POSITIVE REALIZATIONS OF HYBRID LINEAR SYSTEMS DESCRIBED BY THE GENERAL MODEL USING THE STATE VARIABLE DIAGRAM METHOD

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Abstract:

The realization problem for linear hybrid systems described by the general model is formulated and solved. Sufficient conditions for the existence of positive realizations are established. A procedure based on the state variable diagram method for computation of a positive realization of a given transfer matrix is proposed. Effectiveness of the procedure is demonstrated on two examples.

Keywords: *positive realization, hybrid, general model, state variable diagram.*

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behaviour can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs [3], [4]. The realization problem for positive discrete-time and continuous-time without and with delays was considered in [1], [4-9], [15] and for positive fractional linear systems in [13]. The reachability, controllability and minimum energy control of positive linear systems with delays have been considered in [2]. A new class of positive 2D hybrid linear systems described by two vector equations has been introduced in [10] and of fractional positive hybrid systems in [11]. The realization problem for positive linear hybrid systems has been investigated in [12], [16], [17]. Structural decomposition of transfer matrix of positive normal hybrid systems has been proposed in [14].

In this paper a method for computation of positive realizations of linear hybrid system described by the general model will be proposed.

The paper is organized as follows. In section 2 fundamentals of positive hybrid linear systems are recalled and the realization problem is formulated. The main result is presented in section 3. In subsection 3.1 the proposed state variable diagram method is presented for single-input single-output (SISO) linear hybrid systems. An extension of the method for multi-input multi-output (MIMO) systems is presented in subsection 3.2. Concluding remarks are given in section 4.

In the paper the following notation will be used. The set of $n\times m$ real matrices will be denoted by $\Re^{n\times m}$ and $\Re^n = \Re^{n\times 1}$. The set of $n\times m$ real matrices with nonnegative entries will be denoted by and $\Re_+^{n\times m}$ and $\Re_+^n = \Re_+^{n\times 1}$. The $n\times\nu$ identity matrix will be denoted by I_n and the transpose will be denoted by T.

2. Preliminaries and the problem formulation

Consider a hybrid linear system described by the equations [4]

$$\begin{aligned}\dot{x}(t,i+1) &= A_0x(t,i) + A_1\dot{x}(t,i) + A_2x(t,i+1) + \\ &+ B_0u(t,i) + B_1\dot{u}(t,i) + B_2u(t,i+1)\end{aligned} \tag{1a}$$

$$y(t,i) = Cx(t,i) + Du(t,i),\ t\in\Re_+ = [0,+\infty],\ i\in Z_+\ \ Z_+ = \{1,2,...\} \tag{1b}$$

where $\dot{x}(t,i) = \dfrac{\partial x(t,i)}{\partial t}$, $x(t,i)\in\Re^n$, $u(t,i)\in\Re^m$, $y(t,i)\in\Re^p$ are the state, input and output vectors and

$$A_k\in\Re^{n\times n},\ B_k\in\Re^{n\times m},\ k=0,1,2;\ C\in\Re^{p\times n}, D\in\Re^{p\times m}. \tag{1c}$$

Boundary conditions for (1a) have the form

$$x_1(0,i),\ i\in Z_+ \text{ and } x(t,0), \dot{x}(t,0),\ t\in\Re_+ \tag{2}$$

Definition 1. The hybrid system (1) is called internally positive if $x(t,i)\in\Re_+^n$ and $y(t,i)\in\Re_+^p$, $t\in\Re_+$, $i\in Z_+$ for arbitrary boundary conditions

$$x(0,i)\in\Re_+^n,\ i\in Z_+,\ x(t,0)\in\Re_+^n,\ \ \dot{x}(t,0)\in\Re_+^n,\ t\in\Re_+ \tag{3}$$

and any inputs

$$u(t,i)\in\Re_+^m,\ \dot{u}(t,i)\in\Re_+^m,\ t\in\Re_+,\ i\in Z_+. \tag{4}$$

The transfer matrix $T(s,z)$ of the hybrid system (1) is given by

$$\begin{aligned}T(s,z) &= C\left[I_nsz - A_0 - A_1s - A_2z\right]^{-1}(B_0 + B_1s + B_2z) + \\ &+ D\in\Re^{p\times m}(s,z)\end{aligned} \tag{5}$$

where $\Re^{p\times m}(s,z)$ is the set of $p\times m$ real matrices in s and z with real coefficient.

Theorem 1. [4] The hybrid system (1) is internally positive if and only if

$$A_2 \in M_n \tag{6}$$

$$A_0, A_1 \in \Re_+^{n\times n}, \quad A_0 + A_1A_2 \in \Re_+^{n\times n}, \quad B_0, B_1, B_2 \in \Re_+^{n\times m}, \\ C \in \Re_+^{p\times n}, \quad D \in \Re_+^{p\times m} \tag{7}$$

where M_n is the set of $n\times n$ Metzler matrices (with nonnegative off-diagonal entries).

