(\sigma) \quad (6)$$

where X and Y are the maximum values and $x(\sigma), y(\sigma)$ are normalized profiles of the electron density N_e and the magnetic field B along the ray.

The profiles $x(\sigma)$ and $y(\sigma)$ are assumed to be extracted from the Thomson and lidar scattering, and external magnetic measurements.

From Eq. (1) we obtain angular variables ψ and χ as functions of plasma parameters X and Y:

$$\psi = \psi(\bar{X},\bar{Y}), \quad \chi = \chi(\bar{X},\bar{Y}) \quad (7)$$

Equating solutions (7) to the experimental values ψ_{exp} and χ_{exp} we come to the following set of equations:

$$\begin{cases} \psi(\bar{X},\bar{Y}) = \psi_{exp} \\ \chi(\bar{X},\bar{Y}) = \chi_{exp} \end{cases} \quad (8)$$

On the basis of these equations we may construct an algorithm that gives the plasma parameters $\bar{X}$, and $\bar{Y}$ that correspond to the experimental values ψ_{exp} and χ_{exp} for the given normalized profiles $x(\sigma)$ and $y(\sigma)$:

$$\begin{bmatrix} \text{polarimetric data } \psi_{exp}, \chi_{exp} \\ \text{normalized profiles } x(\sigma), y(\sigma) \end{bmatrix} \rightarrow \bar{X}, \bar{Y}$$

In fact, this set of equations may be reduced to the following single equation

$$\Phi(\bar{X},\bar{Y}) = [\psi(\bar{X},\bar{Y}) - \psi_{exp}]^2 + [\chi(\bar{X},\bar{Y}) - \chi_{exp}]^2 = 0 \quad (9)$$

The function $\Phi(\bar{X},\bar{Y})$ will be named hereafter as the "mismatch function".

Equation (9) admits an analytic solution only in the case of small ψ_{exp} and χ_{exp} [4] when the Eq. (1) can be solved using perturbative approach.

In the general case a numerical approach has to be pursued. In section 'Gradient method for inversion of the polarimetry data' we outline the gradient method for inverting the system (8) and obtaining the parameters $\bar{X}$ and $\bar{Y}$.

Gradient method for inversion of the polarimetry data

Let $\mathbf{R} = \mathbf{i}_x\bar{X} + \mathbf{i}_y\bar{Y}$ be a vector in the $(\bar{X},\bar{Y})$ plane. Using this vector, we can rewrite the mismatch function defined by Eq. (9) as follows:

$$\Phi(\mathbf{R}) = [\psi(\mathbf{R}) - \psi_{exp}]^2 + [\chi(\mathbf{R}) - \chi_{exp}]^2 = 0 \quad (10)$$

The numerical procedure for solving this equation by the gradient method [9] is based on successive displacements of the point $\mathbf{R}$ by small a step $\Delta\mathbf{R}$ in the direction opposite to the gradient of the mismatch function

$$\mathbf{G} \equiv \nabla\Phi = \left(\mathbf{i}_x \frac{\partial\Phi}{\partial\bar{X}}, \mathbf{i}_y \frac{\partial\Phi}{\partial\bar{Y}}\right) \quad (11)$$

Let ΔR_G be a displacement in the direction of the gradient $G = \nabla\Phi$:

$$\Delta\mathbf{R}_G = |\Delta\mathbf{R}|\mathbf{G}/|\mathbf{G}| \quad (12)$$

Then a small step in the opposite direction will be

$$\Delta\mathbf{R}_G = -|\Delta\mathbf{R}_G|\mathbf{G}/|\mathbf{G}| \quad (13)$$

The components G_X and G_Y of the vector $\mathbf{G}$ can be estimated numerically via finite differences

$$\Delta_X\Phi = \Phi(\bar{X} + \Delta\bar{X}, \bar{Y}) - \Phi(\bar{X},\bar{Y}) \quad (14)$$
$$\Delta_Y\Phi = \Phi(\bar{X}, \bar{Y} + \Delta\bar{Y}) - \Phi(\bar{X},\bar{Y})$$

so that

$$G_X \approx \frac{\Delta_X\Phi}{\Delta\bar{X}}, \quad G_Y = \frac{\Delta_Y\Phi}{\Delta\bar{Y}} \quad (15)$$

