Shifted Up Cosine Function as Unconventional Model of Probability Distribution

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Abstract:

The shifted up one period of cosine function with field under it normalized to 1 is proposed to be use as the unconventional model of probability density function (PDF). It could also approximate Normal probability distribution in the range ± 2.5 standard deviation with accuracy of about ±0.02, which is fully acceptable in the evaluation of measurement uncertainty. In this paper the properties of the above cosine based PDF are considered. The possibility of its applications in the routine data assessment and in virtual instruments with automatic uncertainty calculations is recommended.

Keywords: *probability density function PDF, cosines approximation of normal distribution.*

1. Introduction

Normal (Gaussian) probability density function - PDF is the most commonly applied as a model of error and uncertainty distributions including recommendations in international *Guide to the expression of uncertainty in measurement*, known under acronym GUM [1]. This application is valid due to the assumption that a large number of independent factors are randomly influencing upon the measured object and/or the measuring chain. In practice, one of the inconveniences of applying Normal PDF is non-limited range of the measurand values ± infinity. In measuring experiments there is a limited number of influencing variables and there are physical limits of collected data values. The additional information that data values are limited allows for using a function of limited width as a model of distribution, which estimates the data dispersion better than unlimited Gauss PDF. This difficulty is not solved good enough by the Gauss distribution of cut off both end tails and of normalized left area under the curve to 1. The description of such tails' truncated Gauss function is rather complex [8]. Then, non-Gaussian different distributions are also applied to uncertainty evaluations. Some of them, which are recommended for uncertainty evaluation by Monte Carlo method, are given in Table 1 of Supplement 1 to GUM [2]. More accurate estimators than the mean value may be also obtained for non-Gaussian distributions [4], [9]. No matter if the Monte Carlo method or the convolution based method is used, the simplification of Normal and t-Student distributions are a great assistance to reduce a number of mathematical operations as in both cases data tables of consume a lot of memory space. Furthermore the authors started developing instruments, where the automatic function of uncertainty calculation is implemented [5], [6]. Different distributions have to be applied for online processing of signal uncertainty. In smaller and simpler intelligent transducers and even in instruments applying DSP the memory is rather limited. Generation of simple PDF model, which approximates Normal distribution, would be considered very helpful in this case.

One of us (ZLW) suggests using as a distribution model of random data of limited values other then Gauss smooth function, e.g. the shifted up cosine in the range of -π to +π. In the simpler case this function may touch horizontal axe "x" - see Fig. 1 or even may be located above this axe. When we elaborated theoretical bases of that unconventional model, a mention of similar type model was found in the short paper by H. Green and D. Rabb published long ago - in 1961 [7]. Green proposed for psychometrics data to use arbitrarily chosen a particular nonoptimal shifted up cosine function in the range ±π as an approximation of Normal distribution. There is neither method how to find the cosine based function of optimal parameters nor the accuracy of approximation has been elaborated. That may be why Green's proposal was not later developed in statistical and metrology publications. Today's computer based possibilities of calculations do not limit estimating the best parameters of any type of PDF curve proposed for approximation of the measurement data distribution.

2. Theoretical bases

2.1. Formula of cosine-based PDF

Proposed is unconventional model of probability density distribution function - PDF in the generalized form

$$f(x) = B + A\cos 2\pi \frac{x}{X_T} \quad \text{for } -0.5X_T \le x \le +0.5X_T \quad (1)$$

Where: x - value of observations, $f(x) > 0$, $X_T=2X$ range equal to one period of cosines, $1/X_T$ - reciprocal of X_T (as frequency in time functions), A, B - constant parameters: cosine amplitude and value of shifting up.

Form of PDF (1) normalized to $X=0.5X_T$ as $f(x/X)$ and its cumulative distribution CPDF are given in Fig. 1 for $A=B$ and $x/X=1$.

Because of the constant positive value B the function $f(x)$ of (1) may be named as **shifted up cosine** function with proposed symbol +COS for general case $A \neq B$. As model of the PDF distribution is valid for $f(x)>0$ only and its field is equal 1, then the function +COS could be applied in the range of one period $\pm(x = 0.5X_T)$.

Function $f(x)$ of (1) generally has three independent parameters (A, B, X). So, to find them three relations are

needed, i.e.:
- field under curve in its range $S=1$, or $S<1$, e.g. as equal for Gauss curve in the same range,
- range of approximation $X_{APROX}<X$ or point to cross Gauss function or particular parameter of +COS function, e.g.: $\pm\sigma$, $\pm 2\sigma, \ldots$ similarly as for Gauss,
- minimum difference between two curves: experimental and +COS distribution, according to some criterion: LSM or LMM - minimum sum of difference squares or modules, Kolgomorov-Smirnov or Chebyshev criterion and others.

