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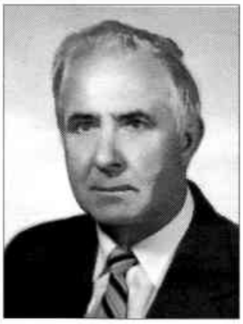
Stabilization of positive linear systems by state-feedbacks



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Abstract

A method based on Gersgorin's theorem of stabilization of positive linear continuous-time and discrete-time systems by state feedbacks is presented. It is shown that the stabilization problems can be reduced to suitable quadratic programming problems with constraints. The method is illustrated by two numerical examples.

Streszczenie

Przedstawiono metodę stabilizacji dodatnich, liniowych układów ciągłych i dyskretnych, za pomocą sprzężeń zwrotnych od wektora stanu, opartą na twierdzeniu Gersgorina. Wykazano, że problemy stabiizacji mogą być sprowadzone do odpowiednich problemów programowania kwadratowego z ograniczeniami liniowymi. Metodę zilustrowano dwoma przykładami liczbowymi.

1. INTRODUCTION

Roughly speaking positive linear systems are dynamical systems in which the input, state and output spaces are spaces over the nonnegative real numbers.

The positive linear systems are used in biomathematics, economics, chemometries and other research areas [1,3,12–23]. The reachability, observability and realizability of continuous-time positive systems were considered in [22,21,6,7]. The realization problem for positive linear systems was investigated in [1,9,12,18,19].

Recently in [23] Trzaska has established a criterion for asymptotic stability of positive standard and singular continuous-time linear systems.

In this paper a method of stabilization of positive linear continuous-time and discrete-time systems by state feedbacks will be presented. It will be shown that the stabilization problems can be reduced to suitable quadratic programming problems with constraints. To the best knowledge of the author this is the first paper on stabilization problem of positive linear systems by state feedbacks.

2. PROBLEM FORMULATION

2.1. Continuous-time systems

Consider a continuous-time linear system described by the equations

$$\dot{x} = Ax + Bu,\ x(t_0) = x_0 \tag{1a}$$
$$y = Cx+Du \tag{1b}$$

where:
$x(t) = x \in R^n$ is the state vector
$u(t) = u \in R^m$ is the input vector
$y(t) = y \in R^p$ is the output vector
A,B,C,D are real matrices of appropriate dimensions.

Let $R_+^{n\times m}$ be the set of real matrices with nonnegative entries and $R_+^n = R_+^{n\times 1}$.

Definition 1. The system (1) is called positive if for any $x_0 \in R_+^n$ and any $u \in R_+^m$ we have $x \in R_+^n$ and $y \in R_+^p$ for $t \geq 0$.

A matrix $A \in R_+^{n\times n}$ is called the Metzler matrix if all its off-diagonal entries are nonnegative.

It is well-know [10,20] that the system (1) is positive if and only if A is a Metzler matrix and $B \in R_+^{n\times m}$, $C \in R_+^{p\times n}$, $D \in R_+^{p\times m}$

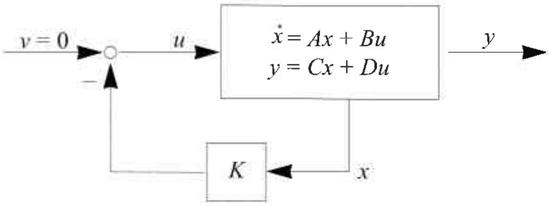

Fig.1

Let us assume that the positive system (1) is not asymptoticaly stable. We are looking for again matrix $K \in R^{m\times n}$ of the state-feedback $u = v - Kx$ (Fig. 1) such that the closed-loop system matrix

$$A_c = A - BK \tag{2}$$

is an asymptotically stable Metzler matrix, i.e. all eigenvalues $\lambda_1,\lambda_2,...,\lambda_n$ of A_c have negative real parts.

2.2. Discrete-time systems

Consider a discrete-time linear system described by the equations

$$x_{i+1} = Ax_i+Bu_i \tag{3a}$$
$$i \in Z_+ := \{0,1,...\}$$
$$y_i = Cx_i+Du_i \tag{3b}$$

where:
$x_i \in R^n$ is the state vector
$u_i \in R^m$ is the input vector,
$y_i \in R^p$ is the output vector
A,B,C,D are real matrices of appropriate dimensions.

Definition 2. The system (3) is called positive if for any $y_i \in R_+^n$ and any $u_i \in R_+^m$ we have $x_i \in R_+^n$ and $y \in R_+^p$ for $i \in Z_+$.

