SUMMATION EQUATION TOOLS FOR SLIDE MICROBEARING SYSTEMS WEAR PROGNOSIS

Krzysztof Wierzcholski

Technical University of Koszalin, Institute of Mechatronics
Nanotechnology and Vacuum Technique
Śniadeckich Street 2, 75-453 Koszalin, Poland
phone: +48 94 3478344, fax: +48 94 3426753
e-mail: krzysztof.wierzcholski@wp.pl

Abstract

Efficient functioning of slide micro-bearings systems especially HDD micro-bearing require to recognition and modulation of the proper values of the friction forces and wear during the exploitation time. Possibility of modulation and control of mentioned problem belong to the artificial intelligence of HDD micro-bearing. This paper presents the some applications of summation equations regard to the calculation prognosis of micro-bearing parameters such as friction forces, ,friction coefficients and wear .Summation equations are presented a new form of difference and recurrence equations where the unknown function occurs as the argument of the reciprocal unified operator of summation (UOS) Presented problem describes not continuous relations hence determines the mathematical and numerical solutions in discrete spaces. Properly in the case of continuous functions, the mentioned summation equations have the same meaning as integral equations. This paper will present the transformation method of summation equations to recurrent equations. Recurrent equations for discrete function correspond to differential equations for the continuous function. The Application of presented theory in this paper contains the numerical solutions referring the wear values of HDD bearing system in the indicated period of operating time.

Keywords: *summation equation, wear prognosis, HDD micro-bearings*

1. General remarks about summation and recurrence equations

In this paper the summation equations are applied for solving the problems connected with the wear prognosis during the HDD -microbearing Seagate Barracuda and Computer Ventilator Xilence Case Fan exploitation [1, 12].

Up to now the information on summation equations and their mechanical applications is very scarce. Many more papers have been published on difference equations. Those mathematicians, physicists and engineers who deal with applied mathematics [3-10] became interested in the early 1960s in the new field: the theory of discrete solutions. This interest became more intense together with a large-scale progress: first in the technology of digital machines and further in the computing technique. The quantity of operations performed was no longer important; what matters is the time of the performance of operations on computers with increasing powers as well as the effectiveness in the convergence process of computational procedures conducted. A number of methods were then put to use for the solution of equations and systems of recurrence and summation equations. These methods could be used and the goals could be achieved in a numerical manner with the application of computers.

The result of an increased interest in recurrence and summation equations is among others the fact that differential equations can be simulated by means of summation and recurrence equations, and some linear summation and recurrence equations can be reduced to those differential equations that are equivalent with respect to solutions to recurrence equations.

It is to be noted that those phenomena are described with summation equations whose results or values change in a discrete manner. A completely different description of phenomena with the aid of differential equations concerns dependencies and values with continuous changes, which as we know are a large approximation of the surrounding reality [4, 11].

Owing to a constant growth of the potential of computers, a development is being observed of those methods that are approximate in the area of partial differential equations. In this research area, recurrence equations are becoming more and more widely applied. Initially, differential methods were developed. Further, the methods by Runge-Kutta, W. Ritz and B. G. Galerkin were created together with many other analytical and numerical methods, where the solutions of recurrence equations were required [9-11]. Another MES finite elements method is to be noted. The advantage of this method is obtaining solutions of differential or recurrence equations in areas with very complex shapes. A disadvantage of this method is obtaining approximate solutions which do not strictly satisfy the equation which governs the process, or its satisfaction with a very low degree of the shape function. The result of an improvement of the MES method with the use of recurrence and summation equations is an application of E. Trefftz functions as base functions in the MES method [7].

2. Definition and some properties of the unified operator of summation

Let S be a general operator of summation. Such operator generalizes up to now needed difference operator Δ, and its reciprocal form i.e. operator denoted by Δ^{-1}.

