NUKLEONIKA 2004;49(Supplement 2):S27–S31

PROCEEDINGS

Sum rule of the correlation function

Stanisław Mrówczyński,
Radosław Maj

Abstract We discuss a sum rule satisfied by the correlation function of two particles with small relative momenta. We first derive the sum rule, which results from the completeness condition of the quantum states of the two particles, and then we discuss it for the case of strongly interacting pair of neutron and proton and for the Coulomb system of two charged particles.

Key words particle correlations • nuclear collisions

S. Mrówczyński[✉]
The Andrzej Sołtan Institute for Nuclear Studies,
69 Hoża Str., 00-681 Warsaw, Poland
and Institute of Physics,
Świętokrzyska Academy,
15 Świętokrzyska Str., 25-406 Kielce, Poland,
Tel./Fax: +48 22/ 621 28 04,
e-mail: mrow@fuw.edu.pl

R. Maj
Institute of Physics,
Świętokrzyska Academy,
15 Świętokrzyska Str., 25-406 Kielce, Poland

Received: 7 January 2004

The correlation functions of two identical or nonidentical particles with small relative momenta have been extensively studied in nuclear collisions for bombarding energies from tens of MeV [4] to hundreds of GeV [6]. These functions provide information about space-time characteristics of particle sources in the collisions. As shown by one of us [11], the correlation function integrated over particle relative momentum satisfies a simple relation due to the completeness of quantum states. The aim of this note is to discuss the sum rule to prove or disprove its usefulness in the experimental studies. A more detailed analysis will be given elsewhere.

The starting point of our considerations is the formula repeatedly discussed in the literature which expresses the correlation function $R(\mathbf{q})$ of two particles with the relative momentum $\mathbf{q}$ as

$$R(\mathbf{q}) = \int d^3r D_r(\mathbf{r}) \left|\phi_{\mathbf{q}}(\mathbf{r})\right|^2 \tag{1}$$

where $\phi_{\mathbf{q}}(\mathbf{r})$ is the wave function of relative motion of the two particles and $D_r(\mathbf{r})$ is the effective source function defined through the probability density $D_r(\mathbf{r},t)$ to emit the two particles at the relative distance $\mathbf{r}$ and the time difference t. Namely, $D_r(\mathbf{r}) = \int dt D_r(\mathbf{r} - \mathbf{v}t, t)$ with $\mathbf{v}$ being the particle pair velocity with respect to the source. The source function $D_r(\mathbf{r})$ is normalized as $\int d^3r D_r(\mathbf{r}) = 1$. We observe that the spherically symmetric single-particle source function provides, in general, the effective source $D_r(\mathbf{r})$ which is elongated along the velocity $\mathbf{v}$. To simplify the analysis, we, however, assume here that the source function $D_r(\mathbf{r})$ is spherically symmetric. Such an assumption makes sense when the particles are emitted instantaneously. Also for simplicity, we treat the formula (1) non-relativistically. The single-particle source function is chosen in the Gaussian form. Then, the effective relative source function is also Gaussian

(2) $$D_r(\mathbf{r}) = \frac{1}{\left(4\pi r_0^2\right)^{2/3}} \exp\left(-\frac{\mathbf{r}^2}{4r_0^2}\right)$$

and the mean radius squared of a single-particle source equals $<\mathbf{r}^2> = 3r_0^2$.

Let us consider the correlation function integrated over the relative momentum. Since $R(\mathbf{q}) \to 1$ when $\mathbf{q} \to \infty$, we rather discuss the integral of $(R(\mathbf{q}) - 1)$. Using eq. (1) and taking into account the normalization condition of $D_r(\mathbf{r})$ one finds after changing order of the **r**- and **q**-integration the expression

(3) $$\int \frac{d^3q}{(2\pi)^3}\left(R(\mathbf{q})-1\right) = \int d^3r D_r(\mathbf{r}) \int \frac{d^3q}{(2\pi)^3}\left(\left|\phi_q(\mathbf{r})\right|^2 - 1\right).$$

It appears that the integral over **q** in the r.h.s. of eq. (3) is determined by the quantum-mechanical completeness condition. Indeed, the wave functions satisfy the well-known closure relation