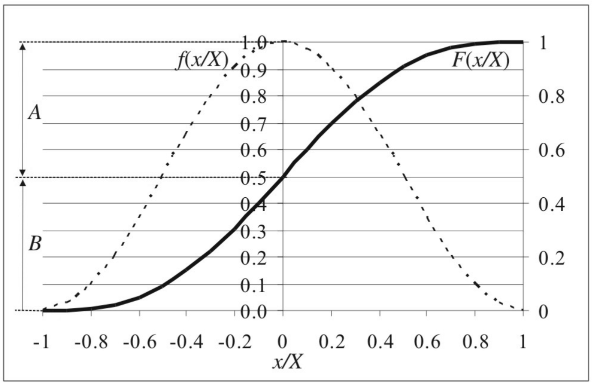

Fig 1. $f(x/X)$ - normalized probability density function (PDF) of +COS distribution and $F(x/X)$ - its cumulative function (CPDF), when mean $\bar{x}=0, A=B$ and $X=0.5X_T$.

2.2. Cosine shifted up by its amplitude A

For unconventional model of PDF to use in practice is proposed the simplest shifted up cosine function when $B=A$ as it is shown on Fig 1. In two its minimum it is tangential to x axe for $x=\pm X$ and from (1) is

$$f(x)=A\left(1+\cos\pi\frac{x}{X}\right) \quad (2)$$

To obtain cumulative distribution CPDF of $f(x)$ the range of integration is $-X=x=+X$. Field for this range $S=1$ and after integration of (2):
$A\cdot 2X=1$ and $2sin(x=\pm X)=0$.

Then:

$$X=1/(2A) \quad (3)$$

and

$$f(x)=A(1+cos 2\pi Ax)=2A\,cos^2\pi Ax \quad (4)$$

Parameters A and X are mutually dependent as in (3) and it is enough to have one of them as given or from histogram of data or from approximation of Normal PDF by the above function according to a given rule.

Function +COS of (4) with $A=B$ is named below as COS^2. It may be so chosen to passing the given point. If such function is e.g. covering the top point of Gauss distribution of standard deviation σ - see curve 2 in Fig. 2, then the amplitude of its cosine component is:

$$A=\frac{1}{2}\frac{1}{\sigma\sqrt{2\pi}} \quad (4a)$$

and as the field under curve (2) is $S=1$, then from (3) half of cosine range

$$X=\sigma\sqrt{2\pi}\approx 2.507\sigma$$

For normalized Gauss distribution, i.e. when its standard deviation $\sigma=1$ it is

$$A=\frac{1}{2}\frac{1}{\sqrt{2\pi}}=0.1995 \text{ and } X=\sqrt{2\pi}\approx 2.507.$$

2.3. Cosine shifted up by $B>A$

In the case when one period of cosine function is shifted more then its amplitude, i.e. by $B>A$, the field under curve $S=2XB$.

From $S=1$ half range X of cosine function is

$$X=1/(2B) \quad (5)$$

Value of X is now smaller than for $A=B$. The function (1) can be expressed as

$$f(x)=B+A\,cos(2\pi xB) \quad (6)$$

Function (6) has two independent parameters, i.e. A and B (or A and X). This function could be taken also for approximation of Gauss PDF, but only if higher accuracy than for $A=B$ is needed.

When $X=1/2B$ the cumulative distribution CPDF as integral of (6) is described by following relation

$$F(x)=\int f(x)\,dx=Bx+\frac{1}{2\pi}\frac{A}{B}\sin\pi\frac{x}{X}+const$$

$$\text{for } -X\le x\le +X \quad (7)$$

In particular case (4) of $A=B$ and then $X=1/(2A)$ it is

$$F(x)=\frac{1}{2}\left(\frac{x}{X}+1\right)+\frac{1}{2\pi}\sin\pi\frac{x}{X}\bigg|_{-X\le x\le X} \quad (7a)$$

Some values of $F(x)$ for and then $X=\frac{1}{2A}$ are given below in Table 1.

2.4. Standard deviation of $B+Acos(\cdot)$ distribution

Standard deviation σ_{+COS} of the PDF model +COS function in general case (6) is

$$\sigma_{+COS}=\sqrt{\int_{-X}^{+X}x^2\left(B+A\cos 2\pi xB\right)dx} \quad (8)$$

In the solution of (7) the generalized formula of $\int x^n\cos cx\,dx$ solution adopted for x^2, is applied.