We define general operator S and its properties in the following form [5, 6]:

$$S_{\varepsilon\rho}^{1}(f_n) \equiv S_0^1(f_n) + \varepsilon\rho(f_n), \quad S_0^1(f_n) \equiv f_{n+1}, \quad \varepsilon = +1 \quad or \quad \varepsilon = -1, \tag{1}$$

we denote:

f_n - Complex functions defined on the natural numbers,

$\varepsilon\rho$ - basis of the unified operator of summation S,

ρ - Complex number of the complex variable,

k=1 - first rank as an upper index of the operator $S_{\varepsilon\rho}^{k}$ of unified summation.

The operation $S_{\varepsilon\rho}^{m}$ on the function f_n is known as the unified summation. The particular case of operation S presented by the relation (1) for ρ=0, has the form S_0^1 and can be denoted in the form S_0 and called as unitary translation operator (UTO). Operation of the UTO on the function f_n was univocal defined by the Eq. (1).

Unified operation (1) for ε= +1, ρ=z describes a new form of summation. Both forms are as follows: S_z^1, S_z^{-1}. At first we show some particular properties of UTO i.e. for ρ=0. It is easy to see that the following operation is true:

$$S_0^1(S_0^1 f_n) = S_0^1(f_{n+1}) = f_{n+2}. \tag{2}$$

If the operator S has rank k, defined in the following recurrence form:

$$S_0^0(f_n) = f_n, \quad S_0^k(f_n) = S_0^1\left[S_0^{k-1}\left(f_n\right)\right] \; for \quad k = 1,2,3,\ldots, \tag{3}$$

then is easy to proof the following dependences:

$$S_0^k(f_n) = f_{n+k}, \quad S_0^k(f_n\, g_n) = f_{n+k}\, g_{n+k}, \quad S_0^k\left(\frac{f_n}{g_n}\right) = \frac{f_{n+k}}{g_{n+k}} \quad for \quad k = 0,1,2,3,\ldots, \tag{4}$$

From equations: (4), follows the multiplicative property of the UTO. Let be k=0. In this case the unitary translation operator has order zero. For arbitrary natural number k the operator has order k. The UTO of the order k, is linear i.e. is simultaneously additively and homogeneity, hence we can write the following equation [5, 6]:

$$S_0^k(\alpha f_n + \beta f_n) = \alpha S_0^k(f_n) + \beta S_0^k(f_n), \tag{5}$$

where α, β are two arbitrary constants independent of n. Moreover the UTO for two arbitrary natural orders s and k satisfies the following iterative summation law:

$$S_0^k S_0^s(f_n) = S_0^{k+s}(f_n). \tag{6}$$

The properties from (3) to (6) refer to the UTO of the rank k where k=0,1,2,3,… Now we go to define the recurrent form of UOS of order k:

$$S_{\varepsilon\rho}^0(f_n) \equiv f_n, \quad S_{\varepsilon\rho}^k(f_n) \equiv S_{\varepsilon\rho}^1\left[S_{\varepsilon\rho}^{k-1}(f_n)\right] \quad for \quad k = 1,2,3,\ldots \tag{7}$$

From the definition (7) implies the following iterative equations:

$$S_{\varepsilon\rho}^k(f_n) \equiv \left(S_0^1 + \varepsilon\,\rho\right)^k f_n \quad for \quad k = 1,2,3,\ldots, \tag{8}$$

which is easy to proof in the mathematical induction way.

3. Definition and some properties of the reciprocal unified operator of summation

Reciprocal UOS regarding to the unified operator of summation defined in Eq.(1) is denoted by the following description:

$$S_{\varepsilon\rho}^{-1}(\ldots). \tag{9}$$

Reciprocal unified operator of summation will be defined in the following form:

$$S_{\varepsilon\rho}^{-1}(f_n) \equiv F_n, \quad because \quad S_{\varepsilon\rho}^{+1}(F_n) \equiv f_n, \tag{10}$$

where F_n, f_n are the functions determined for the natural numbers n=1,2,3,…

Reciprocal unified operator of summation is denoted by the upper index (–1) and is not always univocal. To explain such property we define the following

LEMMA 1

If function F_n presents a result of operation of reciprocal unified operator of summation $S_{\varepsilon\rho}^{-1}$ on the function f_n, then each function:

$$F_n + C \cdot (-\varepsilon\rho)^n, \tag{11}$$

is also a consequence of the operation of reciprocal UOS on the function fn where C is the arbitrary constant (independent of n).