(4) $$\int \frac{d^3q}{(2\pi)^3} \phi_{\mathbf{q}}(\mathbf{r})\phi_{\mathbf{q}}^*(\mathbf{r}') + \sum_\alpha \phi_\alpha(\mathbf{r})\,\phi_\alpha^*(\mathbf{r}') = \delta^{(3)}(\mathbf{r}-\mathbf{r}') \pm \delta^{(3)}(\mathbf{r}+\mathbf{r}')$$

where ϕ_α represents a possible bound state of the two particles of interest. When the particles are not identical the second term in the r.h.s. of eq. (4) should be neglected. This term guarantees a right symmetry of both sides of the equation for the case of identical particles. The upper sign is for bosons, while the lower one for fermions. The wave function of identical bosons (fermions) $\phi_{\mathbf{q}}(\mathbf{r})$ is (anti-)symmetric when $\mathbf{r} \to -\mathbf{r}$, and the r.h.s of eq. (4) is indeed (anti-)symmetric when $\mathbf{r} \to -\mathbf{r}$ or $\mathbf{r}' \to -\mathbf{r}'$. If the particles of interest carry spin, the sum over the spin degrees of freedom in the l.h.s. of eq. (4) is implied.

When the integral representation of $\delta^{(3)}(\mathbf{r} - \mathbf{r}')$ is used, the limit $\mathbf{r} \to \mathbf{r}'$ can be taken in eq. (4), and we get the relation

(5) $$\int \frac{d^3q}{(2\pi)^3}\left(\left|\phi_{\mathbf{q}}(\mathbf{r})\right|^2 - 1\right) = \pm\delta^{(3)}(2\mathbf{r}') - \sum_\alpha \left|\phi_\alpha(\mathbf{r})\right|^2$$

which substituted into eq. (3) provides the desired sum rule

(6) $$\int d^3q\left(R(\mathbf{q})-1\right) = \pm\pi^3 D_r(0) - \sum_\alpha A_\alpha$$

where A_α is the formation rate of the bound state α

(7) $$A_\alpha = (2\pi)^3 \int d^3r D_r(\mathbf{r}) \left|\phi_{\mathbf{q}}(\mathbf{r})\right|^2.$$

A_α relates the cross section to produce the bound state α with the momentum **P** to the cross section to produce the two particles with the momenta **P**/2. The non-relativistic relation reads

$$\frac{d\sigma^\alpha}{d\mathbf{P}} = A_\alpha \frac{d\tilde{\sigma}}{d(\mathbf{P}/2)d(\mathbf{P}/2)}$$

where the tilde means that the short range correlations are removed from the two-particle cross section which is usually taken as a product of the single-particle cross sections.

The completeness condition is, obviously, valid for any inter-particle interaction. It is also valid when the pair of particles interact with the time-independent external field, e.g. the Coulomb field generated by the particle source. Thus, the sum rule (6) holds under very general conditions unless the basic formula (1) is justified. In the case of non-interacting particles described by a plane wave, the sum rule (6) was found in [12], see also [2, 3], by means of explicit integration of $(R(\mathbf{q}) - 1)$ over **q**.

Let us apply the sum rule to the correlation function of strongly interacting system of neutron and proton. The nucleons emitted from a source are usually assumed to be unpolarized, and one considers the spin-averaged correlation function R which is a sum of the singlet and triplet correlation functions $R^{s,t}$ with weight coefficients 1/4 and 3/4, respectively. Here, we consider, however, the singlet and the triplet correlation functions separately. Then, the sum rule (6) reads

(8) $$\int d^3q\left(R^s(\mathbf{q})-1\right) = 0$$

(9) $$\int d^3q\left(R^t(\mathbf{q})-1\right) = -A$$

where A is the deuteron formation rate.

