POINTWISE COMPLETENESS AND POINTWISE DEGENERACY OF STANDARD AND POSITIVE LINEAR SYSTEMS WITH STATE-FEEDBACKS

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Abstract:

The pointwise completeness and pointwise degeneracy of standard and positive linear discrete-time and continuous-time systems with state-feedbacks are addressed. It is shown that: 1) the pointwise completeness and pointwise degeneracy of continuous-time standard systems are invariant under the state and output feedbacks, 2) for standard and positive discrete-time and positive continuous-time systems necessary and sufficient conditions are established for the existence of gain matrices of state-feedbacks such that the closed-loop systems are pointwise complete. Considerations are illustrated by numerical examples.

Keywords: *pointwise completeness, pointwise degeneracy, positive linear systems, state-feedbacks.*

1. Introduction

A dynamical system described by homogenous equation is called pointwise complete if every given final state of the system can by reached by suitable choice of its initial state. A system, which is not pointwise complete, is called pointwise degenerated.

The pointwise completeness and pointwise degeneracy of linear continuous-time systems with delays have been investigated in [3], [9], [12], [15]. The pointwise completeness of linear discrete-time cone-systems with delays has been analyzed in [14].

In positive systems inputs, state variables and outputs take only non-negative values [4], [5]. The pointwise completeness and pointwise degeneracy of positive discrete-time linear systems with delays have been considered in [2].

Mathematical fundamentals of fractional calculus are given in the monographs [10], [11], [13]. The positive fractional linear systems have been introduced in [6], [7] and the pointwise completeness and pointwise degeneracy of fractional linear systems have been investigated in [1], [8].

In this paper the pointwise completeness and pointwise degeneracy of standard and positive linear systems with state-feedbacks will be addressed. The structure of the paper is the following. In section 2 the basic definitions and theorems concerning the pointwise completeness and pointwise degeneracy are recalled and necessary and sufficient conditions are established for the pointwise completeness and pointwise degeneracy of the closed-loop systems. The same problem for positive systems is analyzed in section 3. Concluding remarks are given in section 4.

The following notation will be used in the paper. The set of real $n\times m$ matrices will be denoted by $\Re^{n\times m}$ and with nonnegative entries by $\Re_+^{n\times m}$ and $\Re_+^n = \Re_+^{n\times 1}$. The set of nonnegative integers will be denoted by Z_+ and the $n\times m$ identity matrix will be denoted by I_n.

2. Standard linear systems

2.1. Discrete-time systems

Consider the discrete-time linear system

$$x_{i+1} = Ax_i,\ i\in Z_+ = \{0,1,2,..\} \tag{2.1}$$

where $x_i\in\Re^n$ is the state vector and $A\in\Re^{n\times m}$.

Definition 1. The system (2.1) is called pointwise complete at $i=q$ if for every final state $x_f\in\Re^n$ there exists an initial state $x_q = x_f$ such that

Theorem 1. The system (2.1) is point wise complete if and only if the matrix A is nonsingular i.e $\det A\neq 0$.

Proof. The solution of equation (2.1) at $i=q$ for $x_q = x_f$ is given by $x_f = A^q x_0$ and

$$x_0 = A^{-q}x_f \tag{2.2}$$

if and only if $\det A\neq 0$. ■

Definition 2. The system (2.1) is called pointwise degenerated in the direction v at $i=q$ if there exists a non-zero vector $v\in\Re^n$ such that for all initial states $x_0\in\Re^n$ the solution of (2.1) for $i=q$ satisfies the condition $v^{\mathrm{T}}x_q = 0$ where T denotes the transpose.

Theorem 2. The system (2.1) is pointwise degenerated in the direction v at $i=q$ if and only if the matrix A is singular i.e. $\det A = 0$.

Proof. Note that $\det A^q = (\det A)^q$ and there exists a vector $v\in\Re^n$ such that $v^{\mathrm{T}}A^q = 0$ if and only if $\det A = 0$ and from $x_q = A^q x_0$ we have $v^{\mathrm{T}}x_q = v^{\mathrm{T}}A^q x_0$ for every $x_0 \in\Re^n$ ■

Now let us consider the discrete-time linear system

$$x_{i+1} = Ax_i + Bu_i,\ i\in Z_+ \tag{2.3}$$

with the state-feedback

$$u_i = Kx_i \tag{2.4}$$

where $u_i\in\Re^m$ is the input vector, $B\in\Re^{n\times m}$ and $K\in\Re^{m\times n}$ is a gain matrix.

