Computational Methods for Investigation of Stability of Models of 2D Continuous-Discrete Linear Systems

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Abstract:

The problem of asymptotic stability of models of 2D continuous-discrete linear systems is considered. Computer methods for investigation of asymptotic stability of the Fornasini-Marchesini type and the Roesser type models, are given. The methods proposed require computation of the eigenvalue-loci of complex matrices. Effectiveness of the stability tests are demonstrated on numerical examples.

Keywords: *continuous-discrete system, hybrid system, linear system, stability, computational methods.*

1. Introduction

In continuous-discrete systems both continuous-time and discrete-time components are relevant and interacting and these components can not be separated. Such systems are called the hybrid systems. Examples of hybrid systems can be found in [6], [8], [9], [16]. The problems of dynamics and control of hybrid systems have been studied in [5], [6], [16].

In this paper we consider the continuous-discrete linear systems whose models have structure similar to the models of 2D discrete-time linear systems. Such models, called the 2D continuous-discrete or 2D hybrid models, have been considered in [11] in the case of positive systems.

The new general model of positive 2D hybrid linear systems has been introduced in [12] for standard and in [13] for fractional systems. The realization and solvability problems of positive 2D hybrid linear systems have been considered in [11], [14] and [15], [17], respectively.

The problems of stability and robust stability of 2D continuous-discrete linear systems have been investigated in [1-4], [7], [18 -20].

The main purpose of this paper is to present computational methods for investigation of asymptotic stability of the Fornasini-Marchesini and the Roesser type models of continuous-discrete linear systems.

The following notation will be used: $\Re$ - the set of real numbers, $\Re_+ = [0,\infty]$, Z_+ - the set of non-negative integers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices and $\Re_+^n = \Re_+^{n\times 1}$, $\| x(\cdot) \|$ - the norm of $x(\cdot)$, $\lambda_i\{X\}$ - i-th eigenvalue of matrix X.

2. Preliminaries and formulation of the problem

The state equation of the Fornasini-Marchesini type model of a continuous-discrete linear system has the form [11]

$$\dot{x}(t,i+1) = A_0 x(t,i) + A_1 \dot{x}(t,i) + A_2 x(t,i+1) + Bu(t,i),$$
$$i \in Z_+,\ t \in \Re_+, \tag{1}$$

where $\dot{x}(t,i) = \partial x(t,i)/\partial t$, $x(t,i) \in \Re^n$, $u(t,i) \in \Re^m$, and $A_0, A_1, A_2 \in \Re^{n\times m}$, $B \in \Re^{n\times m}$.

Definition 1. The Fornasini-Marchesini type model (1) is called asymptotically stable (or Hurwitz-Schur stable) if for $u(t,i) \equiv 0$ and bounded boundary conditions

$$x(0,i),\ i \in Z_+,\ x(t,0),\ \dot{x}(t,0),\ t \in \Re_+, \tag{2}$$

the condition $\lim_{i,t\to\infty} \|x(t,i)\| = 0$ holds for $t,i \to \infty$.

The characteristic matrix of the model (1) has the form

$$H(s,z) = szI_n - A_0 - sA_1 - zA_2. \tag{3}$$

The characteristic function

$$w(s,z) = \det H(s,z) = \det[szI_n - A_0 - sA_1 - zA_2] \tag{4}$$

of the model (1) is a polynomial in two independent variables s and z, of the general form

$$w(s,z) = \sum_{k=0}^{n} \sum_{j=0}^{n} a_{kj} s^k z^j,\ a_{nn} = 1. \tag{5}$$

The state equation of the Roesser type model of a continuous-discrete linear system has the form [11]

$$\begin{bmatrix} \dot{x}^h(t,i) \\ x^v(t,i+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h(t,i) \\ x^v(t,i) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t,i),$$
$$t \in \Re_+,\ i \in Z_+, \tag{6}$$

where $\dot{x}^h(t,i) = \partial x^h(t,i)/\partial t$, $x^h(t,i) \in \Re^{n_1}$, $x^v(t,i) \in \Re^{n_2}$ are the vertical and the horizontal vectors, respectively, $u(t,i) \in \Re^m$ is the input vector and $A_{11} \in \Re^{n_1\times n_1}$, $A_{12} \in \Re^{n_1\times n_2}$, $A_{21} \in \Re^{n_2\times n_1}$, $A_{22} \in \Re^{n_2\times n_2}$, $B_1 \in \Re^{n_1\times m}$, $B_2 \in \Re^{n_2\times m}$.

