Motion Planning of the Underactuated Manipulators with Friction in Constrained State Space

Adam Ratajczak, Mariusz Janiak

Abstract:

This paper addresses the constrained motion planning problem for passive joint manipulators with friction. Constraints are imposed on a system state space vector. The dynamics of underactuated manipulators are described by a control–affine system with a drift term. In order to solve the constrained motion planning problem the imbalanced Jacobian algorithm derived from an endogenous configuration space approach is used. The state space constraints are included into the system representation of the manipulator dynamics. The extended system is subject to regularisation because of the Jacobian singularities, then the unconstrained motion planning problem is solved for the regularised system. The solution of the motion planning problem for this system is equivalent to the solution of the constrained motion planning problem for an original system. Performance of the imbalanced Jacobian algorithm has been demonstrated with series of simulation for the two kinds of manipulators with and without friction.

Keywords: *Constrained Motion Planning, Underactuated Manipulators, Endogenous Configuration Space.*

1. Introduction

The typical example of an underactuated robotic system is the manipulator with one or more passive joints. In the classic manipulator every joint has its own actuator, so the number of degrees of freedom (d.o.f.) is equal to the number of control inputs. Contrary, in the underactuated case, the dimension of the state space exceeds the dimension of the control space. The general review of the underactuated manipulators and dedicated control strategies has been presented in [1] and [2]. The underactuated systems could be divided into the models with the gravity terms and the models where the potential terms are neglected. The popular systems belonging to the first group are two double pendulum manipulators: Acrobot [3] with free first joint and Pendubot [4] without an actuator in the second joint. The robot models in which the gravity terms are excluded are e.g. PR manipulator in [5] or RRR manipulator in [6]. Unfortunately, in many publications, the motion equations of underactuated robots are modelled as the frictionless systems. Few examples of the underactuated systems with friction could be found in [7–9]. For the purpose of this paper we enrol two types of manipulators. First, with the presence of gravity, and second one, where the gravity is neglected. For both models we consider two cases, frictionless and with friction.

A mechanical system whose motion is subject to non-integrable position and velocity constraints is called nonholonomic, and such constraints are the first order nonholonomic constraints. When the constraints assume the Pfaffian form, the nonholonomic system is represented by a smooth driftless control system. Examples of such systems are robotics systems, like wheeled mobile platforms or certain systems of dexterous manipulations. In contrast to these systems motion of manipulators with passive joints is subject to the second order nonholonomic constraints [10]. In this case, the system is represented by a control–affine system with non–trivial drift term, what makes the motion planning problem more complicated.

In order to solve the unconstrained motion planning problem for underactuated manipulator, the endogenous configuration space approach [11] has been applied in [10, 12]. With this approach, it is possible to generalise the Jacobian inverse kinematics algorithms, known for classic stationary manipulators, to other robotics systems. The fundamental concept of this method is that the endogenous configuration space includes all admissible controls of the robotic system. Originally, this method has been widely applied to mobile manipulators [11, 13], however, it can be easily adapt to manipulators with passive joints, because of the similarity between the equations of nonholonomic mobile platforms kinematics, and manipulator dynamics model. A different solution of unconstrained motion planning problem has been presented in [14] where the continuation method has been combined with the predictive control scheme. This paper deals with the constrained motion planning problem for underactuated manipulators with friction and bounded state. Such problem for frictionless underactuated manipulators has been solved in [15], where the imbalanced Jacobian algorithm [13] derived from the endogenous configuration space approach has been utilised. Alternative approach to solving the motion planning problem in presence of constraints is offered by optimal control theory, as proposed e.g. in [16], and in particular by the Nonlinear Model Predictive Control approach [17, 18]. The comparison of these two methods applied to solving the constrained motion planning problem for nonholonomic mobile robots has been presented in [19].

This paper extends the approach presented in [15] for underactuated manipulators with friction. Following [15], the constrained motion planning problem is replaced by an unconstrained one, addressed in an extended control system representation. State variable bounds are incorporated into the manipulator motion equations, through adding an extra state variable driven by the plus function depending on the violation of constraints. Then, the expanded system is subject to a regularizing perturbation in order to eliminate the singularities of the Jacobian introduced by the additional state variable and the associated plus function, in the region where the constraints are satisfied. The unconstrained motion planning problem in regularised system is solved by the imbalanced Jacobian algorithm [13, 20], derived from the endogenous configuration space approach. A feature of the imbalanced algorithm is that applies inverse Jacobian operator of the regularised system to the error produced in the extended system. A solution of the unconstrained motion planning problem for regularised system is also a solution of the constrained motion planning problem for basic system. Presented algorithm has been used to solve the constrained motion planning problem for two underactuated manipulators: an inverse pendulum mounted on a movable cart, and a planar double pendulum. The algorithm performance has been illustrated with the computer simulations.

The paper is organised as follows. Section 2 describes derivation of underactuated manipulator dynamics equations. The motion planning algorithm is presented in section 3. Performance of the algorithm is presented with computer simulations in section 4. The paper concludes with section 5.

2. Underactuated Manipulator with Friction

The dynamics of classic manipulator is defined as

$$M(q)\ddot{q} + N(\dot{q}, q) = F, \tag{1}$$

where $q \in \mathbb{R}^n$ is the vector of joint coordinates. The matrix $M(q)$ is the symmetry and positive definite inertia matrix, the vector $N(\dot{q}, q)$ collects centrifugal, Coriolis and possibly gravitational terms. The vector F denotes control forces and other external forces.