Substitution of (2.4) into (2.3) yields the closed-loop system

$$x_{i+1} = A_c x_i,\ i \in Z_+ \tag{2.5}$$

where

$$A = A + BK \tag{2.6}$$

Theorem 3. Let the system (2.3) be pointwise degenerated in the direction v at $i = q$. Then there exists a gain matrix K such that the closed-loop system is point wise complete at $i = q$ if and only if the condition

$$\text{rank}\,[A, B] = n \tag{2.7}$$

is satisfied.

Proof Necessity. The closed-loop system is pointwise complete at $i = q$ if and only if $\det A_c \neq 0$. From the equality

$$A + BK = [A, B]\begin{bmatrix} I_n \\ K \end{bmatrix} \tag{2.8}$$

is follows $\det A_c \neq 0$ implies the condition (2.7).

Sufficiency. Let the nonsingular matrix A_c contain A all r ($r = \text{rank}A$) linearly independent rows of the matrix A. The equation

$$BK = A_c - A \tag{2.9}$$

has a solution K since the condition (2.7) implies

$$\text{rank}B = \text{rank}\,[B, A_c - A]. \ \blacksquare \tag{2.10}$$

Theorem 4. Let the system (2.3) be pointwise degenerated in the direction v at $i = q$. Then there exists a gain matrix K such that the closed-loop system is pointwise complete at $i = q$ if the pair (A, B) of the system (2.3) is reachable.

Proof. The system (2.3) is reachable if and only if

$$\text{rank}\,[Iz - A, B] = n$$

for all $z \in C$ (the field of complex numbers). (2.11)

For $z = 0$ from (2.11) we obtain the condition (2.7).

Therefore, if the pair (A, B) is reachable then there exists K such that the closed-loop system is pointwise complete at $i = q$. $\blacksquare$

Example 1. Consider the system (2.3) with the matrices

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{2.12}$$

It is easy to check that the pair (2.12) is not reachable since

$$\text{rank}\,[Iz - A, B]_{z=1} = \text{rank}\begin{bmatrix} z & 0 & -1 & 0 \\ 0 & z-1 & 0 & 0 \\ 0 & 2 & z-1 & 1 \end{bmatrix}_{z=1} = 2 < n = 3$$

but the condition (2.7) is satisfied

$$\text{rank}\,[A, B] = \text{rank}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 1 \end{bmatrix} = 3.$$

In this case the closed-loop matrix

$$A_c = A + BK = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}[k_1 \quad k_2 \quad k_3] =$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ k_1 & k_2 - 2 & k_3 + 1 \end{bmatrix}$$

is nonsingular for $k_1 \neq 0$ and arbitrary k_2 and k_3.

Therefore, the closed-loop system is pointwise complete for $k_1 \neq 0$ and arbitrary k_2 and k_3.

2.2. Continuous-time systems

Consider the continuous-time linear system

$$\dot{x}(t) = Ax(t) \tag{2.13}$$

where $x(t) \in \Re^n$ is the state vector and $A \in \Re^{n \times n}$.

Definition 3. The system (2.13) is called pointwise complete at $t = t_f$ if for every final state $x_f \in \Re^n$ there exists an initial state $x(0) = x_0$ such that $x(t_f) = x_f$.

Definition 4. The system (2.13) is called pointwise degenerated in the direction v at $t = t_f$ if there exists a non-zero vector $v \in \Re^n$ such that for all initial states $x_0 \in \Re^n$ the solution of (2.13) for $t = t_f$ satisfies the condition $v^T x_f = 0$.

Theorem 5. The system (2.13) is pointwise complete at every $t = t_f$ and for any matrix A.

Proof. The solution of equation (2.13) at $t = t_f$ for $x(t_f) = x_f$ is given by

$$x_f = e^{At_f} x_0 \tag{2.14}$$

and

$$x_0 = e^{-At_f} x_f \tag{2.15}$$

since $\det[e^{At_f}] \neq 0$ for any A and t_f. $\blacksquare$

Therefore, we have the following corollary.