The boundary conditions for (6) are as follows

$$x^h(t,0),\ x^v(t,0),\ t \in \Re_+,\ x^h(0,i),\ x^v(0,i),\ i \geq 1,\ i \in Z_+. \tag{7}$$

Definition 2. The Roesser type model (6) is called asymptotically stable (or Hurwitz-Schur stable) if for $u(t,i) \equiv 0$ and bounded boundary conditions (7) the conditions $\lim_{i,t\to\infty} \| x^h(t,i)\| = 0$ and $\lim_{i,t\to\infty} \| x^v(t,i)\| = 0$ hold for $t,i \to\infty$.

The characteristic matrix of the model (6) has the form

$$H(s,z) = \begin{bmatrix} sI_{n_1} - A_{11} & -A_{12} \\ -A_{21} & zI_{n_2} - A_{22} \end{bmatrix} \tag{8}$$

Using the rules for computing the determinant of block matrices [10], we obtain that the characteristic function $w(s, z) = \det H(s, z)$ of the Roesser type model can be computed from one of the following equivalent formulae

$$w(s,z) = \det(zI_{n_2} - A_{22})\det(sI_{n_1} - A_{11} - A_{12}(zI_{n_2} - A_{22})^{-1}A_{21}), \tag{9a}$$

$$w(s,z) = \det(sI_{n_1} - A_{11})\det(zI_{n_2} - A_{22} - A_{21}(sI_{n_1} - A_{11})^{-1}A_{12}\}. \tag{9b}$$

The characteristic function of the Roesser type model can be written in the form

$$w(s,z) = \sum_{k=0}^{n_1}\sum_{j=0}^{n_2} a_{kj}s^k z^j, a_{n_1n_2} = 1. \tag{10}$$

From [1], [7] we have the following theorem.

Theorem 1. The Fornasini-Marchesini type model (1) with characteristic function (4) (or the Roesser type model (6) with characteristic function (9)) is asymptotically stable if and only if

$$w(s,z) \neq 0, \text{ Re}\, s \geq 0, |z| \geq 1. \tag{11}$$

The polynomial $w(s, z)$ satisfying condition (11) is called continuous-discrete stable (C-D stable) or Hurwitz-Schur stable [1].

The main purpose of this paper is to present computational methods for checking the condition (11) of asymptotic stability of the Fornasini-Marchesini type model (1) and the Roesser type model (6) of continuous-discrete linear systems.

3. Solution of the problem

Theorem 2. The condition (11) is equivalent to the following two conditions

$$w(s, e^{j\omega}) \neq 0, \text{ Re}\, s \geq 0, \forall\omega \in [0, 2\pi], \tag{12}$$

$$w(jy, z) \neq 0, |z| \geq 1, \forall y \in [0, \infty). \tag{13}$$

Proof. From [7] it follows that (11) is equivalent to the conditions

$$w(s,z) \neq 0, \text{ Re}\, s \geq 0, |z| = 1, \tag{14}$$

$$w(s,z) \neq 0, \text{ Re}\, s = 0, |z| \geq 1. \tag{15}$$

It is easy to see that conditions (14) and (15) can be written in the forms (12) and (13), respectively. ■

3.1. Asymptotic stability of the Fornasini-Marchesini type model

From (4) for $z = e^{j\omega}$ we have

$$w(s, e^{j\omega}) = \det[s(I_n e^{j\omega} - A_1) - A_2 e^{j\omega} - A_0]. \tag{16}$$