Let us set that only $m < n$ joints are actuated. Moreover we split the vector q into two parts: the active joints positions $q_a \in \mathbb{R}^m$ and the positions of the passive joints $q_b \in \mathbb{R}^{n-m}$. Then we can rewrite the dynamics (1) as

$$\begin{bmatrix} M_{aa}(q) & M_{ab}(q) \\ M_{ab}^{\mathrm{T}}(q) & M_{bb}(q) \end{bmatrix} \begin{pmatrix} \ddot{q}_a \\ \ddot{q}_b \end{pmatrix} + \begin{pmatrix} N_a(\dot{q}, q) \\ N_b(\dot{q}, q) \end{pmatrix} = \begin{pmatrix} \tau + F_a \\ F_b \end{pmatrix}, \tag{2}$$

where M_{aa}, M_{ab}, M_{bb}, N_a and N_b are elements of matrix $M(q)$ and vector $N(\dot{q}, q)$ from (1) suitably. The vector $\tau \in \mathbb{R}^m$ is the control vector, F_a and F_b denote the external forces acting on active and passive joints respectively.

2.1. Partial feedback linearization

Now, we perform a partial feedback linearization [21] of the system (2). By a substitution $\ddot{q}_b$ from second equation of (2) into first equation of (2) and introducing a new input vector $u \in \mathbb{R}^m$ we arrive with a feedback controller

$$\begin{aligned} \tau = &\big(M_{aa}(q) - M_{ab}(q)M_{bb}(q)^{-1}M_{ab}^{\mathrm{T}}(q)\big)u \\ &+ N_a(\dot{q}, q) - F_a - M_{ab}(q)M_{bb}(q)^{-1}\big(N_b(\dot{q}, q) - F_b\big), \end{aligned} \tag{3}$$

which transforms (2) into

$$\begin{cases} \ddot{q}_a = u, \\ \ddot{q}_b = -M_{bb}(q)^{-1}\big(M_{ab}^{\mathrm{T}}(q)u + N_b(\dot{q}, q) - F_b\big). \end{cases} \tag{4}$$

Finally, we take the new state space vector as $x = (x_1, x_2, x_3, x_4) = (q_a, \dot{q}_a, q_b, \dot{q}_b) \in \mathbb{R}^{2n}$ and the equations (4) takes the form

$$\dot{x} = f(x) + G(x)u, \tag{5}$$

where

$$f(x) = \big(x_2, 0, x_4, -M_{bb}^{-1}(x)(N_b(x) - F_b)\big)^{\mathrm{T}},$$

$$G(x) = \begin{bmatrix} 0 \\ I_m \\ 0 \\ -M_{bb}^{-1}(x)M_{ab}^{\mathrm{T}}(x) \end{bmatrix}.$$

Now, let us define the underactuated manipulator with friction. We consider a viscous friction model

$$F_f = \epsilon \frac{\mathrm{d}q}{\mathrm{d}t},$$

where F_f is a friction force and ϵ is a friction coefficient. Following [7] we take $F_a = 0$ and $F_b = -\epsilon\dot{q}_b$. We assume that the friction appears only in underactuated joints, because the friction in actuated joints could be compensated by the partial feedback linearization (3).

3. Motion Planning Algorithm

Given the control system representation (5), the following constrained motion planning problem will be addressed: starting from an initial state $x_0 = x(0) \in \mathbb{R}^{2n}$, find a control function $u(\cdot) \in \mathcal{L}_m^2[0, T]$, such that for the prescribed state $x_d \in \mathbb{R}^{2n}$ there holds $x(T) = x_d$, while the instantaneous values of state variables are bounded

$$x_i^{ub} \le x_i(t) \le x_i^{lb}, \tag{6}$$

for every $t \in [0, T]$, and for some, perhaps not all, $1 \le i \le 2n$, where x^{ub} and x^{lb} denote state upper and lower bounds respectively. In order to solve this problem we use so-called imbalanced Jacobian algorithm devised in [13, 20], originating from the endogenous configuration space approach [11].

3.1. Endogenous configuration space

In accordance to [11], admissible controls $u(\cdot)$ in the system (5), acting on time interval $[0, T]$, constitute the endogenous configuration space $\mathcal{X} = \mathcal{L}_m^2[0, T]$ of the underactuated manipulator. This space is an infinite dimensional Hilbert space with inner product

$$\big\langle u_1(\cdot), u_2(\cdot)\big\rangle_R = \int_0^T u_1^{\mathrm{T}}(t)R(t)u_2(t)\,\mathrm{d}t,$$

where $R(t)$ is a positive defined weight matrix. To every endogenous configuration $u(\cdot) \in \mathcal{X}$ there corresponds a manipulator state space trajectory $x(t) = \varphi_{x_0,T}\big(u(\cdot)\big)$. We shall assume that this trajectory is well defined for every $t \in [0, T]$, then we can define an end-point map $K_{x_0,T} : \mathcal{X} \longrightarrow \mathbb{R}^{2n}$ of the system (5), transforming endogenous configuration space into manipulator state space

$$K_{x_0,T}\big(u(\cdot)\big) = x(T) = \varphi_{x_0,T}\big(u(\cdot)\big). \tag{7}$$