Corollary. The system (2.13) is not pointwise degenerated for any A and t_f.

Now let us consider the continuous-time linear system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.16}$$

with the state-feedback

$$u(t) = Kx(t) \tag{2.17}$$

where $u(t) \in \Re^m$ is the input vector, $B \in \Re^{n \times m}$ and $K \in \Re^{m \times n}$ is a gain matrix.

Substitution of (2.17) into (2.16) yields the closed-loop system

$$\dot{x}(t) = A_c x(t) \tag{2.18}$$

where

$$A_c = A + BK \tag{2.19}$$

From Theorem 5 and corollary we have the following.

Theorem 6. The pointwise completeness and the pointwise degeneracy of the continuous-time system (2.13) are invariant under the state-feedback (2.17).

Remark. The point wise completeness and the pointwise degeneracy of the continuous-time system (2.13) are also invariant under the output-feedback $u(t) = Fy(t)$ where $C \in \Re^{m\times p}$ is a gain matrix, $y(t) = Cx(t)$ is the output vector and $C \in \Re^{p\times n}$.

In this case the closed-loop system matrix has the form $A_c = A + BFC$.

3. Positive linear systems

3.1. Discrete-time systems

Consider the discrete-time linear system (2.3).

Definition 5 [4], [5]. The system (2.3) is called positive if $x_i \in \Re_+^n$, $i \in Z_+$ for any initial state $x_0 \in \Re_+^n$ and all input sequences $u_i \in \Re_+^m$, $i \in Z_+$.

Theorem 7 [4], [5]. The system (2.3) is positive if and only if

$$A \in \Re_+^{n\times n},\ B \in \Re_+^{n\times m}. \tag{3.1}$$

Definition 6 [1]. The positive system (2.3) is called pointwise complete at $i = q$ if for every final state $x_f \in \Re_+^n$ there exists an initial state $x_0 \in \Re_+$ such that $x_q = x_f$.

A matrix $A \in \Re_+^{n\times n}$ is called monomial if and only if every its row and every its column contains only one positive entry and the remaining entries are zero.

Theorem 8. The positive system (2.1) is pointwise complete at $i = q$ if and only if the matrix A is a monomial matrix.

Proof. It is easy to check that the matrix A^q for $q = 1,2,\ldots$ is monomial if and only if the matrix A is monomial. It is well known [5] that $A^{-q} \in \Re_+^{n\times n}$ if and only if A is a monomial matrix. In this case from (2.2) we have $x_0 \in \Re_+^n$ for every $x_f \in \Re_+^n$. ■

Definition 7. The positive system (2.1) is called point wise degenerated at $i = q$ if there exists at least one final state $x_f \in \Re_+^n$, which is not reachable in q steps from any initial state $x_0 \in \Re_+^n$ i.e. the equality $x_q = x_f$ is not satisfied for any $x_0 \in \Re_+^n$.

Theorem 9. The positive system (2.1) is pointwise degenerated at $i = q$ if and only if the matrix A is not a monomial matrix.

Proof. The equation $x_q = x_f = A^q x_0$ has a solution $x_0 \in \Re_+^n$ for a given $x_f \in \Re_+^n$ if and only the matrix A is a monomial matrix. ■

Now let us consider the positive system (2.3) with the state-feedback (2.4). The closed-loop system (2.5) is positive if and only if $A_c = A + BK \in \Re_+^{n\times n}$.

Let the positive system (2.1) be pointwise degenerated at $i = q$. We are looking for a gain matrix $K \in \Re^{m\times n}$ such that the closed-loop system (2.5) is positive and pointwise complete at $i = q$. A vector (column) is called monomial if only one its component is positive and the remaining components are zero.

Theorem 10. Let the positive system (2.1) be pointwise degenerated at $i = q$. There exists a gain matrix $K \in \Re^{m\times n}$ such that the closed-loop system (2.5) is positive and pointwise complete at $i = q$ if and only if the following conditions are satisfied

$$\operatorname{rank}[A,B] = 0 \tag{3.2a}$$

there exists a monomial matrix A_c such that

$$\operatorname{rank} B = \operatorname{rank}[B, A_c - A]. \tag{3.2b}$$

Proof. From (2.5) it follows that there exists a gain matrix $K \in \Re^{m\times n}$ such that $A_c = A + BK \in \Re_+^{n\times n}$ is monomial matrix only if the matrix (3.2a) holds.