Following [9], we calculate the correlation functions $R^{s,t}$, assuming that the source radius is significantly greater than the n-p interaction range. Then, the wave function of the n-p pair (in a scattering state) can be approximated by its asymptotic form which is

(10) $$\phi_{np}^{s,t}(\mathbf{r}) = e^{i\mathbf{qr}} + f^{s,t}(\mathbf{q})\frac{e^{iqr}}{r}$$

where $q \equiv |\mathbf{q}|$ and $f^{s,t}(\mathbf{q})$, is the scattering amplitude. The amplitude is chosen as

(11) $$f^{s,t}(\mathbf{q}) = \frac{-a^{s,t}}{1 - \frac{1}{2}d^{s,t}a^{s,t}q^2 + iqa^{s,t}}$$

where $a^{s,t}$ ($d^{s,t}$), is the scattering length (effective range) of the n-p scattering; $a^s = -23.7$ fm, $d^s = 2.7$ fm and $a^t = 5.4$ fm, $d^s = 1.7$ fm [10]. The amplitude (11) takes into account only the s-wave scattering. This is justified as long as only small relative momenta are considered. Substituting the wave function (10) into the formula (1) with the source function (2), we get the n-p correlation function. Since the source described by the formula (2) is spherically symmetric the correlation function depends on q only.

In Figs. 1 and 2 we show the singlet and triplet correlation function, respectively, computed for three values of the source size parameter r_0. As seen, the triplet correlation is negative in spite of the attractive neutron-proton interaction. This happens, in accordance with the sum rule (9), because the neutron and proton, which are close to each other in the phase-space, tend to exist in a bound not in a scattering state. And the n-p pairs, which form a deuteron, deplete the sample of n-p pairs and produce a dip of the correlation function at small relative momenta.

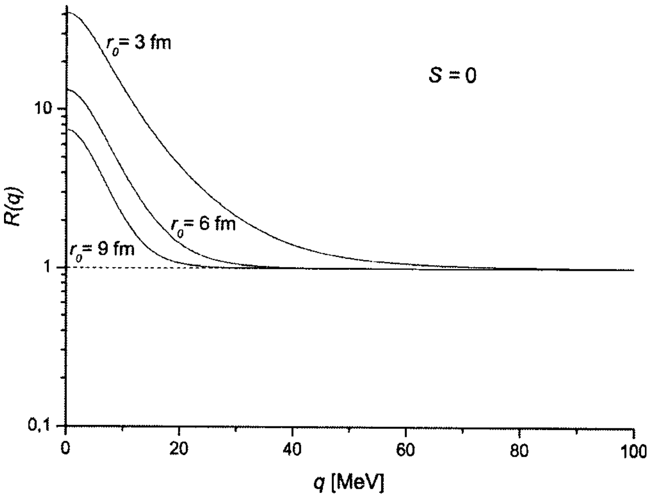

Fig. 1. The singlet correlation function of neutron and proton.

The deuteron formation rate, which enters the sum rule (9), is computed with the deuteron wave function in the Hulthén form

$$\phi_d(\mathbf{r}) = \left(\frac{\alpha\beta(\alpha+\beta)}{2\pi(\alpha-\beta)^2} \right)^{1/2} \frac{e^{-\alpha r} - e^{-\beta r}}{r} \tag{12}$$

with α = 0.23 fm^{-1} and β = 1.61 fm^{-1} [7]. Substituting the wave function (12) and the source function (2) into eq. (7), we get the deuteron formation rate A.

Although the sum rule (6) assumes the integration up to the infinite momentum, one expects that the integral in eq. (6) saturates at sufficiently large q. To discuss the problem quantitatively we define the function

$$S(q_{\max}) = 4\pi \int_0^{q_{\max}} dq\, q^2 \left(R(q) - 1\right). \tag{13}$$

We assume here that the correlation function depends on q only. Then, the angular integration is trivially performed.

In Figs. 3 and 4 we display the function $S(q_{\max})$ found for the singlet and triplet correlation function presented in Figs. 1 and 2, respectively. Although $S(q_{\max})$ at large $q_{\max}$ saturates for both the singlet and for the triplet correlation

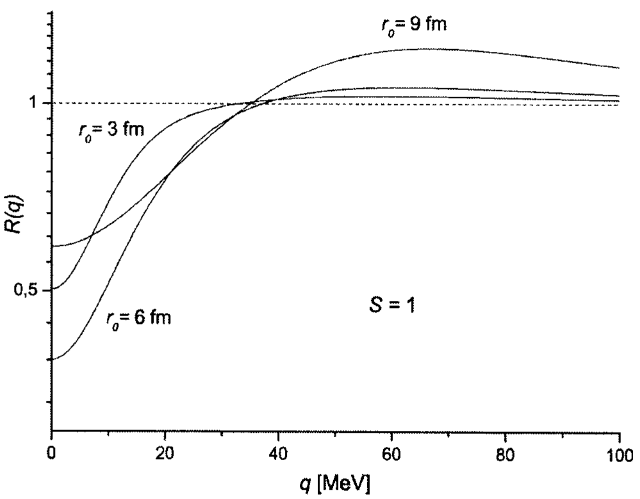

Fig. 2. The triplet correlation function of neutron and proton.