Table 1. Values of COS^2 model cumulative distribution CPDF.

x/X	-1	- 0.9	- 0.8	- 0.7	- 0.6	- 0.5	- 0.4	- 0.3	- 0.2	- 0.1	0
$F(x)$	0	0.001	0.006	0.021	0.049	0.091	0.149	0.221	0.307	0.401	0.5
x/X	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0
$F(x)$	1	0.999	0.994	0.979	0.951	0.909	0.851	0.780	0.694	0.599	0.5

$$\int x^2 \cos cx \, dx = \frac{\sin cx}{c}\left(x^2 - \frac{2}{c^2}\right) + \frac{2x\cos cx}{c^2} \quad (8a)$$

Where: $c=2\pi B$ and for $x=X$, $B=1/2X$ or $X=1/2B$, $B \geq A$.

Two parameters are independent and solution of (8) could be presented in a double form as

$$\sigma_{+\cos} = X\sqrt{\frac{1}{3} - \frac{4}{\pi^2}AX} = \frac{1}{2B}\sqrt{\frac{1}{3} - \frac{2}{\pi^2}\frac{A}{B}} \quad (9)$$

Solution is possible only for limited ratio of A/B as

$$\frac{A}{B} \leq \frac{\pi^2}{6} \approx 1.78$$

In the case of COS^2 when shifted up cosine is tangential to axe x, from (3) $A=B=1/2X$ and then

$$\sigma_{+\cos} = X\sqrt{\frac{1}{3} - \frac{2}{\pi^2}} = \frac{1}{2A}0.3615 \quad (9a)$$

If that standard deviation σ_{+COS} is given, it is possible to find mutually joint values A and X. For example if $\sigma_{+COS}=1$ is: $A=0.1808$, $X=2.766$. Or in opposite, for a given range X, e.g. $X=2.5$ from (9) $\sigma_{+COS}=0.9060$.

For extended uncertainty calculations of COS^2 PDF cover factors k_{PC} for given P and large $n>10$ are put below in Table 2.

Table 2. Extended uncertainty cover factors of COS^2 PDF.

P [%]	50	68.3	90	95	99	99.7	100
$k_{PC}(x/X)$	0.265	0.385	0.596	0.683	0.816	0.878	1

For smaller $n=10$ additional extension than k_{PC} in Table 2, nearer to the Student-Gosset PDF values, should be applied.

3. Approximation of normal distribution by function +COS

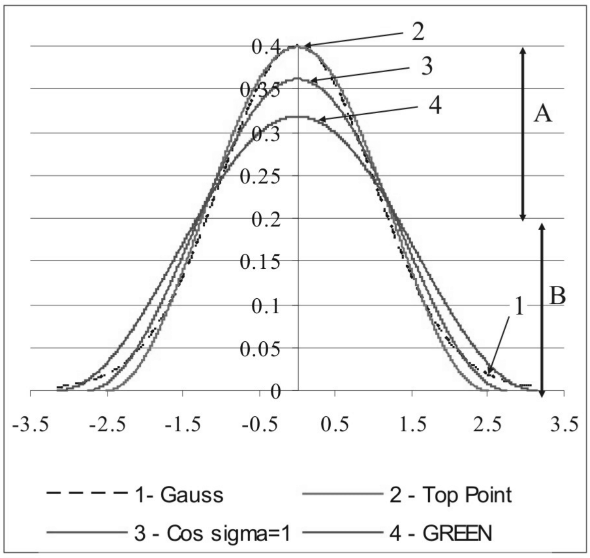

Fig. 2. PDF distributions: 1 - Normalized Gauss N(0,1) (i.e. σ=1) and three tangential to axe x functions $A(1+\cos 2\pi Ax)$: 2 - passing through the top point of Gauss PDF, 3 - of standard deviation $\sigma_{+COS}=1$, 4 - Green proposal $f_{GR}(x)$ [7].

In Fig 2 are shown: curve 1 - normalized Gauss PDF $N(0,1)$, i.e. $f_G(x) = \frac{1}{\sqrt{2\pi}} e^{-0,5x^2}$ of $\bar{x}=0$, $\sigma=1$ and three COS^2 functions $f(x)=A(1+\cos 2\pi Ax)$:
curve 2 - of going through the top point of 1 ,
curve 3 - of standard deviation $\sigma_{+COS}=1$ and
curve 4 - of proposed by Green [7].
Fields S under curves are 1.