From (5) we have

$$D = \lim_{s,z\to\infty} T(s,z) \tag{8}$$

since $\lim_{s,z\to\infty}[I_n sz - A_0 - A_1 s - A_2 z]^{-1} = 0.$

Knowing the matrix D we can find the strictly positive transfer matrix

$$T_{sp}(s,z) = T(s,z) - D \tag{9}$$

Definition 2. The matrices (1c) satisfying the conditions (6), (7) and (5) are called the positive realization of the transfer matrix .

The problem under considerations can be stated as follows.

Given a rational matrix $T(s,z)\in\Re^{p\times m}(s,z)$. Find its positive realization, i.e. a realization (1c) satisfying the conditions (6) and (7).

In this paper sufficient conditions for the existence of a positive realization will be established and a procedure for computation of a positive realization for a given transfer matrix $T(s,z)$ will be proposed.

3. Problem solution

3.1. SISO systems

First we shall solve the problem for SISO hybrid systems using the state variable diagram method [16].

Let a given transfer function of the SISO hybrid system have the form

$$T(s,z) = \frac{n(s,z)}{d(s,z)} = \frac{b_{q_1,q_2}s^{q_1}z^{q_2} + b_{q_1,q_2-1}s^{q_1}z^{q_2-1} + \ldots +}{s^{q_1}z^{q_2} - a_{q_1,q_2-1}s^{q_1}z^{q_2-1} - \ldots -} \\ \frac{+b_{11}sz + b_{10}s + b_{01}z + b_{00}}{-a_{11}sz - a_{10}s - a_{01}z - a_{00}} \in \Re^{p\times m}(s,z) \tag{10}$$

which by definition is the ratio of $Y(s,z)$ and $U(s,z)$ for zero boundary conditions, where , $U(s,z) = \mathcal{Z}\{\mathcal{L}[u(t,i)]\}$, $Y(s,z) = \mathcal{Z}\{\mathcal{L}[y(t,i)]\}$and $\mathcal{Z}$ and $\mathcal{L}$ are the zet and Laplace operators.

Using (8) and (9) we can find

$$D = \lim_{s,z\to\infty} T(s,z) = b_{q_1,q_2} \tag{11}$$

and the strictly proper transfer function

$$T_{sp}(s,z) = \frac{b_{q_1,q_2}s^{q_1}z^{q_2} + b_{q_1,q_2-1}s^{q_1}z^{q_2-1} + \ldots +}{s^{q_1}z^{q_2} - a_{q_1,q_2-1}s^{q_1}z^{q_2-1} - \ldots -} \\ \frac{+b_{11}sz + b_{10}s + b_{01}z + b_{00}}{-a_{11}sz - a_{10}s - a_{01}z - a_{00}} - b_{q_1,q_2} \tag{12}$$
$$= \frac{\bar{b}_{q_1,q_2-1}s^{q_1}z^{q_2-1} + \ldots + \bar{b}_{11}sz + \bar{b}_{10}s + \bar{b}_{01}z + \bar{b}_{00}}{s^{q_1}z^{q_2} - a_{q_1,q_2-1}s^{q_1}z^{q_2-1} - \ldots - a_{11}sz - a_{10}s - a_{01}z - a_{00}}$$

where $\bar{b}_{kl} = b_{kl} + b_{q_1,q_2}a_{kl}$, $k = 0,1,\ldots,q_1$; $l = 0,1,\ldots,q_2$ $(k + l \neq q_1 + q_2)$.

Multiplying the numerator and denominator of (12) by $s^{-q_1}z^{-q_2}$ we obtain

$$T_{sp}(s,z) = \frac{Y(s,z)}{U(s,z)} = \frac{\bar{b}_{q_1,q_2-1}z^{-1} + \ldots + \bar{b}_{11}s^{1-q_1}z^{1-q_2} +}{1 - a_{q_1,q_2-1}z^{-1} - \ldots - a_{11}s^{1-q_1}z^{1-q_2} -} \\ = \frac{+\ldots + \bar{b}_{00}s^{-q_1}z^{-q_2}}{-\ldots - a_{00}s^{-q_1}z^{-q_2}} \tag{13}$$