Derivative of the end-point map

$$J_{x_0,T}\big(u(\cdot)\big)v(\cdot) = \mathrm{D}K_{x_0,T}\big(u(\cdot)\big)v(\cdot) = \left.\frac{\mathrm{d}}{\mathrm{d}\alpha}\right|_{\alpha=0} K_{x_0,T}\big(u(\cdot)+\alpha v(\cdot)\big) = \int_0^T \Phi(T,t)B(t)v(t)\,\mathrm{d}t, \quad (8)$$

will be referred to as the system Jacobian, where

$$A(t) = \frac{\partial}{\partial x}\Big(f(x) + G\big(x(t)\big)u(t)\Big),\ B(t) = G\big(x(t)\big)$$

are matrices of the linear approximation of the system (5) along the configuration $u(\cdot)$. The fundamental matrix $\Phi(t,s)$ fulfils the evolution equation $\frac{\partial}{\partial t}\Phi(t,s) = A(t)\Phi(t,s)$ with initial condition $\Phi(s,s) = I_{2n}$. For the fixed $u(\cdot)$, the Jacobian transforms the endogenous configuration space into manipulator state space $J_{x_0,T}\big(u(\cdot)\big) : \mathcal{X} \longrightarrow \mathbb{R}^{2n}$. Configurations at which this map is surjective are named regular, otherwise they are singular. Equivalently, $u(\cdot)$ is regular if and only if the Gram matrix

$$\mathcal{G}_{x_0,T}(u(\cdot)) = \int_0^T \Phi(T,t)B(t)B^{\mathrm{T}}(t)\Phi^{\mathrm{T}}(T,t)\mathrm{d}t. \quad (9)$$

is full rank $2n$.

3.2. Extension and regularization

One way of dealing with the constrained motion planning problem consists in replacing the constrained problem by an unconstrained one addressed in an extended control system representation. To include the state constraints (6) into the basic system, we shall describe them using the smooth approximation of the plus function $x_+ = \max\{0,x\}$ [22]

$$x_+ \approx p(x,\beta) = x + \frac{1}{\beta}\log(1+e^{-\beta x}).$$

Function $p(x,\beta)$ approaches x_+ when β increases to $+\infty$. The constraints will be satisfied, when the functions $p(x_i(t) - x_i^{ub},\beta)$ and $p(-x_i(t) + x_i^{lb},\beta)$ vanish for every $t \in [0,T]$, and any $\beta > 0$. This will also be held, whenever the sum of integrals over $[0,T]$ of those functions is zero.

The constraints can be included into system (5) by adding an extra state variable as follows

$$\begin{cases} \dot{x} = f(x) + G(x)u, \\ \dot{x}_{2n+1} = \sum\limits_{i=1}^{2n}\Big(p\big(x_i(t) - x_i^{ub},\beta\big) + p\big(-x_i(t) + x_i^{lb},\beta\big)\Big). \end{cases} \quad (10)$$

Let $\bar{x} = (x, x_{2n+1})$ denote the extended state variable, and $E_{\bar{x}_0,T}(u(\cdot))$ denote the end-point map of (10). Now, the solution of the unconstrained motion planning problem $\bar{x}(T) = (x_d, 0)$ for extended system, is also the solution of the constrained motion planning problem for basic system (5). Unfortunately, if the constraints are satisfied, the Jacobian of extended system becomes singular. In order to overcome this difficulty, system (10) will be subject to regularization. The new state variable is perturbed by the regularization function $r(x)$ that prevents the extended system Jacobian of became singular, usually we apply $r(x) = x^{\mathrm{T}}x$. As a result, we obtain a regularized system

$$\begin{cases} \dot{x} = f(x) + G(x)u, \\ \dot{x}_{2n+1} = \sum\limits_{i=1}^{2n}\Big(p\big(x_i(t) - x_i^{ub},\beta\big) + p\big(-x_i(t) + x_i^{lb},\beta\big)\Big) + r(x). \end{cases} \quad (11)$$

Set again $\bar{x} = (x, x_{2n+1})$ and let $R_{\bar{x}_0,T}(u(\cdot))$ denote the end-point map of (11). Then, the motion planning problem for the regularized system consists in determining the control function $u(\cdot)$ that drives $\bar{x}(T)$ to $\bar{x}_d(u(\cdot)) = (xd, \int_0^T r(x)\,\mathrm{d}t)$.

3.3. Imbalanced Jacobian algorithm

The unconstrained motion planning problem in regularised system will be solved using imbalanced Jacobian algorithm [13]. In accordance with endogenous configuration space approach [11], we choose a smooth curve $u_\theta(\cdot) \in \mathcal{X}$, parametrised by $\theta \in \mathbb{R}$, and starting from an arbitrary chosen point $u_0(\cdot)$ in the endogenous configuration space. Along this curve, we define the output error

$$e(\theta) = R_{\bar{x}_0,T}\big(u_\theta(\cdot)\big) - \bar{x}_d(\theta) = E_{\bar{x}_0,T}\big(u_\theta(\cdot)\big) - \bar{x}_d, \quad (12)$$

where $\bar{x}_d(\theta) = \bar{x}_d(u_\theta(\cdot))$ and $\bar{x}_d = (x_d, 0)$ denote the desirable output of the regularized and extended systems, respectively. The curve $u_\theta(\cdot)$ is requested to decrease the error exponentially with a rate $\gamma > 0$, so

$$\frac{\mathrm{d}e(\theta)}{\mathrm{d}\theta} = -\gamma e(\theta). \quad (13)$$

Differentiation of error (12) with respect to θ, and using (13) yields

$$\frac{\mathrm{d}e(\theta)}{\mathrm{d}\theta} = \bar{J}_{\bar{x}_0,T}\big(u_\theta(\cdot)\big)\frac{\mathrm{d}u_\theta(\cdot)}{\mathrm{d}\theta} - \frac{\mathrm{d}\bar{x}_d(\theta)}{\mathrm{d}\theta} = -\gamma e(\theta). \quad (14)$$

where $\bar{J}_{\bar{x}_0,T}\big(u_\theta(\cdot)\big)$ stands for the Jacobian of the regularised system.