PROOF OF LEMMA 1

Let the UOS operates on the expression (11) and we need its linear and multiplicative properties defined by the Eqs. (4), (5). After operation we obtain:

$$S_{\varepsilon\rho}^{+1}\left[F_n + C\cdot(-\varepsilon\rho)^n\right] = S_{\varepsilon\rho}^1(F_n) + S_{\varepsilon\rho}^{+1}\left[C\cdot(-\varepsilon\rho)^n\right] = S_{\varepsilon\rho}^1(F_n) + C\cdot(-\varepsilon\rho)^{n+1} + C\cdot(\varepsilon\rho)(-\varepsilon\rho)^n = f_n. \tag{12}$$

Calculation (12) completes the proof of the Lemma 1. Now we show some selected characteristic properties of reciprocal UOS. At first we assume that the following operations: $S_{\varepsilon\rho}^{1}(\dots), S_{\varepsilon\rho}^{-1}(\dots)$, with the same basis ερ are for the two functions f_n, F_n reciprocal in following sense:

$$S_{\varepsilon\rho}^{+1}\left[S_{\varepsilon\rho}^{-1}(f_n)\right] = f_n, \quad S_{\varepsilon\rho}^{-1}\left[S_{\varepsilon\rho}^{+1}(F_n)\right] = F_n + C\cdot(-\varepsilon\rho)^n. \tag{13}$$

Eqs. (13) follow from the definition (10) and are presented the law of rank reduction.

4. Characteristic examples of reciprocal operator of summation

In Tab. 1 we show some characteristic reciprocal transformations.

Tab. 1. Characteristics reciprocal UOS transformations for arbitrary constants K , a, arbitrary summation constants C and natural numbers k, n

	f_n	$S_{\varepsilon\rho}^{m}(f_n)$	ερ	m
1	K	$\dfrac{K}{1+\varepsilon\rho} + C\cdot(-\varepsilon\rho)^n$	$\varepsilon\rho \neq -1$	-1
2	K	$Kn+C$	$\varepsilon\rho = -1$	-1
3	a^n	$\dfrac{a^n}{a+\varepsilon\rho} + C\cdot(-\varepsilon\rho)^n$	$\varepsilon\rho \neq -a$	-1
4	a^n	$na^{n-1} + C\cdot(a)^n$	$\varepsilon\rho = -a$	-1
5	$n\cdot 2^n$	$(n-2)2^n + C$	$\varepsilon\rho = -1$	-1
6	$n\cdot 2^n$	$-\dfrac{2}{9}\cdot 2^n + \dfrac{1}{3}\cdot 2^n\cdot n + (-1)^n C$	$\varepsilon\rho = +1$	-1
7	n^2	$n^2 J - (2n+1)J^2 - 2J^3 + C\cdot(-\varepsilon\rho)^n,\ for \quad \dfrac{1}{1+\varepsilon\rho} \equiv J$	$\varepsilon\rho \neq -1,$	-1
8	n^2	$\dfrac{n(n-1)(2n-1)}{6} + C$	$\varepsilon\rho = -1$	-1
9	1	2^{-k}	$\varepsilon\rho = +1$	$-k$

5. Linear summation equations with constant coefficients

This section is limited to the consideration of summation equations of first and second kind only. At first we consider the following
THEOREM 1

A non-homogeneous second order summation equation of the second and first kind with constant coefficients:

$$a_2 S_1^{-2}(f_n) + a_1 S_1^{-1}(f_n) + a_0 f_n = A, \tag{14}$$

$$a_2 S_{-1}^{-2}(f_n) + a_1 S_{-1}^{-1}(f_n) + a_0 f_n = A, \tag{15}$$

where: a_1, a_2, a_0, A are real numbers that are independent from n, can always be transformed to equivalent linear second order recurrent equations with constant coefficients.

PROOF OF THEOREM 1.