Lemma 1. The condition (12) for the Fornasini-Marchesini type model (1) with $A_1 \neq \pm I_n$ holds if and only if all eigenvalues of the complex matrix $S_1^{FM}(e^{j\omega})$ have negative real parts for all $\omega \in [0, 2\pi]$, where

$$S_1^{FM}(e^{j\omega}) = (I_n e^{j\omega} - A_1)^{-1}(A_2 e^{j\omega} + A_0). \tag{17}$$

Proof. If $A_1 \neq \pm I_n$ then the matrix $I_n e^{j\omega} - A_1$ is non-singular for all $\omega \in [0, 2\pi]$ and

$$[s(I_n e^{j\omega} - A_1) - A_0 - A_2 e^{j\omega}] = [I_n e^{j\omega} - A_1][s - S_1^{FM}(e^{j\omega})], \tag{18}$$

where $S_1^{FM}(e^{j\omega})$ has the form (17).

From (16) and (18) it follows that

$$w(s, e^{j\omega}) = \det(I_n e^{j\omega} - A_1)\det(sI_n - S_1^{FM}(e^{j\omega})). \tag{19}$$

This means that for $A_1 \neq \pm I_n$ the eigenvalues of the matrix $S_1^{FM}(e^{j\omega})$ are the roots of the polynomial $w(s,e^{j\omega})$. ■

Lemma 2. The condition (13) for the Fornasini-Marchesini type model (1) with $A_2 \neq I_n$ holds if and only if all eigenvalues of the complex matrix $S_2^{FM}(jy)$ have absolute values less than one for all $y \geq 0$, where

$$S_2^{FM}(jy) = (jyI_n - A_2)^{-1}(A_0 + jyA_1). \tag{20}$$

Proof. Substituting $s = jy$ in (4) one obtains

$$w(jy, z) = \det[z(jyI_n - A_2) - A_0 - jyA_1]. \tag{21}$$

If $A_2 \neq I_n$ then the matrix $jyI_n - A_1$ is non-singular for all $y \geq 0$ and

$$[z(jyI_n - A_2) - A_0 - jyA_1] = [jyI_n - A_2][z - S_2^{FM}(jy)], \tag{22}$$

where $S_2^{FM}(jy)$ is defined by (20).

From (21) and (22) it follows that

$$w(jy, z) = \det(jyI_n - A_2)\det(zI_n - S_2^{FM}(jy)). \tag{23}$$

Hence, if $A_2 \neq I_n$ then the eigenvalues of the matrix $S_2^{FM}(jy)$ are the roots of the polynomial $w(jy, z)$. ■

Theorem 3. The Fornasini-Marchesini type model (1) with $A_1 \neq \pm I_n$ and $A_2 \neq I_n$ is asymptotically stable if and only if the conditions of Lemmas 1 and 2 hold, i.e.

$$\text{Re}\,\lambda_i\{S_1^{FM}(e^{j\omega})\} < 0, \forall\omega \in [0, 2\pi], i = 1,2,...,n, \tag{24}$$

and

$$|\lambda_i\{S_2^{FM}(jy)\}| < 1, \forall y \geq 0, i = 1,2,...,n, \tag{25}$$

where the matrices $S_1^{FM}(e^{j\omega})$ and $S_2^{FM}(jy)$ have the forms (17) and (20), respectively.

Proof. It follows from Theorem 2 and Lemmas 1 and 2. ■

From (17) for $\omega = 0$ and $\omega = \pi$ we have

$$S_1^{FM}(1) = (I_n - A_1)^{-1}(A_2 + A_0), \tag{26a}$$

$$S_1^{FM}(-1) = (-I_n - A_1)^{-1}(-A_2 + A_0). \tag{26b}$$

From the theory of matrices it follows that if $(-1)^n \det S_1^{FM}(1) \leq 0$ then not all eigenvalues of the matrix (26a) have negative real parts. Similar condition holds for the matrix (26b). Hence, we have the following remark.