The basic idea of the imbalanced Jacobian algorithm is to omit the term $\frac{\mathrm{d}\bar{x}_d(\theta)}{\mathrm{d}\theta}$ in (14) and to compute $u_\theta(\cdot)$ from the following equation

$$\bar{J}_{\bar{x}_0,T}\big(u_\theta(\cdot)\big)\frac{\mathrm{d}u_\theta(\cdot)}{\mathrm{d}\theta} = -\gamma e(\theta). \quad (15)$$

We use the fact that the error in the regularised system coincides with the error in the extended system, see equation (12). Consequently, this means that we operate on error which is defined in the extended system but we use the Jacobian defined for regularised system $R_{\bar{x}_0,T}\big(u(\cdot)\big)$. Let $\bar{J}^{\#P}_{\bar{x}_0,T}\big(u(\cdot)\big)$ be a pseudo inverse of the Jacobian, so

$$\Big(\bar{J}^{\#P}_{\bar{x}_0,T}\big(u(\cdot)\big)\eta\Big)(t) = B^{\mathrm{T}}(t)\Phi^{\mathrm{T}}(T,t)\mathcal{G}^{-1}_{\bar{x}_0,T}(u(\cdot))\eta.$$

Then, applying this inverse to (15) leads to a dynamic system

$$\frac{\mathrm{d}u_\theta(\cdot)}{\mathrm{d}\theta} = -\gamma \bar{J}^{\#P}_{\bar{x}_0,T}\big(u_\theta(\cdot)\big)e(\theta). \qquad (16)$$

Assume that $u_\theta(\cdot)$ is the solution of (16). After its substitution into (14), we get

$$\frac{\mathrm{d}e(\theta)}{\mathrm{d}\theta} = -\gamma e(\theta) + \pi(\theta), \qquad (17)$$

where $\pi = -\frac{\mathrm{d}\bar{x}_d(\theta)}{\mathrm{d}\theta}$ represents a perturbation. The system (16) will define the imbalanced Jacobian algorithm, if the perturbation $\pi(\theta)$ is such that the error (17) tends to 0 along with θ. If this is a case, solution of the unconstrained motion planning problem in the extended system can be computed as the limit $u_d(t) = \lim_{\theta\to+\infty} u_\theta(t)$ of the trajectory of (16). This is also the solution of the constrained motion planning problem for the basic system. Convergence of the imbalanced algorithm results from a well known property of stable linear systems [13, 20, 23].

For practical and computational reason it is useful to use the finite dimensional representation of the endogenous configuration. We assume that the control functions in (11) are represented by truncated orthogonal series, i.e. $u(t) = P(t)\lambda$, where $P(t)$ is a block matrix comprising basic orthogonal functions in the Hilbert control space $\mathcal{L}^2_m[0,T]$ and $\lambda \in \mathbb{R}^s$ denotes control parameters. The more terms in the series are included, the closer we approach "the true" control function. It follows that the endogenous configuration $u(\cdot)$ is represented by a point $\lambda \in \mathbb{R}^s$. Now, the Jacobian of the regularised system is defined by a matrix

$$\bar{J}_{\bar{x}_0,T}(\lambda) = \int_0^T \Phi_\lambda(T,t)B_\lambda(t)P(t)\,\mathrm{d}t, \qquad (18)$$

where matrices $\Phi_\lambda(T,t)$ and $B_\lambda(t)$ are finite dimensional equivalent of $\Phi(T,t)$ and $B(t)$. In finite dimensional case, the Jacobian pseudo inverse of (18) takes a standard form

$$\bar{J}^{\#P}_{\bar{x}_0,T}(\lambda) = \bar{J}^{\mathrm{T}}_{\bar{x}_0,T}(\lambda)\big(\bar{J}_{\bar{x}_0,T}(\lambda)\bar{J}^{\mathrm{T}}_{\bar{x}_0,T}(\lambda)\big)^{-1}.$$

Consequently, we apply the fixed step size Euler method to (16), so the discrete version of the imbalanced Jacobian algorithm is defined by following dynamic system

$$\lambda(\theta+1) = \lambda(\theta) - \gamma \bar{J}^{\#P}_{\bar{x}_0,T}\big(\lambda(\theta)\big)e(\theta). \qquad (19)$$

In order to guarantee the numerical stability of the algorithm and low value of the discretization error, the parameter γ should be sufficient small. The algorithm (19) will be applied to solve a constrained motion planning problem for two different underactuated manipulators with friction.

4. Simulations

As a testbed we choose two underactuated manipulators with friction. The $P\overline{R}$ (fig. 1) and $R\overline{R}$ (fig. 2). The bar over the joint's symbol denotes the passive joint. Both models have only one underactuated joint, but there is no difficulty to apply the presented algorithm for more complicated models. The simulations are performed in the *Matlab* environment.

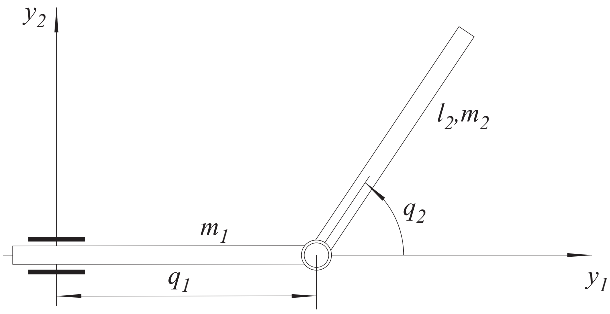

Fig. 1. $P\overline{R}$ Manipulator, where $y_1 = q_1 + l_2\cos(q_2)$, $y_2 = l_2\sin(q_2)$.