Differences Δ_{PDF} between each of COS^2 distributions and Gauss of $\sigma=1$ are in Fig. 3a and of their cumulative distributions CPDF Δ_{CPDF} - in Fig. 3b.

Values of main parameters of above three COS^2 functions are in columns 2 to 4 of Table 3. From Fig. 3a, b and data of Table 3 it is possible to conclude that difference Δ_{PDF} of COS^2 distribution 2 and normalized Gauss PDF, as well Δ_{CPDF} of curves 2 and 3, do not exceed range ±2.1%.

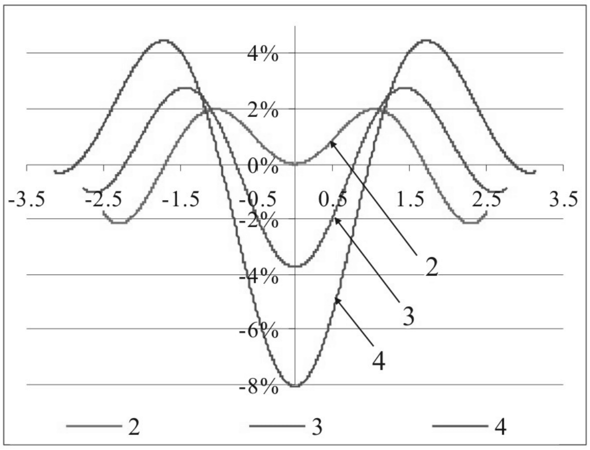

Fig. 3a). Differences Δ_{PDF} of COS^2 no 2, 3, 4 distributions and Normal distribution $N(0,1)$ of $\sigma=1$.

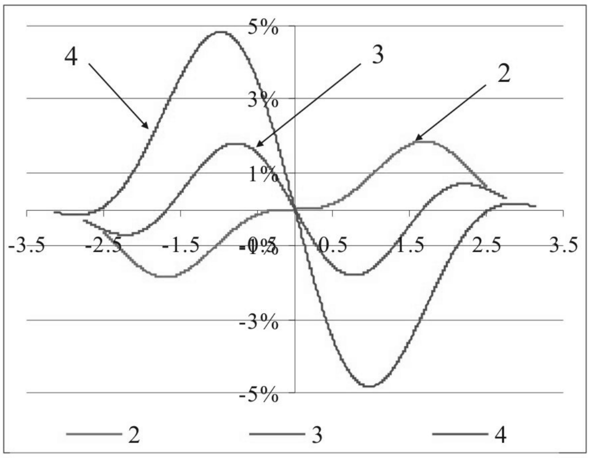

Fig. 3b). Differences Δ_{CPDF} of COS^2 2, 3, 4 and Normal $N(0,1)$ cumulative distributions.

Δ_{PDF} of Gauss and COS^2 function 3, both of standard deviation $\sigma=1$, are changed in the little broader range (+2,8, -3,7)%. Even this accuracy could be fully accepted to the most uncertainty type A evaluations.

In column 4 are also given for comparison parameters of function $f_{GR}(x)$ proposed by Green in [7]. It has $A=1/(2\pi)$, standard deviation $\sigma_{+COS} \approx 1{,}14$. Differences Δ_{PDF} and Δ_{CPDF} of Green function $f_{GR}(x)$ and normalized Gauss distributions are much higher up to (+4.5, -8.1)% then of top point function 3 and is not acceptable for the most research and technical measurements.

Table 3. Parameters of few +COS distributions and of their differences Δ_{PDF}, Δ_{CPDF} to Gauss of $\sigma = 1$.

	TYPE	Top point	σ_{+COS}=1	Green f_{GR}(x)	min $(\Delta f)^2$	min $\lvert\Delta f\rvert$	min $(\Delta f)^2$	min $\lvert\Delta f\rvert$
+COS $X_{APROX}=X$	Curve no	2	3	4	5	6	7	8
	A	given value of A			optimal A		optimal A and B	
		0.200	0.181	0.159	0.186	0.196	0.178	0.179
	B	$B=A$					optimal B and A	
							0.220	0.219
	$X=1/2B$	2.51	2.77	3.14	2.69	2.54	2.27	2.28
	σ_{+COS}	0.906	1	1.136	0.971	0.922	0.937	0.936
Δ PDF $\Delta f = f_{+COS} - f_G$	min Δf	- 0.022	- 0.037	-0.0806	- 0.027	- 0.020	- 0.0015	- 0.0013
	max Δf	0.020	0.028	0.0446	0.024	0.020	0.012	0.012
	mean Δf	0.0024	0.001	0.0003	0.0013	0.0021	0.0050	0.0049
	$\overline{\Delta} f$	0.014	0.020	0.0389	0.016	0.0132	0.0043	0.0044
Δ CPDF $\Delta F = F_{+COS} - F_G$	min ΔF	- 0.019	- 0.018	-0.0483	- 0.012	- 0.016	- 0.012	- 0.011
	max ΔF	0.019	0.018	0.0483	0.012	0.016	0.012	0.011
	mean ΔF	0.000	0.000	0.0000	0.000	0.000	0.000	0.000
	$\overline{\Delta F}$	0.012	0.010	0.0284	0.008	0.010	0.007	0.007