It is well-know [11,15] that the system (3) is positive if and only if $A \in R_+^{n\times n}$, $B \in R_+^{n\times m}$, $C \in R_+^{p\times n}$, $D \in R_+^{p\times m}$

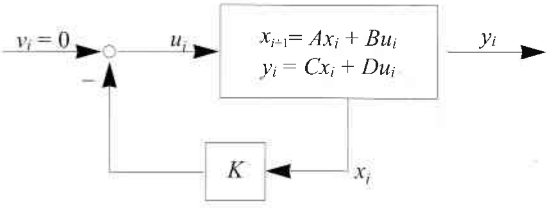

Fig.2

Let us assume that the positive system (3) is not asymptotically stable. We are looking for a gain matrix $K \in R^{m\times n}$ of the state-feedback $u_i = v_i - Kx_i$ (Fig. 2) such that the matrix of positive closed-loop system (2) is asymptotically stable, i.e. all eigenvalues of A_c have modules less than 1.

The main subject of this paper is to established conditions under which the stabilization problem has a solution and to give a procedure for computation of the gain matrix K.

3. PROBLEM SOLUTION

The problem will be solved by the use of the method based on Gersgorin's theorem [8,14] and quadratic programming [4]. To simply the notion we shall consider single input system ($m = 1$) with $B = b$ and $K = k \in R^{1\times n}$.

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix},\ b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \text{ and } k = [k_1\ k_2 \dots k_n] \quad (4)$$

Then

$$A_c = A - bk \begin{bmatrix} a_{11} - b_1k_1, & a_{12} - b_1k_2, & \dots, & a_{1n} - b_1k_n \\ a_{21} - b_2k_1, & a_{22} - b_2k_2, & \dots, & a_{2n} - b_2k_n \\ \dots & \dots & \dots & \dots \\ a_{n1} - b_nk_1, & a_{n2} - b_nk_2, & \dots, & a_{nn} - b_nk_n \end{bmatrix} \quad (5)$$

3.1. Continuous-time systems.

Theorem l. The closed-loop system matrix (5) is asymptotically stable Metzler matrix only if

$$\sum_{i=1}^{n} (a_{ii} - b_ik_i) < 0 \quad (6)$$

Proof. It is well-known that

$$\sum_{i=1}^{n} (a_{ii} - b_ik_i) = \sum_{i=1}^{n} \lambda_i \quad (7)$$

where:
λ_i ($i = 1,\dots,n$) is the ith eigenvalue of (5).

If $\sum_{i=1}^{n} \lambda_i, \geq 0$ then at least one eigenvalue has nonnegative real part and the closed-loop system matrix (5) is not asymptotically stable.

By Geršgorin's theorem the matrix (5) is an asymptotically stable Metzler matrix if

$$a_{ij} - b_ik_j \geq 0 \ \text{ for } i \neq j;\ i,j = 1,\dots,n \quad (8a)$$

and

$$b_ik_i - a_{ii} > \sum_{\substack{j=1 \\ j\neq i}}^{n} (a_{ij} - b_ik_j) \text{ for } i = 1,\dots,n$$
$$(\text{ or } b_jk_j - a_{jj} > \sum_{\substack{i=1 \\ i\neq j}}^{n} (a_{ij} - b_ik_j) \text{ for } j = 1,\dots,n) \quad (8b)$$

Definition 2. A set S_k of all k satisfying the constraints (8) is called the set of feasible gain matrices.

Remark 1. The set S_k may be empty. For example for $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ the set S_k is empty since by (8) $k_1 + k_2 > 3$, $k_1 + k_2 > 7$ and $k_1 \leq 4, k_2 < 2$.

Lemma l. The set S_k is not empty if

$$\frac{b_i}{b_j} a_{jj} < a_{ij} \text{ for } i \neq j \quad (i, j = 1,\dots,n) \quad (9)$$

Proof. From (8) it follows that the set S_k is not empty if $b_ik_j = a_{ij}$ for $i \neq j$ and $b_ik_i > a_{ii}$.

The condition (9) follows immediately from the above relations.

Note that (9) is not satisfied by the matrices A and b in Remarks l.

It is assumed that the S_k is not empty.

The stabilization problem can be formulated as the following quadratic programming problem:

Choose $k = [k_1, k_2, \dots, k_n)$ maximizing the function

$$f(k) = f(k_1, k_2, \dots, k_n) := \sum_{i=1}^{n} (a_{ii} - b_ik_i)^2 \quad (10)$$

subject to $k \in S_k$, i.e.

$$\max_{k\in S_k} f(k) \quad (11)$$

Remark 2. In general case the function (10) may be taken in the form

$$f(k) = \sum_{i=1}^{n} w_i (a_{ii} - b_ik_i)^2 \quad (10)$$

where:
$w_i > 0$, $i = 1,\dots, n$ are some weighting coefficients.