Remark 1. Simple necessary condition for asymptotic stability of the Fornasini-Marchesini type model (1) with $A_1 \neq \pm I_n$ has the form

$$(-1)^n \det(I_n - A_1)\det(A_2 + A_0) > 0 \qquad (27a)$$

$$(-1)^n \det(-I_n - A_1)\det(-A_2 + A_0) > 0. \qquad (27b)$$

Example 1. Consider the Fornasini-Marchesini type model (1) with the matrices

$$A_0 = \begin{bmatrix} -0.4 & 1 & 0 \\ 0 & 0.2 & 0.5 \\ 0 & -0.1 & -0.1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.5 & 0.1 & 0 \\ 0 & 0.1 & -0.4 \\ 0 & 0.2 & -0.2 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -0.4 & -1.8 & 0 \\ 0.1 & -0.4 & 0 \\ 0 & 0 & -0.7 \end{bmatrix}, \qquad (28)$$

It is easy to check that the necessary conditions (27) hold. Computing eigenvalues of the matrices $S_1^{FM}(e^{j\omega})$, $\omega \in [0, 2\pi]$, and $S_2^{FM}(jy)$, $y \in [0, 100]$, one obtains the plots shown in Figures 1 and 2. It is easy to check that eigenvalues of $S_2^{FM}(jy)$ remain in the unit circle for all $y > 100$.

From Figures 1 and 2 it follows that the conditions (24) and (25) of Theorem 3 are satisfied and the system is asymptotically stable.

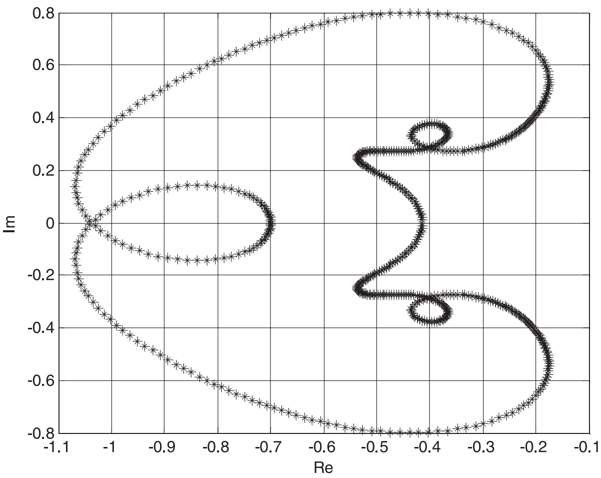

Fig. 1. Eigenvalues of the matrix $S_1^{FM}(e^{j\omega})$, $\omega \in [0, 2\pi]$.

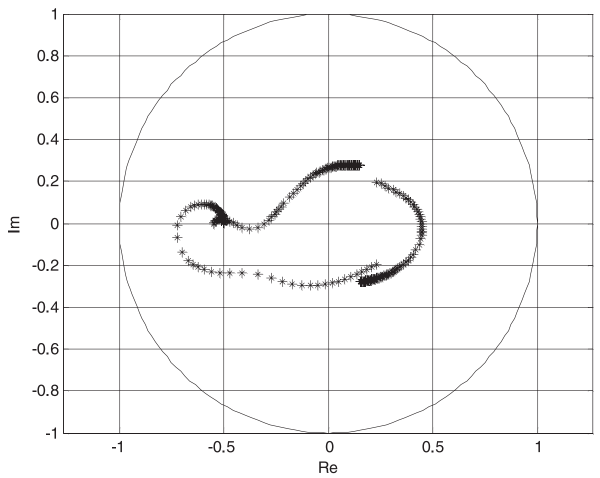

Fig. 2. Eigenvalues of the matrix $S_2^{FM}(jy)$, $y \in [0, 100]$.