Starting with a trial point $\mathbf{R}_0 = (\bar{X}_0,\bar{Y}_0)$ and moving in the direction opposite to $G = \nabla\Phi$ we arrive at the point

$$\mathbf{R}_1 = \mathbf{R}_0 - |\Delta\mathbf{R}|\mathbf{G}/\mathbf{G}| \quad (16)$$

At this point the mismatch function $\Phi(\mathbf{R}_1)$ takes the value

$$\Phi(\mathbf{R}_1) = \Phi(\mathbf{R}_0) - |\Delta\mathbf{R}||\mathbf{G}| \quad (17)$$

Repeating this operation several times we move close to the stationary point $\mathbf{R}_S = (\bar{X}_S,\bar{Y}_S)$, where $\Phi(\mathbf{R}_S) = 0$. This procedure will be convergent if the increment $|\Delta R||\mathbf{G}|$ of the function (9) does not exceed $\Phi(R_0)$:

$$|\Delta\mathbf{R}||\mathbf{G}| \leq \Phi(\mathbf{R}_0) \quad (18)$$

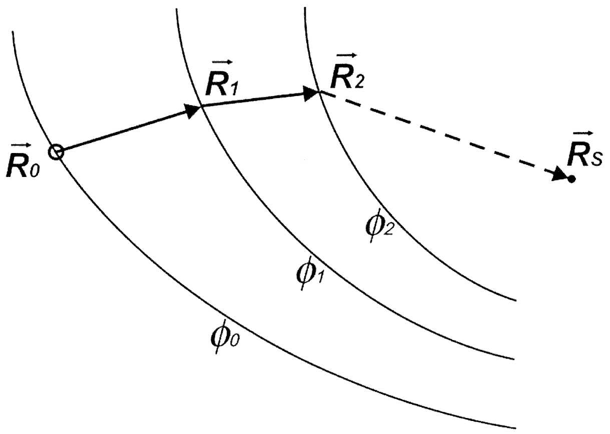

Fig. 1. Piece-wise linear trajectory $R_0 \to R_1 \to R_2 \to \ldots \to R_S$ in the $(\bar{X},\bar{Y})$ plane generated using the gradient method.

In what follows we select the step $|\Delta\mathbf{R}|$ according to condition

$$|\Delta\mathbf{R}||\mathbf{G}| = \tfrac{1}{2}\,\Phi(\mathbf{R}_0) \tag{19}$$

Qualitatively, the trajectory $R_0 \to R_1 \to R_2 \to \ldots \to R_S$ on the $(\bar{X},\bar{Y})$ plane looks as shown in Fig. 1.

According to Fig. 1, the piece-wise trajectory tends to the stationary point $(\bar{X}_{st},\bar{Y}_{st})$, where Eqs. (8) or (9) are satisfied.

$$\mathbf{R}_{st} = \begin{cases} \psi(\bar{X}_{st},\bar{Y}_{st}) = \psi_{\exp} \\ \chi(\bar{X}_{st},\bar{Y}_{st}) = \chi_{\exp} \end{cases} \tag{20}$$

Numerical example

In this Section we demonstrate the efficiency of the suggested method in the simplest magnetic configuration, when the sounding ray lays in the horizontal (equatorial) plane of the toroidal system. In this case the influence of the poloidal field is negligible and it is enough to take into account only the toroidal field (see Fig. 2).