For approximation of the Gauss PDF two other COS[2] functions of the single optimal parameter $A=B$ are given in columns 5 and 6 of Table 3. Value A of curve 5 is calculated from differences Δ_{PDF} by commonly known least square method LSM and of curve 6 by LMM method - min sum of modules. Differences Δ_{PDF} and Δ_{CPDF} of distributions 5 and 6 together with 2 in relation to Gauss PDF are compared in Fig. 4a, b. The accuracy of Gauss approximation by functions 5, 6 in the full period of cosine are very near to the accuracy of the top point curve 2.

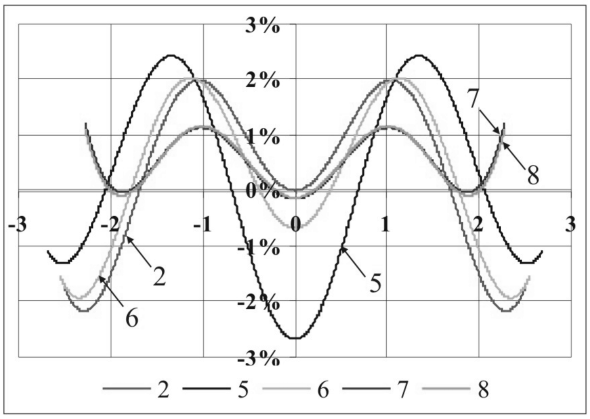

Fig. 4a). Differences between few +COS PDFs of Table 3 and Normal PDF of $\sigma = 1$.

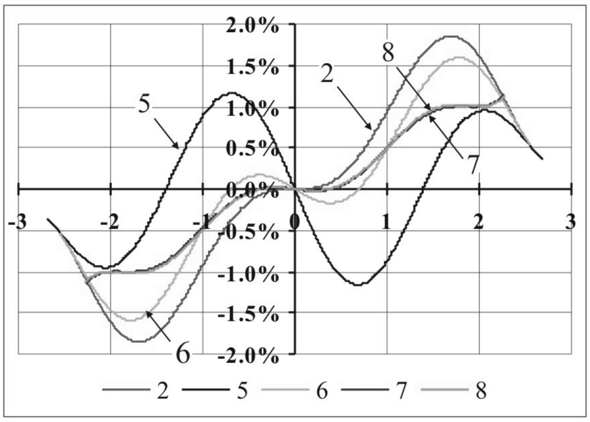

Fig. 4b). Differences between few +COS CPDFs of Table 3 and Normal CPDF of $\sigma = 1$.

For shorter range of approximation it is possible to obtain a little better accuracy. Dependence of optimal values of A and of half range X of the +COS on the range of approximation X_{APROX} is given on Fig 5a. Changes of Δ_{PDF} and Δ_{CPDF} main parameters are in Fig. 5 b, c.

Cosine amplitude A is decreasing with the approximation range X_{APROX} and its period X is increasing according to (3). Min and max differences of Δ_{PDF} and Δ_{CPDF} are changing very little.

The best results is possible to receive for the simultaneous two-parameter optimisation of A and B columns 7 and 8 of Table 3 and curves 7 and 8 in Fig. 4 a, b. Accuracy of PDF and CPDF approximation of Normal distribution by +COS functions (7) or (8) of optimal two parameters A and $B>A$ is twice better ($\approx \pm 0{,}01$) than those of optimal single parameter $A=B$. But these +COS distributions look as Gauss with cut off tails and is not so simple to operate with them as with the first simpler COS[2] function (2), which is tangential to axe x. Mean, extreme values and standard deviations of Δ_{PDF} and Δ_{CPDF} of all +COS functions considered above for approximation of the normalized ($\sigma=1$) Gauss distribution are given together in Table 3 for comparison. Also each Normal distribution with arbitrary σ_i may be similarly approximated by some COS[2] function of $A_i=B_i$ and $X_i=1/(2A_i)$ from columns 2-6 of Table 3. From 9a results