The function (10) may be written as

$$f(k) = kDk^T + ck^T + f_0$$

where:

$$D := diag[b_1^2, b_2^2, \dots, b_n^2],\ c := -2[a_{11}b_1, a_{22}b_2, \dots, a_{nn}b_n],\ f_0 := \sum_{i=1}^{n} a_{ii}^2$$

Using (12) we may formulate the stabilization problem as follows. Given A, b, find k which maximizes the quadratic function (12) and satisfies the constraints (8). To solve the problem we may use any known method of quadratic programming [4].

Note that D is a positive defined matrix if $b_i \neq 0$ for $i = 1,\dots,n$. Therefore the stabilization problem has a unique solution if S_k is not empty. Taking into account Lemma 1 we have the following.

Theorem 2. The stabilization problem has a unique solution if $b_i \neq 0$ for $i = 1\dots.n$ and (9) holds.

Remark 3. Note that it is possible to stabilize a positive system (1) even if it is not controllable.

For example the system (1) with $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is uncontrollable, $\det[b, Ab] = \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$, but a gain matrix $k = [k_1, k_2]$ may be found (for example $k_1 = k_2 = 1.75$) such that A_c is an asymptotically stable Metzler matrix $A_c = \begin{bmatrix} -0.75 & 0.25 \\ 0.25 & -0.75 \end{bmatrix}$

Example 1. Consider the positive continuous-time system (1) with $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Find $k = [k_1, k_2]$ such that the closed-loop system matrix

$$A_c = A - bk = \begin{bmatrix} 1 - k_1, & 3 - k_2 \\ 4 - k_1, & 2 - k_2 \end{bmatrix} \quad (13)$$

is an asymptotically stable Metzler matrix.

In this case the set S'_k is defined by the constraints

$$k_1 \leq 4,\ k_2 \leq 3,\ k_1 + k_2 > 4,\ k_1 + k_2 > 6 \quad (14)$$

and is not empty (Fig. 3).

To solve the problem we have to find k_1, k_2 which maximize the function

$$f(k) = (1 - k_1)^2 + (2 - k_2)^2 = [k_1\ k_2] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} + [-2, -4] \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} + 5 \quad (15)$$

and satisfy the constraints (14). It is easy to show that the optimal solution is $k_1 = 4$, $k_2 = 3$ (see Fig. 3) and (13) has the form

$$A_c = \begin{bmatrix} -3, & 0 \\ 0, & -1 \end{bmatrix}$$

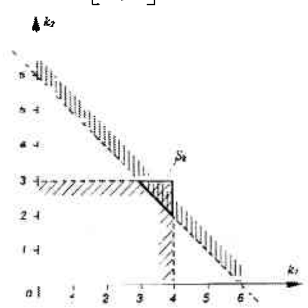

Fig.3

3.2. Discrete-time system

A matrix $A \in R^{n\times n}$ is called positive if $A \in R_+^{n\times n}$ at least one its entry is positive.

Theorem 3. The closed-loop system matrix (5) is asymptotically stable positive matrix only if

$$\sum_{i=1}^{n} (a_{ii} - b_i k_i) < n \qquad (16)$$

Proof. From (7) it follows that if $\sum_{i=1}^{n}\lambda_i \geq n$ then at least one eigenvalue has modulus greater or equal to 1 and the closed-loop matrix (5) is not asymptotically stable.

By Geršgorin's theorem the matrix (5) is an asymptotically stable positive matrix if

$$a_{ij} - b_i k_j \geq 0 \text{ for } i, j = 1,...,n \qquad (17a)$$

and

$$\sum_{j=1}^{n} (a_{ij} - b_i k_j) < 1 \text{ for } i=1,...,n$$
$$(\text{or } \sum_{i=1}^{n} (a_{ij} - b_i k_j) < 1 \text{ for } j=1,...,n\ ) \qquad (17b)$$

Definition 2. A set S'_k of all k satisfying the constraints (17) is called the set of feasible gain matrices.

Remark 4. The set S'_k may be empty. For example for

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

the set S'_k is empty since by (17) $k_1 < 1, k_2 < 1, k_1 + k_2 > 3$ and $k_1 + k_2 > 2$.