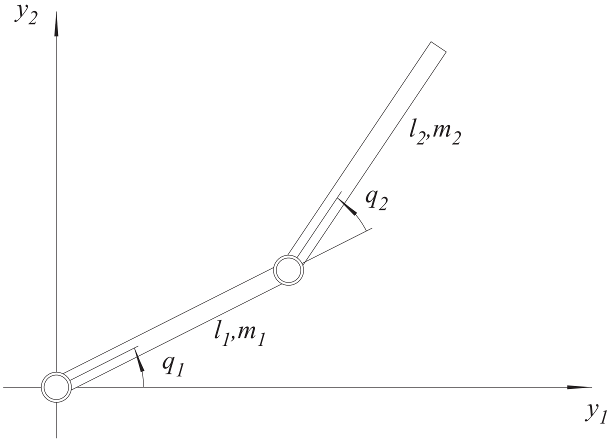

Fig. 2. $R\overline{R}$ Manipulator, where $y_1 = l_1\cos(q_1) + l_2\cos(q_1+q_2)$, $y_2 = l_1\sin(q_1) + l_2\sin(q_1+q_2)$.

4.1. $P\overline{R}$ manipulator

The first model is the $P\overline{R}$ manipulator, in which only the first, prismatic joint is actuated. The second, revolute joint has no actuator but there is a friction force. This underactuated manipulator is a planar robot moving with the presence of the gravity. The model corresponds to a popular cart–pole system.

Dynamics The dynamics of the $P\overline{R}$ manipulator is defined as

$$\begin{bmatrix} m_1+m_2 & -m_2l_2\sin(q_2) \\ -m_2l_2\sin(q_2) & m_2l_2^2 \end{bmatrix}\ddot{q} + \begin{pmatrix} -m_2l_2\cos(q_2)\dot{q}_2^2 \\ m_2l_2g\cos(q_2) \end{pmatrix} = \begin{pmatrix} \tau \\ -\epsilon\dot{q}_2 \end{pmatrix}, \qquad (20)$$

where m_1 and m_2 denote the masses of first and second link respectively, and the l_2 is the length of the pendulum. The τ is the control force in the first joint, ϵ stands for the friction coefficient and g denotes the gravitational acceleration. When we use the partially feedback linearization (3) and change the state space vector $x = (x_1,x_2,x_3,x_4) = (q_1,\dot{a}_1,q_2,\dot{q}_2)$ the equation (20) takes the form

$$\dot{x} = \underbrace{\begin{pmatrix} x_2 \\ 0 \\ x_4 \\ -\frac{g}{l_2}\cos(x_3) - \frac{\epsilon x_4}{l_2^2 m_2} \end{pmatrix}}_{f(x)} + \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{1}{l_2}\sin(x_3) \end{pmatrix}}_{G(x)} u.$$

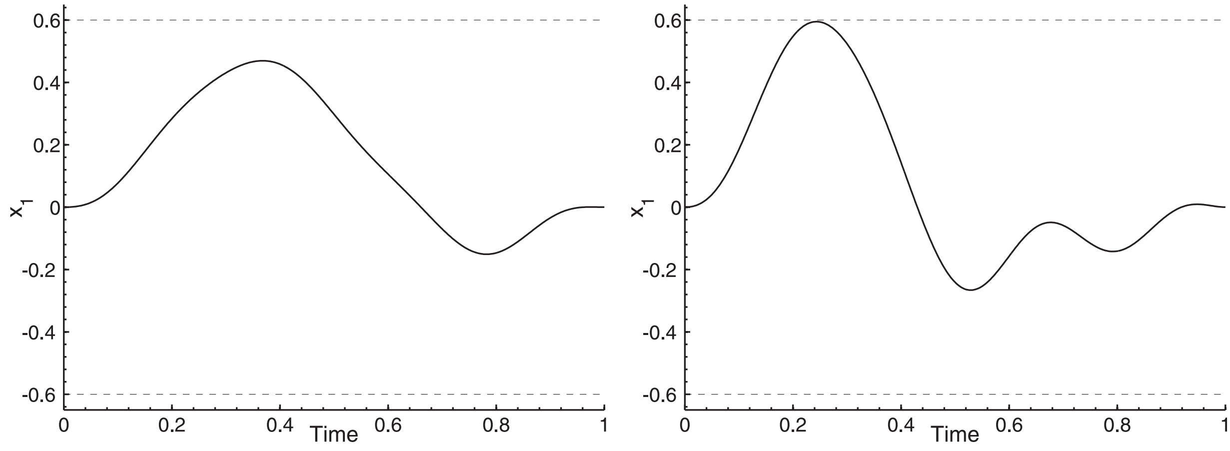

Fig. 3. x_1 trajectory (left side – frictionless, right side – with friction).