A summation equation is equivalent to the recurrence equation if both equations have the same particular solutions. At first we assume the summation equation of the first kind (14). On both sides of Eq.(14), we operate with a unified operator of summation S_1^1 with basis one, hence we obtain:

$$a_2 S_1^{-1}(f_n) + a_1 f_n + a_0\left(f_{n+1} + f_n\right) = A + A. \tag{16}$$

On both sides of Eq.(16), we again operate with a unified operator of summation S_1^1 with basis one, thus after ordering we obtain finally:

$$a_0 f_{n+2} + \left(2a_0 + a_1\right) f_{n+1} + \left(a_0 + a_1 + a_2\right) f_n = 4A. \tag{17}$$

Equation (17) obtained presents a linear, non-homogeneous recurrent equation of the second order with constant coefficients. Symbol f_n denotes the discrete unknown function.

Now we assume the summation equation of the second kind (15). On both sides of Eq. (15), we operate with the unified operator of summation S_{-1}^1 with basis minus one, hence we obtain:

$$a_2 S_{-1}^{-1}(f_n) + a_1 f_n + a_0\left(f_{n+1} - f_n\right) = A - A. \tag{18}$$

On both sides of Eq.(18), we again operate with the unified operator of summation S_{-1}^1 with basis minus one, thus after ordering we obtain finally:

$$a_0 f_{n+2} + \left(-2a_0 + a_1\right) f_{n+1} + \left(a_0 - a_1 + a_2\right) f_n = 0. \tag{19}$$

Eq. (19) presents a linear homogeneous second order recurrent equation with constant coefficients. Symbol fn describes the discrete unknown function. Expressions presented in Eqs.(17), (19) completes the proof of Theorem 1.

It can be easily seen that an analytical solution of n-order linear recurrent equations with constant coefficients is always possible and feasible. Such problems will be applied in the next intersection.

It is easy to prove that if summation equation has variable coefficients in the form of polynomials, then we can transform such equation to the recurrence equation too.

6. Uniform mega- algorithm of solutions for ordinary recurrence equations

The results of applied mathematical achievements are presented in the form of Uniform Mega-Algorithm elaboration to linear independent particular solutions $z_n^{[1]},\ldots,z_n^{[n]}$ determination of ordinary, linear n-order homogeneous and non-homogeneous following difference or recurrent equation [5]:

$$p_k(n) z_{n+k} + p_{k-1}(n) z_{n+k-1} + \ldots + p_2(n) z_{n+2} + p_1(n) z_{n+1} + p_0(n) z_n = b(n), \tag{20}$$

where coefficients p_j, b for j=0, 1, …, k depend on variable n in neighborhood of regular or non-regular points.The linear independent particular solutions of recurrent equation are presented in the sequence form and are satisfying the imposed boundary conditions.

If coefficients p_j are independent of n and b equals zero then recurrent equation (20) has the following general solution [5]:

$$z_n = \sum_{s=1}^{r} x_s^n \left(\sum_{m=0}^{\upsilon_s - 1} C_{sm} \cdot n^m \right), \text{ Csm–summation constants.} \tag{21}$$

Symbol χ_s for s=1,2,3,…,r; r≤k denotes the successive different, roots of characteristic algebraic equation:

$$p_k \chi^k + p_{k-1} \chi^{k-1} + ... + p_1 \chi + p_0 = 0, \qquad (22)$$

with multiple ν_s attributed to the roots χ_s whereas the sum of manifolds of roots is equal to the order of the recurrence equation namely:

$$\nu 1 + \nu 2 + \ldots + \nu r - 1 + \nu r = k. \qquad (23)$$

COROLLARY. 1

Linear, non-homogeneous, first order recurrent equations with variable coefficient a_n and variable free term b_n:

$$u_{n+1} + a_n u_n = b_n, \qquad (24)$$

has the following general solution:

$$u_n = (-1)^{n-1} \cdot \prod_{j=1}^{n-1} a_j \left\{ C + \sum_{k=1}^{n-1} \left[\frac{b_k}{(-1)^k \prod_{s=1}^{k} (a_s)} \right] \right\}, \quad for \quad n = 2,3,4,... \qquad (25)$$

$$u_1 = C,$$

where:

u_n - unknown discrete function,

C - arbitrary constant, indexes j and k belong to the set 1,2,…,n–1 whereas s=1,2,…,k.