$$A_i\sigma_i = A\sigma \quad \text{and} \quad X_i\sigma = X\sigma_i \tag{10}$$

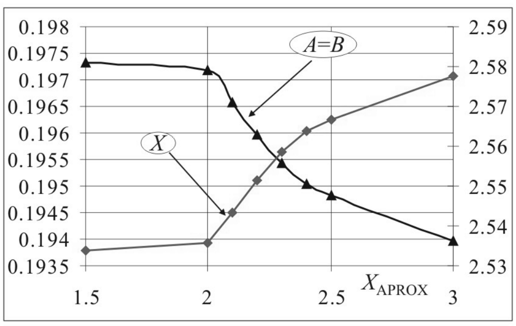

Fig. 5. a) Changes of A and X on range of approximation X_{APROX}.

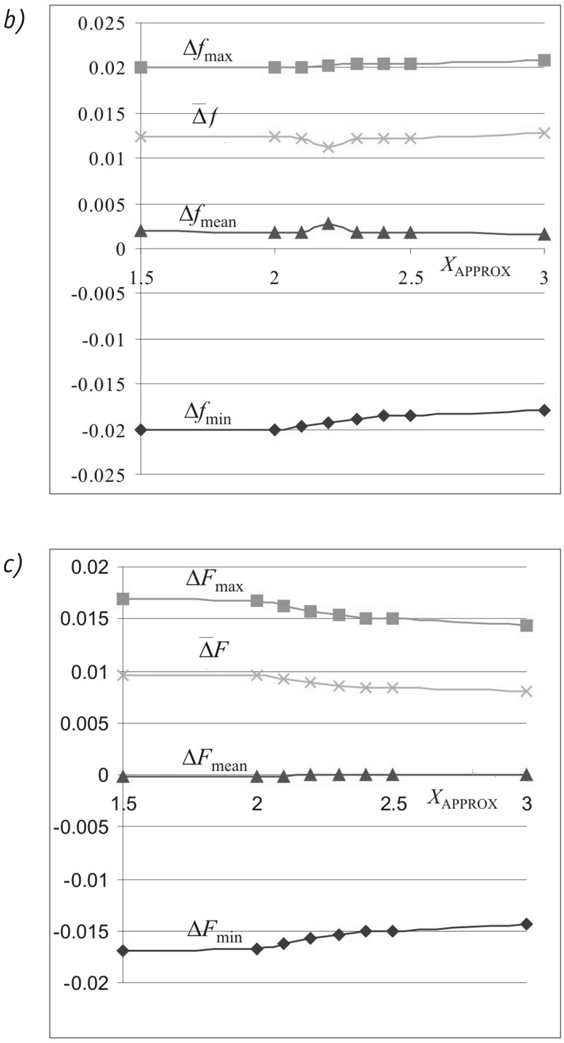

Fig. 5. b) -Δ_{PDF} and c) Δ_{CPDF} parameters of ranges X_{APROX} of approximation the Gauss distribution by +COS of LSM-optimal A.

Three examples of +COS approximations for Gauss PDFs of $\sigma=(0.5, 1, 2)$ are given in Fig. 6.

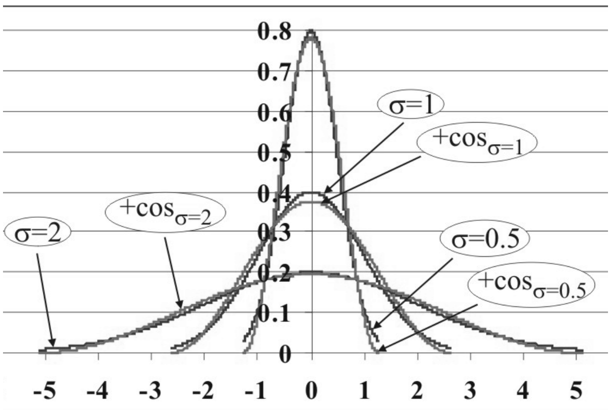

Fig. 6. Gauss PDFs of and σ =(1/2, 1, 2) and their +COS approximates.

4. Short note about convolusions of +COS function

Result of convolution of two similar +COS distributions $f(x)=A(1+cos\,2\pi xA)$ is shown in Fig 7a.