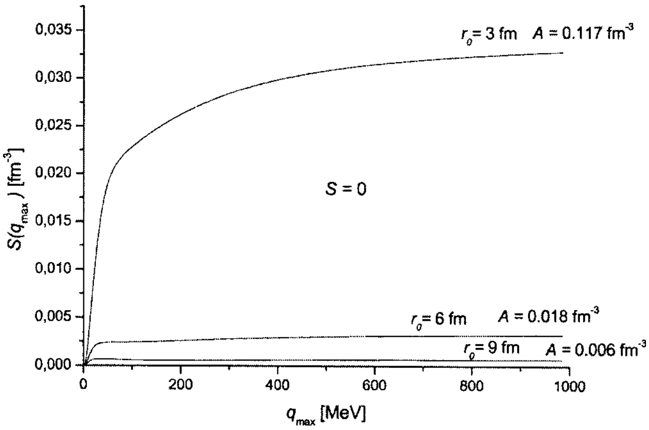

Fig. 3. The function $S(q_{\max})$ for the singlet correlation function. To set the scale, the corresponding values of the deuteron formation rate are given.

function, neither the sum rule (8) nor (9) is satisfied. The singlet $S(q_{\max})$ does not vanish at large $q_{\max}$, while the triplet $S(q_{\max})$ is not negative. However, comparing the numerical values of $S(q_{\max})$ to the corresponding deuteron formation rate, which sets the characteristic scale, one sees that with growing r_0 the sum rule is violated less and less dramatically. It is not surprising as the asymptotic form of the wave function (10) and the s-wave approximation become then more accurate. The formula (13) shows that due to the factor q^2 even small deviations of the correlation function from unity at large q generate sizable contribution to S. Therefore, one should take into account higher partial waves to satisfy the sum rule.

As well known, the Coulomb problem is exactly solvable within the non-relativistic quantum mechanics. The exact wave function of two non-identical particles interacting due to the repulsive Coulomb force is given as

$$\phi_q(\mathbf{r}) = e^{-\frac{\pi\lambda}{2q}} \Gamma\left(1 + i\frac{\lambda}{q}\right) e^{iqz/2} F\left(-i\frac{\lambda}{q}, 1, iq\eta\right) \tag{14}$$

where $q \equiv |\mathbf{q}|$, $\lambda = \mu e^2/8\pi$ with μ being the reduced mass of the two particles and $\pm e$ is the charge of each of them; F denotes the hypergeometric confluent function, and η is

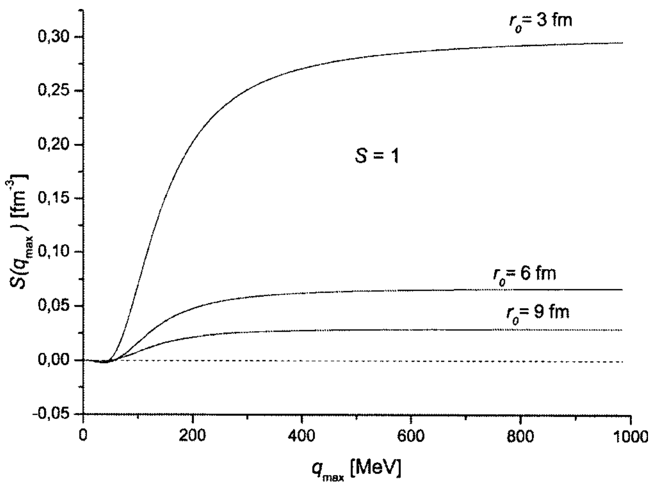

Fig. 4. The function $S(q_{\max})$ for the triplet correlation function.