ANALYTICAL PROOF OF COROLLARY 1.

The proof is presented in [5].

7. Wear prognosis for HDD bearing system

EXAMPLE 7.1.

Differences of the surface wear of the some HDD micro-bearing system between the next and foregoing year of operating time during the succeeding years numbered by n=1,2,3... are as follows:

$$F_n = f_{n+1} - f_n, \quad for \quad n = 1,2,3,... \qquad (26)$$

Determine the differences of wear F_n if we know, that the wear (increases of journal diameter) in succeeding years are described by the sequence $\{f_n\}$ for n=1,2,… . The wear in each year f_n, increased by wear difference function $g(F_n)=BF_n$, equal to the balance of the journal diameter function $h(n) = AD^n$. For this problem are imposed boundary condition $F_1=f_2-f_1$ in pm describing the difference of wear between second and first years of the exploitation. Value F_1 denotes measured increases of journal diameter value in the first two year of exploitation. Experimental parameters A[pm], B, D depend on material properties, exploitation time during the year, rotation velocity of the journal respectively.

SOLUTION OF EXAMPLE 7.1

From Eq.(26) follows, that wear surface we can present in the form:

$$S_{-1}^{-1} F_n = S_{-1}^{-1}(f_{n+1} - f_n) = S_{-1}^{-1} [S_{-1}^{+1}(f_n)] = f_n. \qquad (27)$$

The solved problem we can describe by the following, non-homogeneous, first order summation equation with variable free term (compare intersection 5):

$$S_{-1}^{-1}F_n + BF_n = A \cdot D^n \quad for \quad n = 1,2,3,... \tag{28}$$

The unknown of his summation equation is the sequence with the general term F_n.

Imposing the UOS operator S_{-1}^{+1} on the both sides of Eq. (28), we obtain:

$$F_n + BS_{-1}^{+1}(F_n) = A \cdot S_{-1}^{+1}(D^n) \quad for \quad n = 1,2,3,... \tag{29}$$

After simple and known operations performed in Eq. (29) we attain the following, non-homogeneous, first order recurrence equation with constant coefficient and variable free term:

$$F_{n+1} + a_n F_n = b_n \quad for \quad n = 1,2,3,..., \; a_n \equiv \tfrac{1-B}{B}, \quad b_n \equiv \frac{A(D-1)}{B} \cdot D^n. \tag{30}$$

It is easy to see, that the recurrence equation (30) has the form (24). Utilizing the solution (25), we can write general solution of the equation (30) in following form:

$$F_n = C \cdot (-1)^{n-1} \cdot \left(\frac{1-B}{B}\right)^{n-1} + \frac{A(D-1)}{B} \sum_{k=1}^{n-1} (-1)^{n-k-1} \left(\frac{1-B}{B}\right)^{n-k-1} \cdot D^k, \; for \quad n = 2,3,...F_1 = C, \tag{31}$$

where C denotes the arbitrary summation constant. After terms ordering Eq.(31) attain the form:

$$F_n = C \cdot \left(\frac{B-1}{B}\right)^{n-1} + \frac{A(D-1)}{B-1}\left(\frac{B-1}{B}\right)^{n} \cdot \sum_{k=1}^{n-1}\left(\frac{DB}{B-1}\right)^{k}, \; for \quad n = 2,3,...F_1 = C. \tag{32}$$

Taking into account the sum of n–1 terms of geometrical sequence on the r.h.s. of Eq. (32), we obtain:

$$F_n = C \cdot \left(\frac{B-1}{B}\right)^{n-1} + \left(\frac{B-1}{B}\right)^{n-1} XD\left[\left(\frac{DB}{B-1}\right)^{n-1} - 1\right], \; for \quad n = 1,2,3,...X \equiv \frac{A(D-1)}{(DB-B+1)}, \tag{33a}$$

After simple transformations Eq. (33a) tends finally to the form:

$$F_n = \left(\frac{B-1}{B}\right)^{n-1}(C - XD) + XD^n, \; for \quad n = 1,2,3,...,. \tag{33b}$$

Imposing the boundary condition (30) for n=1 on the general solution (33b), we obtain $C=F_1$.