Defining

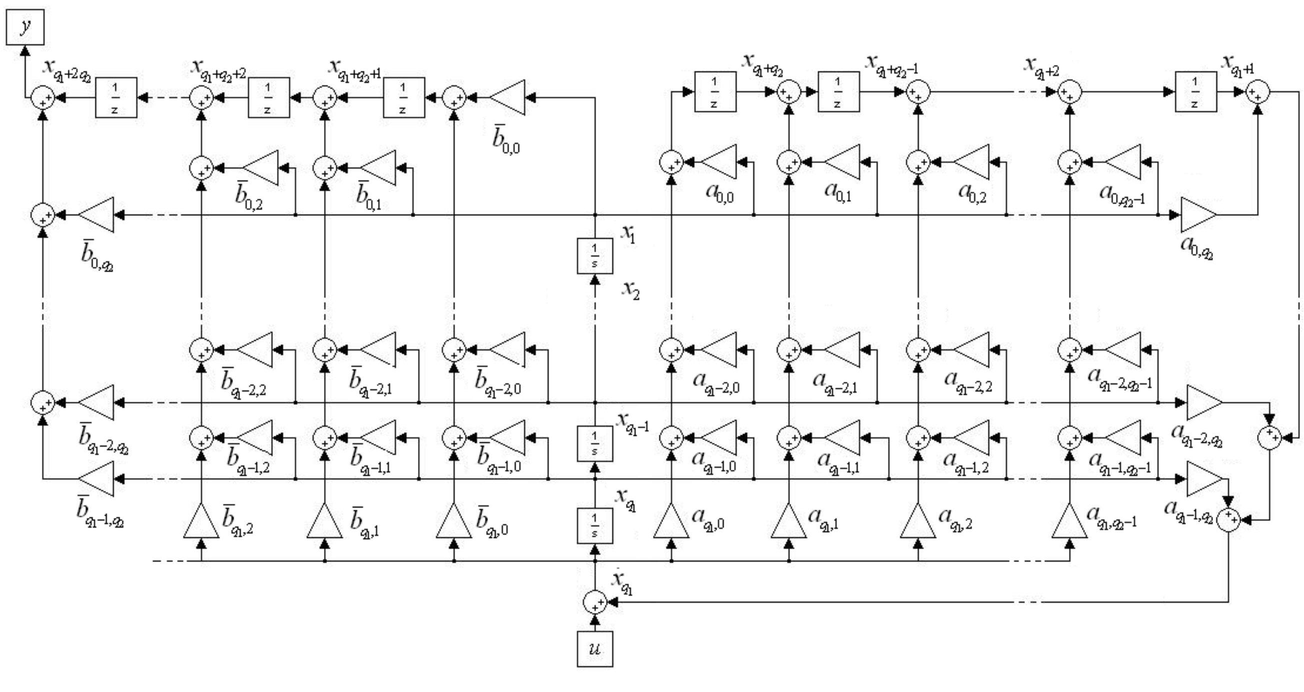

Fig. 1. State variable diagram for transfer function (13).

$$E(s,z)=\frac{U(s,z)}{1-a_{q_1,q_2-1}z^{-1}-\ldots-a_{11}s^{1-q_1}z^{1-q_2}-\ldots-a_{00}s^{-q_1}z^{-q_2}} \tag{14}$$

From (14) and (13) we have

$$E(s,z)=U(s,z)+(a_{q_1,q_2-1}z^{-1}+\ldots+a_{11}s^{1-q_1}z^{1-q_2}+\ldots+a_{00}s^{-q_1}z^{-q_2})E(s,z) \tag{15}$$

and

$$Y(s,z)=(\bar{b}_{q_1,q_2-1}z^{-1}+\ldots+\bar{b}_{11}s^{1-q_1}z^{1-q_2}+\ldots+\bar{b}_{00}s^{-q_1}z^{-q_2})E(s,z) \tag{16}$$

Using (15) and (16) we may draw the state variable diagram shown in Fig. 1.

The number of integration elements $1/s$ is equal to q_1 and the number of delay elements $1/z$ is equal to $2q_2$. The outputs of the integration elements are chosen as the state variables $x_1(t,i),\ldots,x_{q_1}(t,i)$ and the outputs of the delay elements as the state variables $x_{q_1+1}(t,i),\ldots,x_{q_1+q_2}(t,i),x_{q_1+q_2+1}(t,i),\ldots,x_{q_1+2q_2}(t,i)$. Using the state variable diagram we may write the equations

$$\begin{aligned}
&\dot{x}_1(t,i)=x_2(t,i)\\
&\dot{x}_2(t,i)=x_3(t,i)\\
&\vdots\\
&\dot{x}_{q_1-1}(t,i)=x_{q_1}(t,i)\\
&\dot{x}_{q_1}(t,i)=a_{0,q_2}x_1(t,i)+a_{1,q_2}x_2(t,i)+\ldots+a_{q_1-1,q_2}x_{q_1}(t,i)+x_{q_1+1}(t,i)+u(t,i)
\end{aligned} \tag{17a}$$