We solve Eq. (1) with the following values of parameters, typical for the ITER project [11]:

i. The maximum value of magnetic field is assumed to be B_0 = 5.3 T, and the magnetic profile $y(\sigma)$ is supposed to correspond to the field of a toroidal solenoid

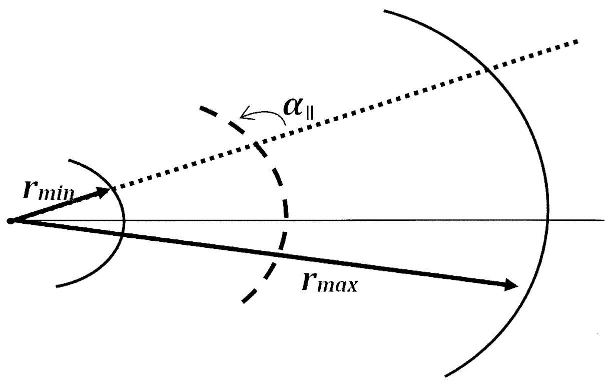

Fig. 2. The path of the sounding ray (bold dashed line) in the equatorial plane of a tokamak. Here $r_{\min}$ and $r_{\max}$ are the inner and the external radii of the toroidal chamber.

$$y(\sigma) = \frac{r_{\min}}{r_{\min} + \sigma} \tag{21}$$

where $r_{\min}$ is the minor radius of the toroidal chamber.

ii. The sounding frequency is assumed to be ω = 1.5 × 10^{13} Hz. The corresponding wavelength λ = 125 μm is comparable to the wavelength of 195 μm that had been used on JET.
iii. The angle between the ray and the magnetic field vector **B** is assumed to be 88.2°, so that $\cos\alpha$ = 0.031.
iv. The maximum electron density $N_e^{\max}$ is assumed to be 10^{14} cm^{-3}. The density profile $x(\sigma)$ is approximated by the Gaussian function

$$x(\sigma) = \exp[-(\sigma - \sigma_0)^2/g^2] \tag{22}$$

with g = 3. We have chosen the Gaussian profile (22) for illustrative purpose only. In a real plasma the profile $x(\sigma)$ should be extracted from the Thomson or lidar scattering data. In principle, the profile $x(\sigma)$ could be bimodal, that is with two local maxima.
v. The values of azimuthal and ellipticity angles, simulating the results of experimental measurements are obtained from Eq. (1) with the initial values $\psi(\sigma = 0) = \pi/4$ and $\chi(\sigma = 0) = 0$, as it is frequently done in polarimetry.
vi. The inner radius of the toroidal chamber is assumed to be $r_{\min}$ = 2 m and the external radius is equal to $r_{\max}$ = 8 cm, so that the length of the ray path inside the chamber is 6 m. The ray crosses the axis of the toroidal chamber at an angle α = 88.2°, so that $\cos\alpha$ = 0.031. Under these conditions the contributions from the Faraday and the Cotton-Mouton effects are comparable. Thus, the method of inversion is valid even if neither the Faraday nor the Cotton-Mouton effect are small.
vii. The parameters $\bar{X}_{st}$ and $\bar{Y}_{st}$, corresponding to the maximum values of magnetic field B = 5.3 T and the electron density $N_e = 10^{14}$ cm^{-3}, are found to be $\bar{X}_{st}$ = 0.0014, $\bar{Y}_{st}$ = 0.063. The starting values $\bar{X}_0$ and $\bar{Y}_0$ were approximately 30–40% smaller than the stationary values X_{st} and Y_{st}.

The values obtained for the parameters X, Y and the mismatch function $\Phi(\bar{X},\bar{Y})$ in this numerical procedure are presented in Table 1, and the first three steps of the iteration on the (X,Y) plane are shown in the Fig. 3.

After the third step we are so close the real values that points corresponding to further steps would be practically invisible in the scale of the figure.

According to Table 1, the iterative procedure converges to the stationary point $\bar{X}_{st}$ = 0.0014, $\bar{Y}_{st}$ = 0.063, relatively fast, so that the relative deviations $|\bar{X} - \bar{X}_{st}|/\bar{X}_{st}$ and $|\bar{Y} - \bar{Y}_{st}|/\bar{Y}_{st}$ become smaller than 2–3% already after 3–5 iterations.

Thus the gradient method used jointly with the knowledge-based model of the toroidal plasma provides an effective method for inversion of polarimetric data.