By Kronecker-Cappely theorem the equation $BK = A_c - A$ has a solution K if and only if the condition (3.2b) is satisfied for a monomial matrix $A_c \in \Re_+^{n\times n}$. ■

Example 2. Consider the positive system (2.3) with the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix},\quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{3.3}$$

It is easy to check that the pair (3.3) satisfies the conditions (3.2).

We are looking for a gain matrix $K = [k_1\ k_2\ k_3]$ such that the closed-loop system matrix

$$A + BK = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1+k_1 & 2+k_2 & 3+k_3 \end{bmatrix} \in \Re_+^{3\times 3} \tag{3.4}$$

is a monomial matrix. From (3.4) it follows that the closed-loop system matrix is a monomial one for $k_1 = -1$, $k_2 = -2$ and $k_3 > -3$.

Example 3. Consider the positive system (2.3) with the matrices

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix},\quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{3.5}$$

The pair (3.5) satisfied the condition (3.2a) since

$$\operatorname{rank}[A,B] = \operatorname{rank}\left[\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right] = 3$$

but the pair does not satisfy the condition (3.2b). It is easy to see that does not exist a monomial matrix $A_c \in \Re_+^{3\times 3}$ such that for (3.5) $\text{rank}\,[B, A_c - A] = 1$.

In this case we have

$$A + BK = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ k_1 & k_2 & 2 + k_3 \end{bmatrix}. \tag{3.6}$$

From (3.6) it follows that does not exist a gain matrix $K = [k_1 \ \ k_2 \ \ k_3]$ such that the closed-loop system matrix is a monomial one.

3.2. Continuous-time systems

Consider the continuous-time linear system (2.16).

Definition 8. The system (2.16) is called positive if $x(t) \in \Re_+^n$, $t \geq 0$ for any initial state $x(0) = x_0 \in \Re_+^n$ and all input vectors $u(t) \in \Re_+^m$, $t \geq 0$.
A matrix $A = [a_{ij}] \in \Re^{n\times n}$ is called Metzler matrix if $a_{ij} \geq 0$ for $i \neq j$, $i,j = 1,\ldots,n$. Let M_n be the set of $n\times n$ real Metzler matrix.

Theorem 11 [4], [5]. The system (2.16) is positive if and only if

$$A \in M_n \text{ and } B \in \Re_+^{n\times m}. \tag{3.7}$$

Definition 9 [2]. The positive system (2.13) is called pointwise complete at $t = t_f$ if for every final state $x_f \in \Re_+^n$ there exists an initial state $x_0 \in \Re_+^n$ such that $x(t_f) = x_f$.

Theorem 12 [2]. The positive system (2.13) is pointwise complete at $t = t_f$ if and only if the matrix A is diagonal.

Proof. From (2.15) it follows that for any $x_f \in \Re_+^n$ there exists $x_0 \in \Re_+^n$ if and only if e^{-At_f} is monomial matrix. Taking into account that

$$e^{-At_f} = \sum_{k=0}^{\infty} \frac{(-At_f)^k}{k!} = I_n - \frac{At_f}{1!} + \frac{(At_f)^2}{2!} - \ldots \tag{3.8}$$

we see that the matrix (3.8) is monomial if and only if A is diagonal. ■

Definition 10 [2]. The positive system (2.13) is called pointwise degenerated at $t = t_f$ if there exists at least one final state $x_f \in \Re_+^n$ which is not reachable at $t = t_f$ from any initial state $x_0 \in \Re_+^n$, i.e. the equality $x(t_f) = x_f$ is not satisfied for any $x_0 \in \Re_+^n$.

Theorem 13. The positive system (2.13) is pointwise degenerated at $t = t_f$ if and only if the matrix A is not diagonal.

Proof. For a given $x_f \in \Re_+^n$ there exists $x_0 \in \Re_+^n$ satisfying (2.15) if and only if the matrix e^{-At_f} is monomial and this holds if and only if the matrix A is diagonal. ■

Now let us consider the continuous-time system (2.16) with the state-feedback (2.17). The closed-loop system is positive if and only if the closed-loop system matrix (2.19) is a Metzler matrix.