3.2. Asymptotic stability of the Roesser type model

From (8) for $z = e^{j\omega}$ we have

$$w(s, e^{j\omega}) = \det \begin{bmatrix} sI_{n_1} - A_{11} & -A_{12} \\ -A_{21} & I_{n_2}e^{j\omega} - A_{22} \end{bmatrix} \qquad (29)$$

Lemma 3. The condition (12) for the Roesser type model (6) with $A_{22} \neq \pm I_{n_2}$ holds if and only if all eigenvalues of the complex matrix $S_1^R(e^{j\omega})$ have negative real parts for all $\omega \in [0, 2\pi]$, where

$$S_1^R(e^{j\omega}) = A_{11} + A_{12}(I_{n_2}e^{j\omega} - A_{22})^{-1} A_{21}. \qquad (30)$$

Proof. If $A_{22} \neq \pm I_{n_2}$ then the matrix $I_{n_2}e^{j\omega} - A_{22}$ is non-singular for all $\omega \in [0, 2\pi]$ and from (9a) it follows that

$$w(s, e^{j\omega}) = \det(I_{n_2}e^{j\omega} - A_{22})\det(sI_{n_1} - S_1^R(e^{j\omega})), \qquad (31)$$

where $S_1^R(e^{j\omega})$ has the form (30). This means that for $A_{22} \neq \pm I_{n_2}$ the eigenvalues of the matrix $S_1^R(e^{j\omega})$ are the roots of the polynomial $w(s, e^{j\omega})$. ■

Lemma 4. The condition (13) for the Roesser type model (6) with $A_{11} \neq I_{n_1}$ holds if and only if all eigenvalues of the complex matrix $S_2^R(jy)$ have absolute values less than one for all $y \geq 0$, where

$$S_2^R(jy) = A_{22} + A_{21}(jyI_{n_1} - A_{11})^{-1} A_{12}. \qquad (32)$$

Proof. If $A_{11} \neq I_{n_1}$ then the matrix $jyI_{n_1} - A_{11}$ is non-singular for all $y \geq 0$. From (9b) for $s = jy$ we have

$$w(jy, z) = \det(jyI_{n_1} - A_{11})\det(zI_{n_2} - A_{22} - A_{21}(jyI_{n_1} - A_{11})^{-1} A_{12}\}. \qquad (33)$$

From (32) and (33) it follows that

$$w(jy, z) = \det(jyI_{n_1} - A_{11})\det(zI_{n_2} - S_2^R(jy), \qquad (34)$$

where $S_2^R(jy)$ is defined by (32).

If $A_{11} \neq I_{n_1}$ then the eigenvalues of the matrix $S_2^R(jy)$ are the roots of the polynomial $w(jy, z)$. ■

Theorem 4. The Roesser type model (6) with $A_{22} \neq \pm I_{n_2}$ and $A_{11} \neq I_{n_1}$ is asymptotically stable if and only if the conditions of Lemmas 3 and 4 hold, i.e.

$$\mathrm{Re}\,\lambda_i\{S_1^R(e^{j\omega})\} < 0, \forall \omega \in [0, 2\pi], i = 1,2,...,n_1, \qquad (35)$$

and

$$|\lambda_i\{S_2^R(jy)\}| < 1, \forall y \geq 0, i = 1,2,...,n_2, \qquad (36)$$

where matrices $S_1^R(e^{j\omega})$ and $S_2^R(jy)$ have the forms (30) and (32), respectively.

Proof. The proof follows from Theorem 2 and Lemmas 3 and 4. ■

From (30) for $\omega = 0$ and $\omega = \pi$ it follows that

$$S_1^R(1) = A_{11} + A_{12}(I_{n_2} - A_{22})^{-1} A_{21}, \qquad (37a)$$

$$S_1^R(-1) = A_{11} + A_{12}(-I_{n_2} - A_{22})^{-1} A_{21}. \qquad (37b)$$

From the above and theory of matrices we have the following remark.