Discussion

1. Accounting for the poloidal magnetic field. By neglecting the poloidal field in the equatorial plane

Table 1. Evolution of the parameters $\bar{X}$, $\bar{Y}$, the values of azimuthal and ellipticity angles and the function $\Phi(\bar{X}, \bar{Y})$, as obtained in the gradient method

Step	X/Y	ϕ	ψ	χ
0	0.03/0.03	2.03	–0.772	0.0062
1	0.01/0.04	0.137	0.056	0.285
2	0.005/0.05	0.104	0.365	0.276
3	0.002/0.06	0.0051	0.591	0.175
4	0.0015/0.065	0.0008	0.630	0.155
5	0.0014/0.062	1.31×10^{-6}	0.647	0.132
Experiment		0	0.648	0.132

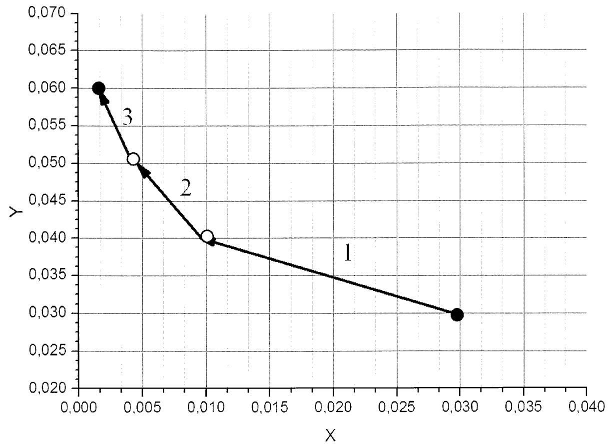

Fig. 3. First three steps of the numerical iterative procedure in the (X,Y) plane.

of the toroidal system we introduce only an insignificant error in polarimetry effect: as was shown in [3], the poloidal field is responsible only for 3–6% of the Cotton-Mouton effect and does not affect the Faraday effect in the geometry selected for this analysis.

2. The helical structure of magnetic lines, arising from the interaction of toroidal and poloidal fields. Experimental measurements of this structure can hardly be performed at present time. A reasonable solution to this problem might be to combine the experimental and theoretical values in the spirit of EFIT program [1, 6–8], which fits theoretical model of the equilibrium plasma to all accessible data. The inversion procedure for polarimetry data outlined above might be one of the steps in this direction.
3. Application of the Stokes vector formalism (SVF) in the inversion of polarimetry data. The method proposed in [4, 5] and the numerical procedure presented here are based on the angular variables technique (AVT), introduced in [2]. In principle, the authors hope that a similar approach can be implemented also in framework of the SVF approach with the squared difference $[s(\bar{X},\bar{Y}) - s_{\exp}]^2$ between theoretical and experimental values of the Stokes vector playing similar role as the mismatch function in the gradient method.
4. Sensitivity to experimental uncertainties. The measured angular parameters $\psi_{\exp}$ and $\chi_{\exp}$ are known with finite accuracy, reflected by the experimental uncertainties $\delta\psi_{\exp}$ and $\delta\chi_{\exp}$. The values δN_e and δB obtained from the inversion procedure depend also on the uncertainties in the Faraday and Cotton-Mouton effects, which are of comparable magnitude, and we may expect that the same is true for the relative uncertainties $\delta N_e/N_e$ and $\delta B/B$:

$$\frac{\delta N_e}{N_e} \sim \frac{\delta B}{B} \sim \frac{\delta\psi_{\exp}}{\psi_{\exp}} \sim \frac{\delta\chi_{\exp}}{\chi_{\exp}} \qquad (23)$$

However, the situation will change if one of these effects is much weaker than other one. The uncertainties in inversion problem deserve a more detailed analysis in future studies both in the context of polarimetry and in the context of more general aspects of the EFIT approach.

Conclusions

In this paper we discussed properties of an unconventional procedure for inversion of the polarimetry data for physical conditions characteristic for the ITER project. The fitting of the theoretical model to the experimental data is done numerically by the gradient method. It is shown that this gradient method provides acceptable accuracy of order 1–3% for 3–4 iterations. It is pointed out that this approach might be a helpful amendment to the EFIT approach.

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