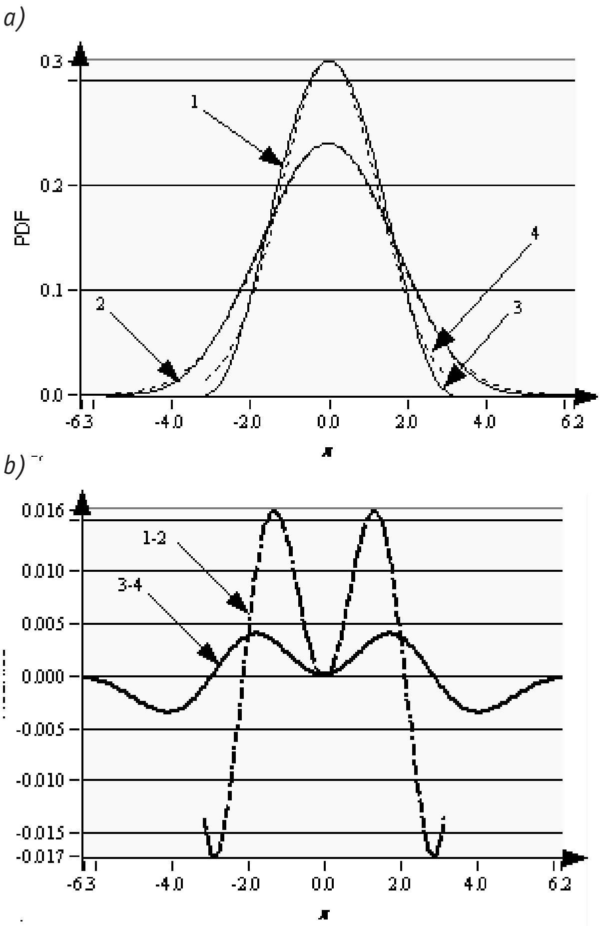

Fig. 7. a) PDF of: 1 - COS² distribution and 3 - convolution of two identical COS²; 2 - and 4 - their best fitting Gauss PDFs (dotted lines); b. Differences Δ_{PDF}: of Gauss and COS² - curve 1, of their convolutions PDFs - curve 2.

For this example of +COS = COS² function $A=B=0.2$, $X=2.5$, $\sigma_{+COS}=0.904$ $\sigma_G=0.997$. Excess $=2.4$ (as for triangular PDF).

Convolution result is not COS² function, it is nearer to Normal PDF. Standard deviation of convoluted two similar non-correlated functions type +COS is

$$\sigma_{2(+COS)*} = \sqrt{\sigma^2_{+COS} + \sigma^2_{+COS}} = \sqrt{2}\sigma_{+COS}$$

Its SD $\sigma_{2(+COS)}=1.28$. Excess $=2.71$. Full (±) range is $4X=10$, $\sigma_{Best\,Gauss}=1.33$.

Convolution of COS² with various PDF gives results very near to the same convolutions of Normal PDF. Differences of convolution function of two +COS PDF and its Gaussian function (Fig. 7b) are twice as smaller than those of single COS², $\bar{\Delta}_{PDF}=(0.0011, 0.0022)$.

5. Example of uncertainty calculations

Given is the sample of 200 x_i values of repeated measurement observations obtained by regular sampling of the simulated random population. Trend was removed from collected data similarly as in [3], [5]. Accuracy of their mean value as estimator has to be found. It will be done with application of Gauss and COS² PDF models and then the results should be compared.

Histogram of deviations from the sample mean value $\bar{x}$ of 17 subranges is shown as 1 in Fig. 8. There are also: Gauss PDF - 2 and two +COS PDF, i.e. 3 - of the half-range

$X_1 = 2.35$ equal distance from 0 to extreme data x_{MAX} and 4 - of X_2 a little wider then found by Fig. 9a) for the calculated sample SD $S(x_i) = 0.978$.

Half-ranges X of COS² and extended uncertainties U of various confidence levels k_P are given in Table 4. Matching of Gauss and COS² PDF to histogram data is tested by SD of their differences - last two lines. Commonly used compliance test $\chi 2$ is not proper for COS² PDF.

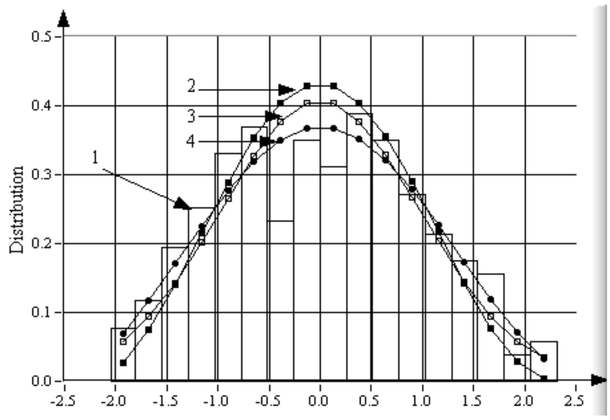

Fig. 8. 1 - histogram h_i of the sample of simulated data, 2 - sample Gauss PDF and two COS² PDF models: 3 - of the range $2X_1$ between extreme x_1, x_n observations, and 4 - of the range $2X_2$ obtain from SD of sample by (9a).