$$\begin{aligned}
&x_{q_1+1}(t,i+1)=\bar{a}_{0,q_2-1}x_1(t,i)+\bar{a}_{1,q_2-1}x_2(t,i)+\ldots+\bar{a}_{q_1-1,q_2-1}x_{q_1}(t,i)+a_{q_1,q_2-1}x_{q_1+1}(t,i)+x_{q_2+2}(t,i)+a_{q_1,q_2-1}u(t,i)\\
&\vdots\\
&x_{q_1+q_2-1}(t,i+1)=\bar{a}_{0,1}x_1(t,i)+\bar{a}_{1,1}x_2(t,i)+\ldots+\bar{a}_{q_1-1,1}x_{q_1}(t,i)+a_{q_1,1}x_{q_1+1}(t,i)+x_{q_1+q_2}(t,i)+a_{q_1,1}u(t,i)\\
&x_{q_1+q_2}(t,i+1)=\bar{a}_{0,0}x_1(t,i)+\bar{a}_{1,0}x_2(t,i)+\ldots+\bar{a}_{q_1-1,0}x_{q_1}(t,i)+a_{q_1,0}x_{q_1+1}(t,i)+a_{q_1,0}u(t,i)\\
&x_{q_1+q_2+1}(t,i+1)=\hat{a}_{0,0}x_1(t,i)+\hat{a}_{1,0}x_2(t,i)+\ldots+\hat{a}_{q_1-1,0}x_{q_1}(t,i)+\bar{b}_{q_1,0}x_{q_1+1}(t,i)+\bar{b}_{q_1,0}u(t,i)\\
&x_{q_1+q_2+2}(t,i+1)=\hat{a}_{0,1}x_1(t,i)+\hat{a}_{1,1}x_2(t,i)+\ldots+\hat{a}_{q_1-1,1}x_{q_1}(t,i)+\bar{b}_{q_1,1}x_{q_1+1}(t,i)+x_{q_1+q_2+1}(t,i)+\bar{b}_{q_1,1}u(t,i)\\
&\vdots\\
&x_{q_1+2q_2}(t,i+1)=\hat{a}_{0,q_2-1}x_1(t,i)+\hat{a}_{1,q_2-1}x_2(t,i)+\ldots+\hat{a}_{q_1-1,q_2-1}x_{q_1}(t,i)+\bar{b}_{q_1,q_2-1}x_{q_1+1}(t,i)+x_{q_1+2q_2-1}(t,i)+\bar{b}_{q_1,q_2-1}u(t,i)
\end{aligned} \tag{17b}$$

$$y(t,i)=\bar{b}_{0,q_2}x_1(t,i)+\bar{b}_{1,q_2}x_2(t,i)+\ldots+\bar{b}_{q_1-1,q_2}x_{q_1}(t,i)+x_{q_1+2q_2}(t,i) \tag{17c}$$

where

$$\begin{aligned}
&\bar{a}_{kl}=a_{kl}+a_{q_1l}a_{kq_2}\\
&\hat{a}_{kl}=\bar{b}_{kl}+\bar{b}_{q_1l}a_{kq_2}\\
&k=0,1,\ldots,q_1-1;\quad l=0,1,\ldots,q_2-1
\end{aligned} \tag{17d}$$

Substituting in the equations (17a) i by $i+1$ and differentiating with respect to t the equations (17b) we obtain the equations (1) with

$$\begin{aligned}
&A_0=0,\quad A_1=\begin{bmatrix}0&0&0\\A_{21}^{(1)}&A_{22}^{(1)}&0\\A_{31}^{(1)}&A_{32}^{(1)}&A_{33}^{(1)}\end{bmatrix}\in\Re^{(q_1+2q_2)\times(q_1+2q_2)},\quad A_2=\begin{bmatrix}A_{11}^{(2)}&A_{12}^{(2)}&0\\0&0&0\\0&0&0\end{bmatrix}\in\Re^{(q_1+2q_2)\times(q_1+2q_2)},\\
&B_0=0,\quad B_1=\begin{bmatrix}0\\B_{12}\\B_{13}\end{bmatrix}\in\Re^{(q_1+2q_2)\times1},\quad B_2=\begin{bmatrix}B_{21}\\0\\0\end{bmatrix}\in\Re^{(q_1+2q_2)\times1},\quad C=[C_1\quad 0\quad C_3]\in\Re^{1\times(q_1+2q_2)}
\end{aligned} \tag{18a}$$

where

$$A_{21}^{(1)}=\begin{bmatrix}\bar a_{0,q_2-1} & \bar a_{1,q_2-1} & \dots & \bar a_{q_1-1,q_2-1}\\ \vdots & \vdots & \dots & \vdots\\ \bar a_{0,1} & \bar a_{1,1} & \dots & \bar a_{q_1-1,1}\\ \bar a_{0,0} & \bar a_{1,0} & \dots & \bar a_{q_1-1,0}\end{bmatrix}\in\Re^{q_2\times q_1},\quad A_{22}^{(1)}=\begin{bmatrix}a_{q_1,q_2-1} & 1 & 0 & \dots & 0\\ a_{q_1,q_2-2} & 0 & 1 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ a_{q_1,1} & 0 & 0 & \dots & 1\\ a_{q_1,0} & 0 & 0 & \dots & 0\end{bmatrix}\in\Re^{q_2\times q_2},$$

$$A_{31}^{(1)}=\begin{bmatrix}\hat a_{0,0} & \hat a_{1,0} & \dots & \hat a_{q_1-1,0}\\ \hat a_{0,1} & \hat a_{1,1} & \dots & \hat a_{q_1-1,1}\\ \vdots & \vdots & \dots & \vdots\\ \hat a_{0,q_2-1} & \hat a_{1,q_2-1} & \dots & \hat a_{q_1-1,q_2-1}\end{bmatrix}\in\Re^{q_2\times q_1},\quad A_{32}^{(1)}=\begin{bmatrix}\bar b_{q_1,0} & 0 & 0 & \dots & 0\\ \bar b_{q_1,1} & 0 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \bar b_{q_1,q_2-2} & 0 & 0 & \dots & 0\\ \bar b_{q_1,q_2-1} & 0 & 0 & \dots & 0\end{bmatrix}\in\Re^{q_2\times q_2},$$