the parabolic coordinate (see below). The wave function for the attractive interaction is obtained from (14) by means of the substitution $\lambda \to -\lambda$. When one deals with identical particles, the wave function $\phi_{\mathbf{q}}(\mathbf{r})$ should be replaced by its (anti-)symmetrized form. The modulus of the wave function (14) equals

$$\left|\phi_{\mathrm{q}}(\mathbf{r})\right|^2 = G(q)\left|F\left(-i\frac{\lambda}{q},1,iq\eta\right)\right|^2$$

where $G(q)$ is the so-called Gamov factor defined as

$$G(q) = \frac{2\pi\lambda}{q}\frac{1}{\exp\left(\frac{2\pi\lambda}{q}\right)-1}\,. \tag{15}$$

As seen, the modulus of the wave function of non-identical particles solely depends on the parabolic coordinate η. Therefore, it is natural to calculate the Coulomb correlation function in the parabolic coordinates: $\eta \equiv r - z$, $\xi \equiv r + z$ and ϕ with ϕ being the azimuthal angle. Then, the correlation function computed with the Gaussian source function (2) equals

$$R(q) = \frac{G(q)}{2\sqrt{\pi}r_0}\int_0^\infty d\eta \exp\left(-\frac{\eta^2}{16r_0^2}\right)\left|F\left(-i\frac{\lambda}{q},1,iq\eta\right)\right|^2 \tag{16}$$

where the integration over ξ has been performed.

In Fig. 5 we demonstrate the correlation function of non-identical repelling particles given by eq. (16). Figure 6 shows the function $S(q_{max})$ defined in eq. (13) which is computed for the correlation functions presented in Fig. 5. According to the sum rule (6), $S(q_{max})$ should vanish for sufficiently large q_{max}. As seen in Fig. 6, the function $S(q_{max})$ does not seem to saturate at large q_{max} and the sum rule is badly violated.

What is wrong here? The derivation of the sum rule (6) implicitly assumes that the integral in the l.h.s. of eq. (6) exists i.e. it is convergent. Otherwise interchanging of the integrations over $\mathbf{q}$ and over $\mathbf{r}$, which leads to eq. (3), is mathematically illegal. Unfortunately, the Coulomb correlation functions appear to decay too slowly with q, and consequently the integral in the left hand side of eq. (6) diverges.

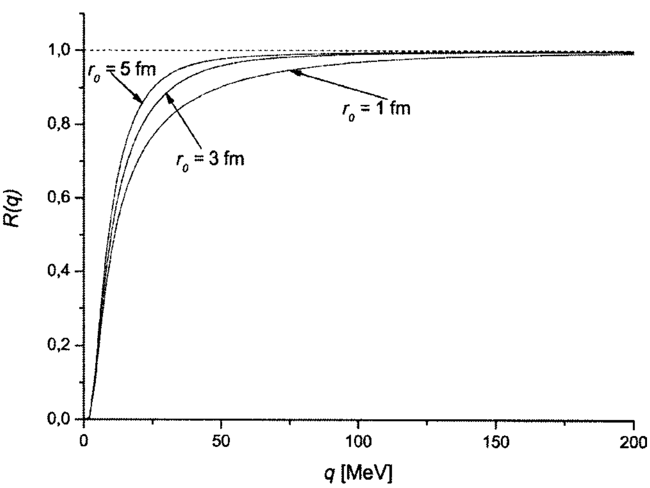

Fig. 5. The Coulomb correlation function of non-identical repelling particles.

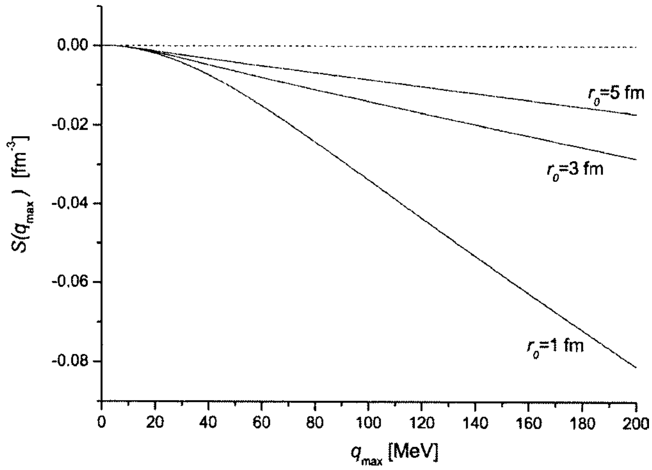

Fig. 6. The function $S(q_{max})$ for the Coulomb correlation function of non-identical repelling particles.