Table 4. Mean $\bar{x}$ extended uncertainty of confidence levels k_P of COS²(0, X_i) and Normal $N(0, \bar{\sigma})$ distributions.

Confidence level k_P (cover factor)	$COS^2(0, X_i)$			$N(0, \bar{\sigma})$
	X=1	X_1=2.35	X_2=2.77	$S(x_i)$=0.978
	k_{PC}	Extended uncertainty ±U		
0.500	0.265	0.043	0.052	0.047
0.683	0.385	0.063	0.076	(u_A) 0.069
0.900	0.596	0.097	0.117	0.114
0.950	0.683	0.112	0.134	0.136
0.990	0.816	0.134	0.160	0.178
0.997	0.878	0.143	0.172	0.205
1	1	0.164	0.196	∞
$u_A(U) = \sqrt{\sum (f(x_i) - h_{xi})^2} / \sqrt{n}$		0.019	0.012	0.016
$u_A(U)$ %		13.3%	7.0%	7.8%

Results of measurand $\bar{x} = x \pm U$ with the extended uncertainty U of probability $P = k_P$ are:

Gauss model: $x = \bar{x} \pm k_{PG} \dfrac{S(x_i)}{\sqrt{n}} = \bar{x} \pm 0.205 \qquad P = 0.997$

+COS model: $x = \bar{x} \pm k_{PC} \dfrac{X_1}{\sqrt{n}} = \bar{x} \pm 0.143 \qquad P = 0.997$

or $x = \bar{x} \pm k_{PC} \dfrac{X_2}{\sqrt{n}} = \bar{x} \pm 0.172 \qquad P = 0.997$

Extended uncertainty with confidence level 0.997 of the Gauss model is of 43.4 % or 33.1% higher than of the COS² model of X_1 or X_2 half-ranges. The mean value in the case of COS² model is inside the range ±(2.38 or 2.49)u_A of Gauss PDF with the probability nearly 1.

6. FINAL CONCLUSIONS

The additional information that experimental data values of the sample are distributed in limited range allows using the non-Gaussian distributions of limited width as model of their histogram. Some of them are given in Table 1 of Supplement 1 to GUM [2].

To achieve that the unconventional model not commonly used is also proposed. This suggested model of measurand probability density function is a shifted up (upward moved) one period of cosine function. This is the two-parameter model only as the Gauss distribution is and provides a useful tool for evaluation of uncertainties. This function of properly chosen parameters approximates very well the central region of Gauss PDF in the range up to about ± 2.5 standard deviation and gives better estimated uncertainty of the data of limited dispersion than unlimited Gauss PDF.

An especially useful form of cosine distribution is when the amplitude A of cosine and shifting value B are equal. It means that the range of the random variable density has limits in points, where cosine is touching the horizontal axe. In radians it is ±π. It is also equal to cos².

The proposed distribution does not have such inconvenience, which is characteristic for Gaussian distribution of which its left and right borders of tails are not limited up to ±∞. In practice it is impossible for empirical observations to get values so widely distributed even with very small probability.

The unconventional cosine based model of distribution of proposed symbol COS² is expressed by a well known, easy to generate trigonometric function.

The COS² distribution is very similar to Normal distribution and for the most cases the precision of fitting about in the range of its one full period is obtained. For more narrow ranges the better accuracy of approximation ±1% can be achieved.

The best fitting, near of approximation of Gaussian distribution by cosine PDF, is possible to achieve if cosine function is upward moved just a bit more than the value of cosine amplitude A. However, in such a case the cosine distribution is bounded similarly to a Normal distribution with cut off tails.

Distribution COS² has value of kurtosis nearly equal to triangular one, so it is also possible to use for it more accurate two-component estimator of PDF [9].

The convolutions of cosine-based distributions approached Normal distribution; some operations are easier, use less time and memory consuming.

As the generation and simulations of cosine based distribution function are simpler, so they can be implemented in single chip computers or in small micro-calculators.

The discussed cosine based distribution would be implemented in transducers and simpler instruments equipped with automatic uncertainty estimation to indicate results on line with required confidence [5], [6].

It may be also worth including the cosine-based distribution to basic PDF distributions used in the routine data assessment.

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