$$A_{33}^{(1)}=\begin{bmatrix}0 & 0 & \dots & 0 & 0\\ 1 & 0 & \dots & 0 & 0\\ 0 & 1 & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & 1 & 0\end{bmatrix}\in\Re^{q_2\times q_2},\ A_{11}^{(2)}=\begin{bmatrix}0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & 1\\ a_{0,q_2} & a_{1,q_2} & a_{2,q_2} & \dots & a_{q_1-1,q_2}\end{bmatrix}\in\Re^{q_1\times q_1},\quad A_{12}^{(2)}=\begin{bmatrix}0 & 0 & \dots & 0\\ \vdots & \vdots & \dots & \vdots\\ 0 & 0 & \dots & 0\\ 1 & 0 & \dots & 0\end{bmatrix}\in\Re^{q_1\times q_2},$$

$$B_{12}=\begin{bmatrix}a_{q_1,q_2-1}\\ \vdots\\ a_{q_1,1}\\ a_{q_1,0}\end{bmatrix}\in\Re^{q_2\times 1},\quad B_{13}=\begin{bmatrix}\bar b_{q_1,0}\\ \bar b_{q_1,1}\\ \vdots\\ \bar b_{q_1,q_2-1}\end{bmatrix}\in\Re^{q_2\times 1},\quad B_{21}=\begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix}\in\Re^{q_1\times 1},$$

$$C_1=[\bar b_{0,q_2}\quad \bar b_{1,q_2}\quad \dots\quad \bar b_{q_1-1,q_2}]\in\Re^{1\times q_1},\quad C_3=[0\quad \dots\quad 0\quad 1]\in\Re^{1\times q_2} \tag{18b}$$

Theorem 2. There exists a positive realization (18) of the transfer function (10) if the following conditions are satisfied

1) $a_{kl}\ge 0$ for $k=0,1,\dots,q_1;\ \ l=0,1,\dots,q_2;\ \ k+l\ne q_1+q_2$
2) $b_{kl}\ge 0$ for $k=0,1,\dots,q_1;\ \ l=0,1,\dots,q_2$

Proof. If the condition 2) is met then $D=b_{q_1,q_2}\ge 0$ and the coefficients of the strictly proper transfer function (12) are nonnegative. From (18) it follows that if the conditions 1) and 2) are satisfied then $A_k\in\Re_+^{n\times n}$, $B_k\in\Re_+^{n\times m}$, $C\in\Re_+^{p\times n}$, $k=0,1,2;\ D\ge 0$ and by Theorem 1 the realization (18) is positive. □
From (18) we have the following corollary.

Corollary 1. If the conditions 1) and 2) of Theorem 2 are satisfied then there exists a positive realization of the transfer function (10) with $A_0=0$ and $B_0=0$ and $A_2\in\Re_+^{n\times n}$.

Example 1. Find a positive realization of the transfer function

$$T(s,z)=\frac{s^2z^2+s^2z+s^2+z^2+z+2}{s^2z^2-2s^2z-s^2-z^2-2z-1} \tag{19}$$

Using (8) and (9) we obtain

$$D=\lim_{s,z\to\infty}T(s,z)=1 \tag{20}$$

and the strictly proper transfer function

$$T_{sp}(s,z)=T(s,z)-1=\frac{3s^2z+2s^2+2z^2+3z+3}{s^2z^2-2s^2z-s^2-z^2-2z-1}=\frac{3z^{-1}+2z^{-2}+2s^{-2}+3s^{-2}z^{-1}+3s^{-2}z^{-2}}{1-2z^{-1}-z^{-2}-s^{-2}-2s^{-2}z^{-1}-s^{-2}z^{-2}} \tag{21}$$

In this case (15) and (16) have the form

$$E(s,z)=U(s,z)+(2z^{-1}+z^{-2}+s^{-2}+2s^{-2}z^{-1}+s^{-2}z^{-2})E(s,z) \tag{22}$$

and

$$Y(s,z)=(3z^{-1}+2z^{-2}+2s^{-2}+3s^{-2}z^{-1}+3s^{-2}z^{-2})E(s,z) \tag{23}$$

Using (22) and (23) we may draw the state variable diagram shown in Fig. 2.