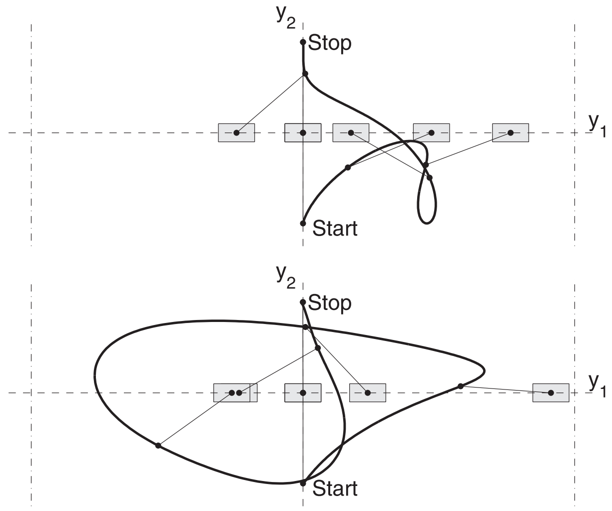

Fig. 4. Task–space path (top – frictionless, bottom – with friction).

Simulation results In the simulations we take the pendulum length equal $l_2 = 0.2$, first link mass $m_1 = 3$, second link mass $m_2 = 0.1$, and the gravitational acceleration $g = 9.81$. We will proceed the swing–up manoeuvre, so the initial point $x_0 = (0, 0, -\pi/2, 0)$ and the desirable $x_d = (0, 0, \pi/2, 0)$. We choose the representation of the control function as the truncated Fourier series, so the matrix $P(t)$ collects the following basic function $\{1, \sin(i\omega t), \cos(i\omega t)\}$ for $i = 1, \ldots, 4$. The time horizon is $T = 1$, the convergence coefficient $\gamma = 0.15$, and the initial control parameter vector $\lambda = (1, 0_{1\times8})$. We simulate two cases: first, without friction $\epsilon = 0$, and second, with friction coefficient equal to $\epsilon = 0.02$. For both cases we set the limits for the first state element as $-0.6 < x_1 < 0.6$. The fig. 3 presents the trajectory of the state variable x_1. We also mark the limits with dotted lines. In fig. 4 we show the task–space path. The posture of the manipulator is sampled every 0.2 time units. As one can expect, the control function in the frictionless case has smaller amplitude (fig. 5). The algorithm convergence for two value of the friction coefficient ϵ is demonstrated in fig. 6.

When we consider a swing–up manoeuvre, for the $P\overline{R}$ robot, the energy needed to perform the motion is greater for the manipulator with friction than for the robot without friction. This fact can be observed in figs. 3–5. The model with friction needs more oscillations and higher amplitude than the frictionless manipulator. The algorithm takes more steps to converge for the model with friction (fig. 6). In this case, the slower convergence arises from the fact that the

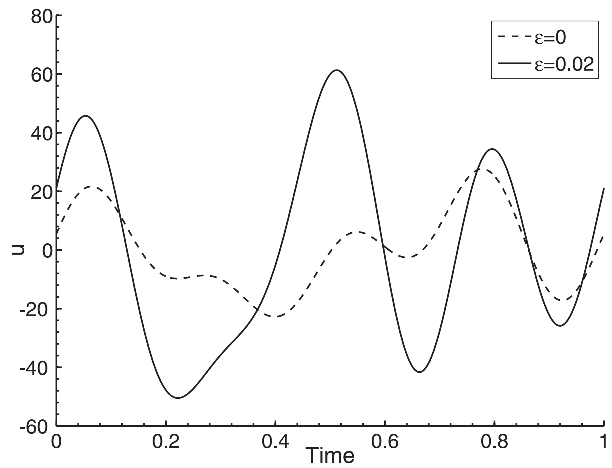

Fig. 5. Control function u.

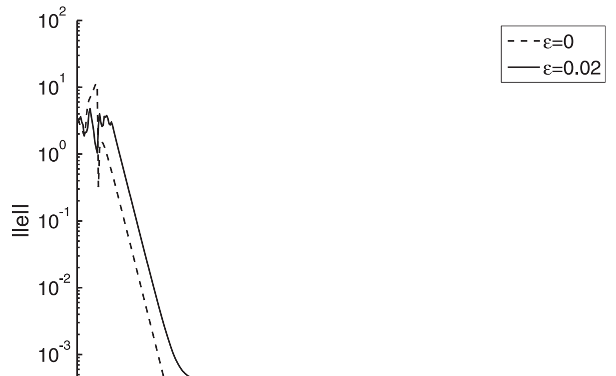

Fig. 6. Algorithm convergence.

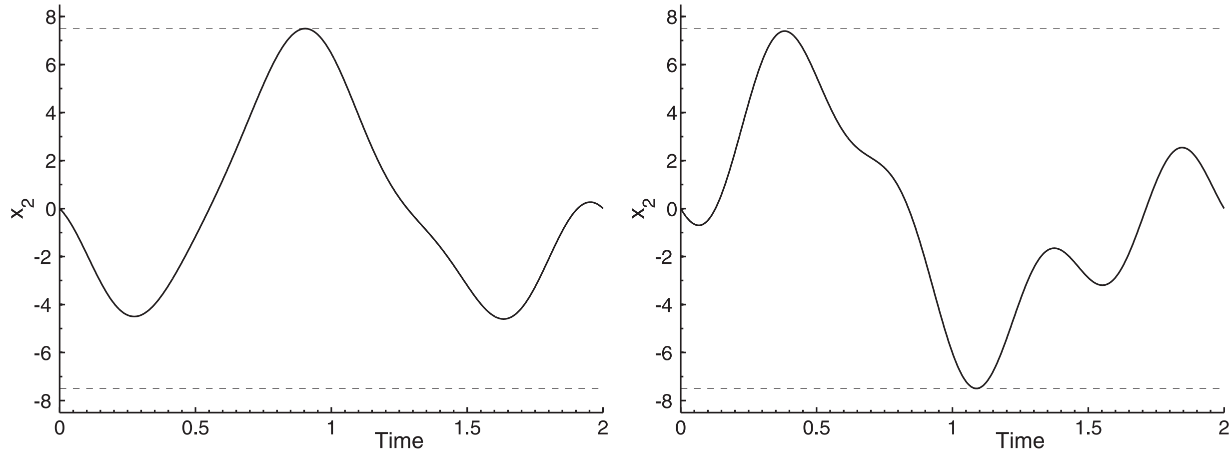

Fig. 7. x_2 trajectory (left side – frictionless, right side – with friction).