Let the positive system (2.16) be pointwise degenerated at $t = t_f$. We are looking for a gain matrix $K \in \Re^{m\times n}$ such that the closed-loop system matrix (2.19) is a diagonal matrix.

Theorem 14. Let the positive system (2.16) be pointwise degenerated at $t = t_f$. There exists a gain matrix $K \in \Re^{m\times n}$ such that the closed-loop system is positive and pointwise complete if and only if there exists a diagonal matrix $\overline{A}_c \in \Re^{n\times n}$ such that the condition

$$\text{rank}\, B = \text{rank}\,[\overline{A}_c - A, B]. \tag{3.8}$$

is satisfied.

Proof. By Kronecker-Cappely theorem the equation

$$BK = \overline{A}_c - A. \tag{3.9}$$

has a solution K for any diagonal matrix $\overline{A}_c$ and $B \in \Re_+^{n\times n}$ if and only if the condition (3.8) is satisfied. ■

Note that in particular case the matrix $\overline{A}_c$ can be chosen as the diagonal matrix with the same diagonal entries as in the matrix A. In this case all diagonal entries of $\overline{A}_c - A$ are zero.

Example 4. Consider the positive system (2.16) with the matrices

$$A = \begin{bmatrix} -1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.10}$$

We are looking for a gain matrix

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{bmatrix} \tag{3.11}$$

such that the closed-loop system matrix (2.19) is diagonal.

Let

$$\overline{A}_c = \text{diag}[k_1 \ \ k_2 \ \ k_3 \ \ k_4]. \tag{3.12}$$

The condition (3.8) for (3.10) and (3.12) takes the form

$$\text{rank} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} k_1 + 1 & -2 & 0 & -1 & 0 & 1 \\ 0 & k_2 & 0 & 0 & 0 & 0 \\ -2 & -1 & k_3 & -2 & 1 & 0 \\ 0 & 0 & 0 & k_4 - 2 & 0 & 0 \end{bmatrix} = 2$$

and it is satisfied for $k_1 = 0$ and $k_4 = 2$.

In this case the equation (3.9) has the form

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} K = \begin{bmatrix} k_1 + 1 & -2 & 0 & -1 \\ 0 & k_2 & 0 & 0 \\ -2 & -1 & k_3 & -2 \\ 0 & 0 & 0 & k_4 - 2 \end{bmatrix} \tag{3.13}$$

and its solution is

$$K = \begin{bmatrix} -2 & -1 & k_3 & -2 \\ k_1+1 & -2 & 0 & -1 \end{bmatrix} (k_1, k_3 \text{ are arbitrary}) \quad (3.14)$$

for $k_2=0$ and $k_4=2$.

It is easy to check that for

$$\overline{A}_c = \mathrm{diag}[-1 \quad 0 \quad 0 \quad 2]$$

the matrix K has the form

$$K = \begin{bmatrix} -2 & -1 & 0 & -2 \\ 0 & -2 & 0 & -1 \end{bmatrix}$$

which can be obtained from (3.14) for $k_1=-1$ and $k_3=0$.

4. Concluding remarks

The pointwise completeness and pointwise degeneracy of standard and positive linear discrete-time and continuous-time systems with state-feedbacks have been addressed. It has been shown that:

1) The pointwise completeness and pointwise degeneracy of continuous-time standard systems are invariant under the state and output feedbacks (Theorem 6).
2) If the discrete-time linear system is pointwise degenerated then there exists a gain matrix of the state-feedback such that the closed-loop system is pointwise complete if and only if the condition (2.7) is satisfied.
3) If the positive discrete-time linear system is pointwise degenerated then there exists a gain matrix of the state-feedback such that the closed-loop system is positive and pointwise complete if and only if the conditions (3.2) are satisfied (Theorem 10).
4) If the positive continuous-time linear system is point wise degenerated then there exists a gain matrix of the state-feedback such that the closed-loop system is positive and pointwise complete if and only if the pair satisfies the condition of Theorem 14.

The considerations have been illustrated by numerical examples.

The considerations can be easily extended for:

1) linear discrete-time and continuous-time systems with delays
2) linear fractional systems without and with delays.

Extensions of these considerations for standard positive and fractional 2D linear systems are open problem.

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