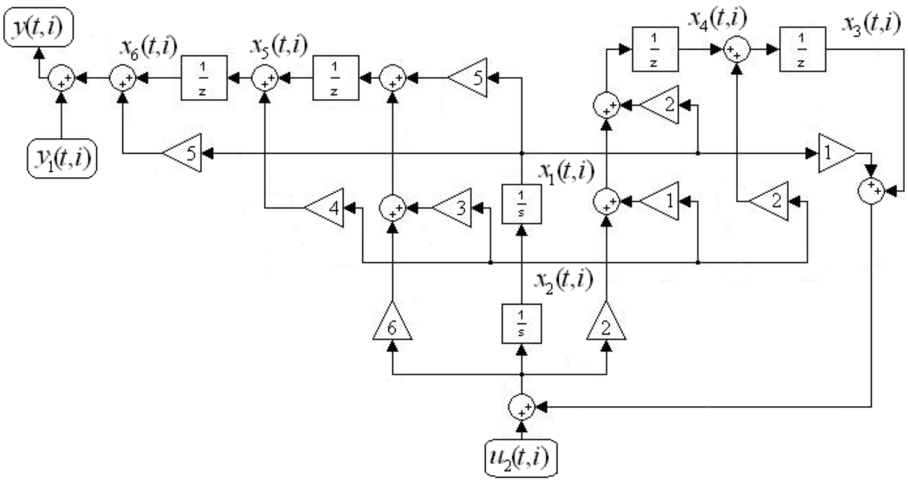

Fig. 2. State variable diagram for transfer function (21).

The outputs of the integration elements are chosen as the state variables $x_1(s,z), x_2(s,z)$ and the outputs of the delays elements as the state variables $x_3(s,z),\ldots,x_6(s,z)$. From the state variable diagram we have the equations

$$\begin{aligned}
&\dot{x}_1(t,i+1) = x_2(t,i+1)\\
&\dot{x}_2(t,i+1) = x_1(t,i+1) + x_3(t,i+1) + u(t,i+1)\\
&\dot{x}_3(t,i+1) = 4\dot{x}_1(t,i) + 2\dot{x}_3(t,i) + \dot{x}_4(t,i) + 2\dot{u}(t,i)\\
&\dot{x}_4(t,i+1) = 2\dot{x}_1(t,i) + \dot{x}_3(t,i) + \dot{u}(t,i)\\
&\dot{x}_5(t,i+1) = 5\dot{x}_1(t,i) + 2\dot{x}_3(t,i) + 2\dot{u}(t,i)\\
&\dot{x}_6(t,i+1) = 6\dot{x}_1(t,i) + 3\dot{x}_3(t,i) + \dot{x}_5(t,i) + 3\dot{u}(t,i)
\end{aligned} \tag{24}$$

and

$$y(t,i) = 2x_1(t,i) + x_6(t,i) \tag{25}$$

The equations (24) and (25) can be written in the form (1), where

$$\begin{aligned}
&x(t,i) = [x_1(t,i) \quad x_2(t,i) \quad x_3(t,i) \quad x_4(t,i) \quad x_5(t,i) \quad x_6(t,i)]^T\\
&A_0 = 0,\ \ B_0 = 0,\ \ C = [2 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]\\
&A_1 = \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 4&0&2&1&0&0\\ 2&0&1&0&0&0\\ 5&0&2&0&0&0\\ 6&0&3&0&1&0 \end{bmatrix},\ \ A_2 = \begin{bmatrix} 0&1&0&0&0&0\\ 1&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix},\ \ B_1 = \begin{bmatrix}0\\0\\2\\1\\2\\3\end{bmatrix},\ \ B_2 = \begin{bmatrix}0\\1\\0\\0\\0\\0\end{bmatrix}
\end{aligned} \tag{26}$$

The desired positive realization of (19) is given by (20) and (26).

3.2. MIMO systems

First we shall consider linear hybrid m-inputs and one-output systems with the transfer matrix

$$T(s,z) = [T_1(s,z) \quad \ldots \quad T_m(s,z)] \quad \in R^{1\times m}(s,z) \tag{27}$$

where

$$T_k(s,z) = \frac{n_k(s,z)}{d_k(s,z)},\ k = 1,\ldots,m \tag{28}$$

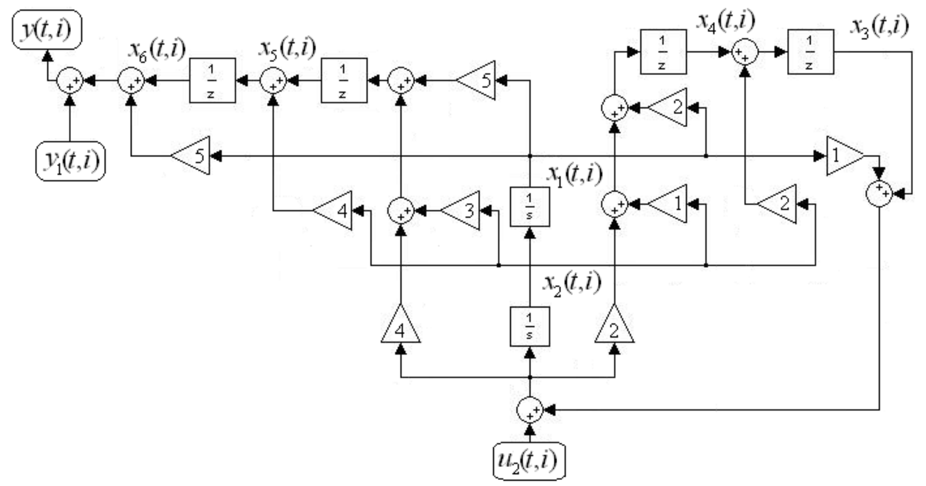

Fig. 3. State variable diagram for transfer function (36).