Hence, the sequence of the differences of wear surface in succeeding years, has the following form:

$$F_n = \left(\frac{B-1}{B}\right)^{n-1}(F_1 - XD) + XD^n, \; for \quad n = 1,2,3,... \tag{34}$$

After N-years the wear (i.e. decreasing of the journal diameter) attain the value:

$$\sum_{n=1}^{N} F_n = (1-B)(F_1 - XD)\left[\left(\frac{B-1}{B}\right)^{N} - 1\right] + DX\,\frac{D^N - 1}{D-1}. \tag{35}$$

In particular calculations we assume: A=1400pm, B=3/2, D=3/2, F_1=843pm. Hence we obtain X=400pm and the particular solution (34) has the form:

$$F_n = \frac{243\,pm}{3^{n-1}} + 400\,pm\left(\frac{3}{2}\right)^{n}, \quad for \quad n = 1,2,3,... \tag{36}$$

From the formula (34) follows, that in succeeding years the differences of wear surfaces for the considered HDD micro-bearing system, between the each next and foregoing year, are as follows in nano-meters:

$$\{F_n\}_{n=1}^{\infty} = 0.843nm, \quad 0.981nm, \quad 1.377nm, \quad 2.034nm, \quad 3.040nm,... \tag{37}$$

From Eq. (35) follows, that after N=10 years exploitation the journal diameter decreases 68.32nm.

8. Final results

Corollary 1. Taking into account influences of variable sometimes mutually depended impulses on the behavior of HDD slide journal bearing system, we can describe how the design variables of mentioned bearings affect the bearing stiffness and the natural frequencies of the bearing shaft to obtain the optimum wear values during the exploitation time.

Corollary 2. This research also shows that the supporting structure which includes the stator, housing and base plate plays an important role in determining the natural frequencies and mode shapes of slide journal bearing system.

9. Conclusions

The construction of a certain unified mega-algorithm of summation equations, which generalizes the difference operators that have been used so far, and in particular cases also describes certain new forms of summation and differentiation, which constitute a useful tool for the solution of the calculation problems applied for the wear prognosis of HDD micro-bearings.

Acknowledgement

This paper was supported by Polish Ministerial Grant 3475/B/T02/2009/36 in years 2009-2012.

References

[1] Bharat, B., *Nano-tribology and nanomechanics of MEMS/NEMS and BioMEMS, BioNEMS materials and devices*, Microelectronic Engineering, 84, pp. 387-412, 2007.
[2] Jang, G. H., Seo, C.H., Ho Scong, Lee, *Finite element model analysis of an HDD considering the flexibility of spinning disc-spindle, head-suspension-actuator and supporting structure*. Microsystem Technologies, 13, pp. 837-847, 2007.
[3] Jury, E. J., *The Theory of Applications of the Z Transform Method*. J. Willey Co.,New York 1964.
[4] Kącki, E., *Równania różniczkowe cząstkowe w zagadnieniach fizyki i techniki*. WNT, Warszawa 1989.
[5] Koźniewska, I., *Równania rekurencyjne*. PWN, Warszawa 1973.
[6] Levy, H., Lessman, F., *Równania różnicowe skończone*, PWN, Warszawa 1966.
[7] Maciąg, A., *Wave polynomials in elasticity problems*, Engineering Transactions, 55 (20), pp.129-153, 2007.
[8] Mikusinski, J., *Rachunek operatorów*. PWN, Warszawa 1957.
[9] Miller, K.S., *Linear difference equations*, New York 1968.
[10] Miller, K.S., *An introduction to the cal90culus of finite difference and difference equations*. MCGraw-Hill Book Co. INC, New York 1960.
[11] Ralston, A., *A First Course in Numerical Analysis*, (in Polish), PWN, Warszawa 1971.
[12] Wierzcholski, K., *Enhancement of memory capacity in HDD micro- bearing with hyperbolic journals*, Journal of KONES powertrain and Transport, Vol.15, No.3, pp.555-560, Warsaw 2008.