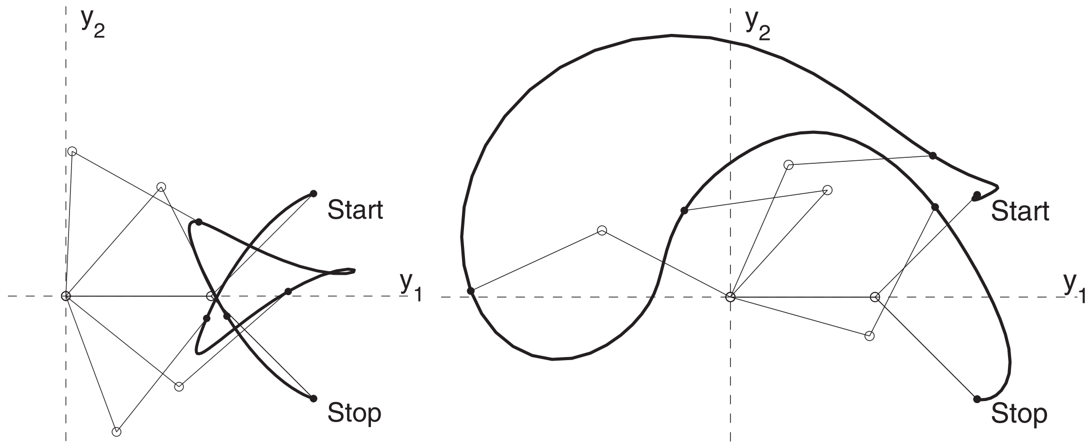

Fig. 8. Task–space path (left side – frictionless, right side – with friction).

state variable x_1 is closer to the selected limits than in the model without friction (see fig. 3).

4.2. $R\overline{R}$ manipulator

As a second model we choose the double pendulum, $R\overline{R}$ planar manipulator with actuated first joint. We assume that the model moves without the gravity. Similar to previous system, the friction force appears in the second, passive joint.

Dynamics The dynamics of the $R\overline{R}$ manipulator takes the form

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix} \ddot{q} + \\ + \begin{pmatrix} -\frac{1}{2} m_2 l_1 l_2 \sin(q_2)(2\dot{q}_1\dot{q}_2 + \dot{q}_2^2) \\ \frac{1}{2} m_2 l_1 l_2 \sin(q_2)\dot{q}_1^2 \end{pmatrix} = \begin{pmatrix} \tau \\ -\epsilon \dot{q}_2 \end{pmatrix},$$

where

$$M_{11} = \frac{1}{3} l_1^2 m_1 + m_2 \left(l_1^2 + \frac{1}{3} l_2^2 + l_1 l_2 \cos(q_2) \right),$$

$$M_{12} = l_2 m_2 \left(\frac{1}{2} l_1 \cos(q_2) + \frac{1}{3} l_2 \right),$$

$$M_{22} = \frac{1}{3} l_2^2 m_2.$$

The symbols m_1, m_2, l_1 and l_2 stand for masses and lengths of the first and second link respectively. Again, τ is the control moment and ϵ denotes the friction coefficient. After the same transformation like in the $P\overline{R}$ model we obtain the dynamic system

$$\dot{x} = \underbrace{\begin{pmatrix} x_2 \\ 0 \\ x_4 \\ -\frac{3 l_1 x_2^2 \sin(x_3)}{2 l_2} - \frac{3 \epsilon x_4}{l_2^2 m_2} \end{pmatrix}}_{f(x)} + \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{3 l_1 \cos(x_3) + 2 l_2}{2 l_2} \end{pmatrix}}_{G(x)} u.$$

Simulation results The parameters of the second testbed we choose as $l_1 = 0.5$, $l_2 = 0.5$, $m_1 = 1$, and $m_2 = 0.5$. We take the initial point as $x_0 = (0, 0, \pi/4, 0)$, and we want to drive to a desirable point $x_d = (0, 0, -\pi/4, 0)$. Identically to the simulation of the $P\overline{R}$ manipulator, we choose the representation of the control function as the truncated Fourier series. We take the vector $P(t) = (1, \sin(\omega t), \cos(\omega t), \sin(2\omega t), \cos(2\omega t), \sin(3\omega t), \cos(3\omega t), \sin(4\omega t), \cos(4\omega t))$. The time horizon for the $R\overline{R}$ manipulator is $T = 2$, the decay rate $\gamma = 0.5$, and the initial control vector $\lambda = (1, 0_{1\times 8})$. Similarly to the previous model we also simulate two cases. First, frictionless $\epsilon = 0$, and second with friction coefficient $\epsilon = 0.15$. Contrary to the $P\overline{R}$ manipulator, here we limit the velocity in the actuated joint $-7.5 < x_2 < 7.5$. The trajectory of the state x_2 with marked limits are presented in fig. 7. The fig. 8 shows the task-space path. The manipulator posture is sampled with the frequency of 0.4 time units. Here, likewise to the $P\overline{R}$ manipulator, the control function for the nonzero friction coefficient is larger than for the frictionless case (fig. 9). The comparison of the algorithm convergence for the two cases is presented in the fig. 10