It is assumed that the minimal common denominator $d(s,z)$ satisfies the assumption

$$d(s,z) = \prod_{k=1}^{m} d_k(s,z) \tag{29}$$

Using (8) and (9) we can find the matrix D and the strictly proper transfer matrix $T_{sp}(s,z)$. Applying the approach presented above for SISO systems to MIMO system with (27) we may find a realization of each transfer function (28). A realization of the transfer function (27) can be found by the use of the following theorem.

Theorem 3. Let

$$A_{0k} = 0, A_{1k}, A_{2k}, B_{0k} = 0, B_{1k}, B_{2k}, C,\ k = 1,...,m \tag{30}$$

be a realization of the transfer function (28). Then a realization of the strictly proper transfer matrix

$$T_{sp}(s,z) = T(s,z) - D = [T_1(s,z) - D_1 \quad ...$$

$$T_m(s,z) - D_m],\quad D_k = \lim_{s,z\to\infty} T_k(s,z) \tag{31}$$

is given by

$$\begin{aligned} &A_1 = \text{blockdiag}\,[A_{11} \quad ... \quad A_{1m}],\\ &A_2 = \text{blockdiag}\,[A_{21} \quad ... \quad A_{2m}],\\ &B_1 = \text{blockdiag}\,[B_{11} \quad ... \quad B_{1m}],\\ &B_2 = \text{blockdiag}\,[B_{21} \quad ... \quad B_{2m}],\\ &C = [C_1 \quad ... \quad C_m] \end{aligned} \tag{32}$$

Proof. Using (8), (31) and (32) we obtain

$$\begin{aligned} T_{sp}(s,z) &= [C_1\ ...\ C_m]\{\text{blockdiag}\,[I_n sz - A_1 s - A_2 z]\}^{-1}\\ &\quad \{\text{blockdiag}\,[B_{11}s + B_{21}z \quad ... \quad B_{1m}s + B_{2m}z]\}\\ &= [C_1 \quad ... \quad C_m]\{\text{blockdiag}\,[I_n sz - A_1 s - A_2 z]^{-1}\}\\ &\quad \{\text{blockdiag}\,[B_{11}s + B_{21}z \quad ... \quad B_{1m}s + B_{2m}z]\}\\ &= \big[C_1\,[I_n sz - A_1 s - A_2 z]^{-1}(B_{11}s + B_{21}z) \quad ...\\ &\quad C_m\,[I_n sz - A_m s - A_m z]^{-1}(B_{1m}s + B_{2m}z)\big]\\ &= [T_1(s,z) - D_1 \quad ... \quad T_m(s,z) - D_m] \end{aligned}$$

□

Theorem 4. There exists a positive realization (32) of the transfer matrix (27) if all coefficients of the numerator $n_k(s,z), k = 1,\ldots,m$ are nonnegative and all coefficient of the denominators $d_k(s,z),\ k = 1,\ldots,m$ are nonpositive except the leading coefficient equal to 1.

Proof. If the assumptions are satisfied then by Theorem 2 the realization (30) of the transfer function (27) is a positive one. From (32) it follows that in this case all matrices (32) have nonnegative entries and by Theorem 1 the realization of the transfer matrix is positive. □

Example 2. Given the transfer matrix

$$T(s,z) = [T_1(s,z) \quad T_2(s,z)] \tag{33}$$

where $T_1(s,z)$ is given by (19) and

$$T_2(s,z) = \frac{2s^2z^2 + 2s^2 + 3z^2 + s + 1}{s^2z^2 - 2s^2 - z^2 - 2sz - s - 2} \tag{34}$$

Using (8) and (9) from (33), (19) and (34) we have

$$D = \lim_{s,z\to\infty} T(s,z) = [1 \quad 2] \tag{35}$$

and

$$T_{sp}(s,z) = T(s,z) - D = \left[\frac{3s^2z + 2s^2 + 2z^2 + 3z + 3}{s^2z^2 - 2s^2z - s^2 - z^2 - 2z - 1}\right.$$

$$\left.\frac{6s^2 + 5z^2 + 4sz + 3s + 5}{s^2z^2 - 2s^2 - z^2 - 2sz - s - 2}\right] =$$

$$= \left[\frac{3z^{-1} + 2z^{-2} + 2s^{-2} + 3s^{-2}z^{-1} + 3s^{-2}z^{-2}}{1 - 2z^{-1} - z^{-2} - s^{-2} - 2s^{-2}z^{-1} - s^{-2}z^{-2}}\right.$$

$$\left.\frac{6z^{-2} + 5s^{-2} + 4s^{-1}z^{-1} + 3s^{-1}z^{-2} + 5s^{-2}z^{-2}}{1 - 2z^{-2} - s^{-2} - 2s^{-1}z^{-1} - s^{-1}z^{-2} - 2s^{-2}z^{-2}}\right] \tag{36}$$