To clarify how the integral diverges one has to find the asymptotics of the correlation function at large q. It is rather difficult task as one has to determine the behavior of wave function (14) at large q for any $\mathbf{r}$, and then one has to perform the integration over $\mathbf{r}$. Up to our knowledge, the problem of large q asymptotics of the Coulomb correlation function has not been satisfactorily solved although it has been discussed in several papers [1, 5, 8, 13, 14]. We have not found a fully satisfactory solution but our rather tedious analysis, which will be published elsewhere, suggests that $(R(q) - 1) \sim 1/q^2$ when $q \to \infty$. Then, the integral in the left hand side of eq. (6) linearly diverges. We observe that the Gamov factor (15), which represents a zero size source and decays as $1/q$ at large q, leads to the quadratic divergence of the integral (6). We also note here that the asymptotics $1/q^2$ of the correlation function does not have much to do with the well known classical limit of the correlation function [1, 5, 8, 13, 14]. Since the large q limit of the correlation function corresponds to the small separation of the charged particles, which at sufficiently large q is smaller than the de Broglie wave length, the classical approximation breaks down.

The results presented here are not very encouraging. Although the sum rule (6) provides a rigorous relationship it is not very useful. The model calculations of the n-p correlation function do not satisfy the sum rule as the s-wave approximation, which is sufficient to properly describe a general shape of the correlation function, fails to correctly reproduce its tail. Unfortunately, the sum rule is very sensitive to the tail. The situation with the Coulomb interaction is a real disaster. Due to the strong electrostatic repulsion at small distances the correlation function too slowly decays at large momenta, and, consequently, the sum-rule integral is divergent. Then, the sum rule is not applicable at all. However, the sum rule, being rather unpractical, explains some qualitative features of the correlation function. In particular, it shows that in spite of the attractive interaction the correlation can be negative as it happens in the triplet channel of neutron and proton due to the deuteron formation.

Acknowledgment We are very grateful to W. Florkowski for a generous help in performing numerical calculations, and to

C. Manuel, R. Lednický and S. Pratt for fruitful discussions and correspondence.

References

1. Baym G, Braun-Munzinger P (1996) Physics of Coulomb corrections in Hanbury-Brown-Twiss interferometry in ultrarelativistic heavy ion collisions. Nucl Phys A 610:286c–296c
2. Bertsch GF (1994) Meson phase space density in heavy ion collisions from interferometry. Phys Rev Lett 72:2349–2350
3. Bertsch GF (1996) Meson phase space density in heavy ion collisions from interferometry (Errata). Phys Rev Lett 77:789
4. Boal DH, Gelbke CK, Jennings BK (1990) Intensity interferometry in subatomic physics. Rev Mod Phys 62:553–602
5. Gyulassy M, Kauffmann SK (1981) Coulomb effects in relativistic nuclear collisions. Nucl Phys A 362:503–533
6. Heinz UW, Jacak BV (1999) Two-particle correlations in relativistic heavy-ion collisions. Annu Rev Nucl Part Sci 49:529–579
7. Hodgson PE (1971) Nuclear reactions and nuclear structure. Clarendon Press, Oxford
8. Kim YD, de Souza RT, Gelbke CK, Gong WG, Pratt S (1992) Final-state Coulomb interactions for intermediate-mass fragment emission. Phys Rev C 45:387–397
9. Lednický R, Lyuboshitz VL (1982) Final-state interaction effect on pairing correlations between particles with small relative momenta. Sov J Nucl Phys 35:770–784
10. McCarthy IE (1968) Introduction to nuclear theory. John Wiley & Sons, New York
11. Mrówczyński S (1995) Sum rule of the correlation function. Phys Lett B 345:393–396
12. Podgoretsky MI (1991) Two-particle correlations of identical pions. Sov J Nucl Phys 54:891–893
13. Pratt S, Petriconi S (2003) Alternative size and lifetime measurements for high-energy heavy-ion collisions. Phys Rev C 68:054901
14. Sinyukov YuM, Lednický R, Akkelin SV, Pluta J, Erazmus B (1998) Coulomb corrections for interferometry analysis of expanding hadron systems. Phys Lett B 432:248–257