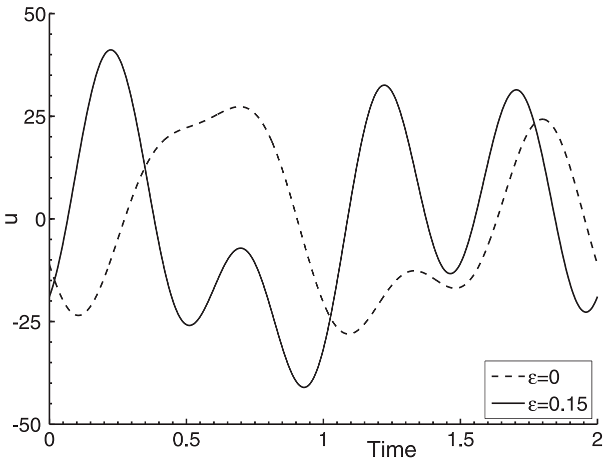

Fig. 9. Control function u.

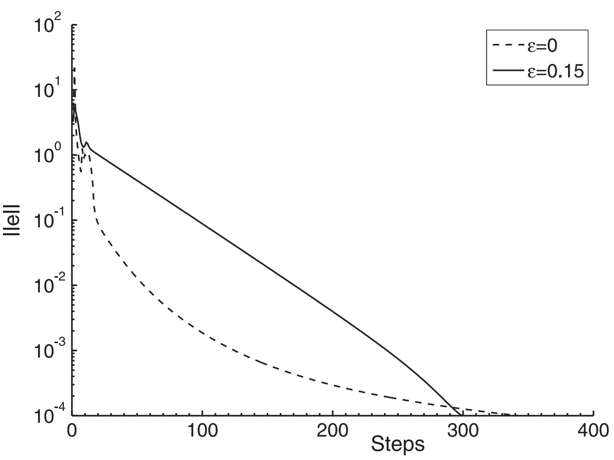

Fig. 10. Algorithm convergence.

The velocities x_2 for the $R\overline{R}$ manipulator (fig. 7) for both friction cases have comparable amplitude. However, the path of the end–effector is much longer (fig. 8) and the velocity profile has more oscillations (fig. 7) for the model with friction. Fig. 10 draws a conclusion that the algorithm needs almost equal number of steps to solve the problem with and without friction.

5. Conclusions

In this paper we have presented the imbalanced Jacobian algorithm applied to the constrained motion planning problem for underactuated manipulators with friction. This algorithm derived from an endogenous configuration space approach, originally has been dedicated to mobile manipulators. Following the standard procedure, the state constraints are included into robot's dynamics. Next, the expanded systems are regularized in order to avoid the algorithm singularity. Finally, using the motion planning Jacobian algorithm for the extended system one can obtain the solution of the constrained problem for the original system.

The efficiency of the algorithm has been presented in simulations for two kinds of underactuated manipulators with and without friction. For the $P\overline{R}$ manipulator we have bounded the position of the first joint x_1, by the reason that in real robots the range of the prismatic joint is limited. Also, for the second, $R\overline{R}$ manipulator, the limitation of the velocity x_2 has the practical reason. In real manipulators, the velocity of revolute link depends on used actuator. For both manipulators and for both friction cases the presented algorithm has solved the constrained motion planning problem correctly. In all cases the desirable points have been reached with assumed accuracy and the state constraints have been kept.

The presented result could be improved in several ways. There is no problem to constrain more than one state variable, or even the whole state vector. Using the similar way of reasoning, we can also limit the control functions. It should be taken into account that the motion could be impossible to perform with badly chosen boundaries. Additionally, too narrow limits effect in slower algorithm convergence. Also, the underactuated models could be more complicated. Firstly, the inequality between the dimension of control space and state space could be more than one. Secondly, there are many other underactuated models, which can be considered, e.g. dynamics of wheeled mobile robots, surface and underwater vehicles or flying systems.

Finally, we present some advantages of our approach contrary to the other methods for underactuated manipulators. First of all, the analogy of the system (5) to other underacutated systems yields a conclusion that the presented algorithm could be used to other models which can take the form of control–affine systems with or without drift. Comparing our method with methods using linear algorithm based on a linearization around the operation point. In the presented approach we use the linearization along the trajectory, so the area of implementation is larger. The method presented in [1] consists in two phases, in first phase the actuated joints are positioned, and next using the specific methods the passive joints converge to the desired position. Our algorithm is one-phase and the motion and control forces are continuous and smooth over the time horizon. Other method [6] relays on the segmented trajectory. For specific models the two kinds of reference trajectory are constructed. Using such approach one can move the underactuated manipulators in translational or rotational way. The disadvantage of this algorithm, comparing to our, is that here the manipulator must be at least 3 d.o.f. In the presented method there is no restriction on the system dimension.

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AUTHORS
Adam Ratajczak* – Institute of Computer Engineering, Control and Robotics, Wrocław University of Technology, 50-372 Wrocław, Janiszewskiego 11/17, e-mail: adam.ratajczak@pwr.wroc.pl
Mariusz Janiak – Institute of Computer Engineering, Control and Robotics, Wrocław University of Technology, 50-372 Wrocław, Janiszewskiego 11/17, e-mail: mariusz.janiak@pwr.wroc.pl

* Corrresponding author

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