The state variable diagram corresponding to the transfer function $T_{sp1}(s,z)$ is shown in Fig. 2.
and the positive realization is given by (26) i.e.

$$A_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 5 & 0 & 2 & 0 & 0 & 0 \\ 6 & 0 & 3 & 0 & 1 & 0 \end{bmatrix},$$

$$A_{12} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_{11} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$C_1 = [2 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0] \tag{37}$$

The state variable diagram corresponding to $T_{sp2}(s,z)$ is shown in Fig. 3.

Using this state variable diagram we can write the equations

$$\begin{aligned}
&\dot{x}_1(t,i+1) = x_2(t,i+1) \\
&\dot{x}_2(t,i+1) = x_1(t,i+1) + x_3(t,i+1) + u_2(t,i+1) \\
&\dot{x}_3(t,i+1) = 2\dot{x}_2(t,i) + \dot{x}_4(t,i) \\
&\dot{x}_4(t,i+1) = 4\dot{x}_1(t,i) + \dot{x}_2(t,i) + 2\dot{x}_3(t,i) + 2\dot{u}_2(t,i) \\
&\dot{x}_5(t,i+1) = 11\dot{x}_1(t,i) + 3\dot{x}_2(t,i) + 6\dot{x}_3(t,i) + 6\dot{u}_2(t,i) \\
&\dot{x}_6(t,i+1) = 4\dot{x}_2(t,i) + \dot{x}_5(t,i) \\
&y(t,i) = 5x_1(t,i) + x_6(t,i)
\end{aligned} \tag{38}$$

From those equations we have the realization of $T_{sp2}(s,z)$ in the form

$$A_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 4 & 1 & 2 & 0 & 0 & 0 \\ 11 & 3 & 6 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$A_{22} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_{21} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 6 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$C_2 = [5 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1] \tag{39}$$

The state variable diagram corresponding to the transfer matrix (36) can be obtained as the connection of the state variable diagrams shown in Fig. 2 and Fig. 3 (see Fig. 4).

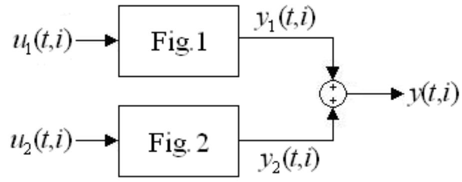

Fig. 4. Connection of state variable diagrams.

By Theorem 3 the desired realization of the transfer matrix (33) is given by

$$A_1 = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{12} \end{bmatrix}, \quad A_2 = \begin{bmatrix} A_{21} & 0 \\ 0 & A_{22} \end{bmatrix},$$

$$B_1 = \begin{bmatrix} B_{11} & 0 \\ 0 & B_{12} \end{bmatrix}, \quad B_2 = \begin{bmatrix} B_{21} & 0 \\ 0 & B_{22} \end{bmatrix},$$

$$C = [C_1 \quad C_2], \quad D = [1 \quad 2] \tag{40}$$

where the submatrices $A_{11}, A_{12}, B_{11}, A_{12}, C_1$ are given by (37) and submatrices $A_{21}, A_{22}, B_{21}, A_{22}, C_2$ are given by (39). The realization is positive since all entries of the matrices (40) are nonnegative.

Remark 1. If the assumption (29) is not satisfied and

$$\deg_s d(s,z) < \prod_{k=1}^{m} \deg_s d_k(s,z)$$

and

$$\deg_z d(s,z) < \prod_{k=1}^{m} \deg_z d_k(s,z) \tag{41}$$

then to decrease the dimension of a realization of (27) it is recommended to find $d(s,z)$ and write the transfer matrix (27) in the form

$$T(s,z) = \frac{1}{d(s,z)} [\bar{n}_1(s,z) \quad \dots \quad \bar{n}_m(s,z)] \tag{42}$$

where $deg_s d(s,z)$ ($deg_z d(s,z)$) denotes the degree of the minimal common denominator with respect to $s(z)$.

Note that the m-inputs and p-outputs systems can be considered as the sequence of p m-inputs and one-output systems. In this way the presented approach can be extended for m-inputs and p-outputs linear systems.

4. Concluding remarks

The problem of computation of positive realizations of hybrid linear systems described by the equations (1) by the use of the state variable diagram method has been addressed. It has been shown that there exists a positive realization of a given transfer matrix if all coefficients of the numerator of each transfer function are nonnegative and all coefficients of the denominator are nonpositive except the leading one equal to 1. The presented method enable us to find a positive realization with zero A_0, B_0 matrices. If the condition (41) is satisfied then it is recommended to find first minimal common denominator for each row of the transfer matrix. Those considerations can be extended to linear hybrid systems with delays and to linear fractional hybrid systems.

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