Remark 2. Simple necessary condition for asymptotic stability of the Roesser type model (6) with $A_{22} \neq \pm I_{n_2}$ are as follows: $(-1)^n \det S_1^R(1) > 0$ and $(-1)^n \det S_1^R(-1) > 0$.

Example 2. Consider the Roesser type model (6) with the matrices

$$A_{11} = \begin{bmatrix} -1 & 0 \\ 0.1 & -5 \end{bmatrix},\ A_{12} = \begin{bmatrix} -0.5 & 0 \\ -1 & 0 \end{bmatrix},$$

$$A_{12} = \begin{bmatrix} -0.5 & -1 \\ 0 & -1 \end{bmatrix},\ A_{22} = \begin{bmatrix} -0.5 & 0.8 \\ 0.2 & 0.4 \end{bmatrix}. \quad (38)$$

Eigenvalues of the matrices $S_1^R(e^{j\omega})$, $\omega \in [0, 2\pi]$ and $S_2^R(jy)$, $y \in [0, 100]$, are shown in Figures 3 and 4. If is easy to check that eigenvalues of $S_2^R(jy)$ remain in the unit circle for all $y > 100$.

From Figures 3 and 4 it follows that the conditions (35) and (36) of Theorem 4 are satisfied and the system is asymptotically stable.

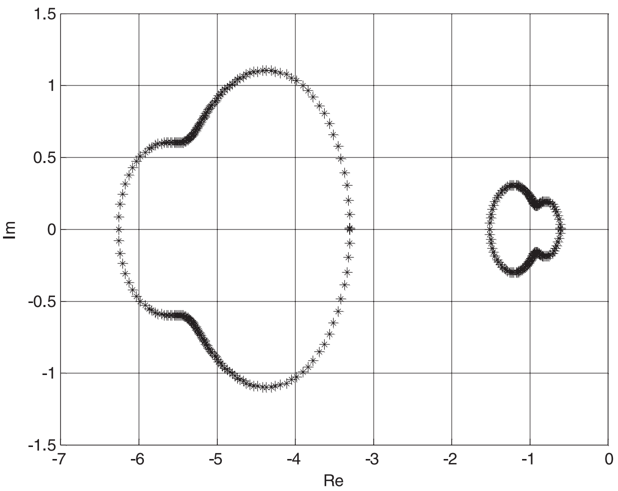

Fig. 3. Eigenvalues of the matrix $S_1^R(e^{j\omega})$, $\omega \in [0, 2\pi]$.

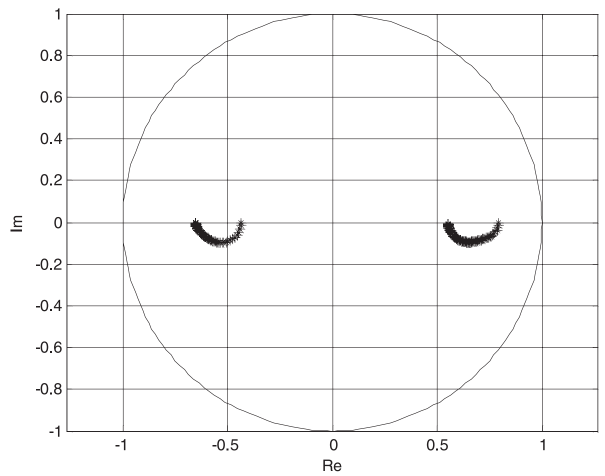

Fig. 4. Eigenvalues of the matrix $S_2^R(jy)$, $y \in [0, 100]$.

4. Concluding remarks

Computational methods for investigation of asymptotic stability of the Fornasini-Marchesini type model (1) (Theorem 3) and the Roesser type model (6) (Theorem 4) of continuous-discrete linear systems have been given. These methods require computation of eigenvalue-loci of complex matrices (17) and (20) for the Fornasini-Marchesini type model and complex matrices (30) and (32) for the Roesser type model.

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