Lemma 2. The set S'_k is not empty if

$$\sum_{j=1}^{n} a_{ij} - 1 < b_i \sum_{j=1}^{n} \frac{a_{ij}}{b_j} \quad \text{for } i = 1,...,n \qquad (18)$$

Proof. From (17) it follows that the set S'_k is not empty if (17a) holds and

$$\sum_{j=1}^{n} a_{ij} - 1 < b_i \sum_{j=1}^{n} k_j \qquad (19)$$

From (17a) we have $\frac{a_{ij}}{b_j} \geq k_j$ and $\sum_{j=1}^{n} k_j \leq \sum_{j=1}^{n} \frac{a_{ij}}{b_j}$ Substitution of the last inequality into (19) yields (18).

Note that (18) is not satisfied by the matrices A and b in Remark 4.

The stabilization problem can be formulated as the following quadratic programming problem.

Choose $k = [k_1, k_2,..., k_n]$ minimizing the function

$$f(k) = f(k_1,k_2,...,k_n) := \sum_{i=1}^{n} (a_{ii} - b_i k_i)^2 = kDk^T + ck^T + f_0 \qquad (20)$$

subject to $k \in S'_k$, i.e. $\min_{k\in S'_k} f(k)$

where:

D, c and f_0 are defined in the same way as for (12).

In a similar way as for continuous-time systems the following theorem can be proved.

Theorem 4. The stabilization problem has a unique solution if $b_i \neq 0$ for $i = 1,...,n$ and (18) holds.

Example 2. Consider the positive discrete-time system (3) with

$$A = \begin{bmatrix} 0.8 & 1.2 \\ 1.2 & 1.4 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Find $k = [k_1\ k_2\ ]$ such that the closed-loop system matrix

$$A_c = A - bk = \begin{bmatrix} 0.8 - k_1 & 1.2 - k_2 \\ 1.2 - 2k_1 & 1.4 - 2k_2 \end{bmatrix} \qquad (21)$$

is an asymptotically stable positive matrix.

In this case the set $S'k$ is defined by the constraints

$$k_1 \leqslant 0.6, k_2 \leqslant 0.7,\ k_1 + k_2 > 1 \qquad (22)$$

and in not empty (Fig. 4).

To solve the problem we have to find k_1, k_2 which minimize the function

$$f(k_1, k_2) = (0.8 - k_1)^2 + (1.4 - 2k_2)^2$$
$$= [k_1\ k_2]\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} + [-1.6, -5.6]\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} + 2.6$$

and satisfy the constraints (22)

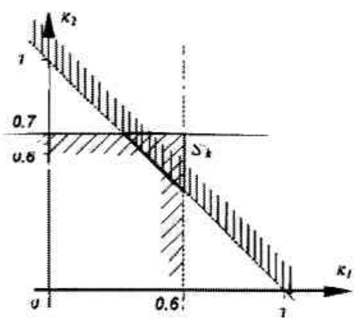

Fig.4

The optimal solution is given by $k_1 = 0.6$, $k_2 = 0.7$ (Fig. 4) and the matrix (21) has the form

$$A_c = \begin{bmatrix} 0{,}2 & 0{,}5 \\ 0 & 0 \end{bmatrix}$$

4. EXTENSIONS AND CONCLUDING REMARKS.

A method based on Geršgorin's theorem of stabilization of positive continuous–time and discrete-time linear systems by state feedbacks has been presented. It has been shown that the stabilization problems can be reduced to suitable quadratic problems with constraints. The idea of presented method can be extended for multi-input ($m > 1$) continuous-time and discrete-time linear systems with state and output feedbacks.

It is well-known [13] that the matrices A and

$$\bar{A} = RAR^{-1} = \begin{bmatrix} a_{11} & \frac{r_1}{r_2} a_{12} & .... & \frac{r_1}{r_n} a_{1n} \\ \frac{r_2}{r_1} a_{21} & a_{22} & .... & \frac{r_2}{r_n} a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{r_n}{r_1} a_{n1} & \frac{r_2}{r_2} a_{n2} & .... & a_{nn} \end{bmatrix}$$

have the same spectrum (eigenvalues) for any

$$R = diag[r_1, r_2, ..., r_n],\ r_i \neq 0$$

Therefore, instead of the conditions (8) the following conditions can be used

$$\frac{r_i}{r_j}\ (a_{ij} - b_i k_j) \geqslant 0 \text{ for } i \neq j;\ i, j = 1,...,n$$

and

$$b_i k_i - a_{ii} > \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{r_i}{r_j}\ \ (a_{ij} - b_i k_j) \text{ for } i = 1,...,n$$

for some $r_i > 0$, $i = 1,...,n$.

Similarly instead of the conditions (17) we may use

$$\frac{r_i}{r_j}(a_{ij} - b_i k_j) \geq 0 \text{ for } i, j = 1,...,n$$

$$\sum_{j=1}^{n} \frac{r_i}{r_j} (a_{ij} - b_i k_j) < 1 \text{ for } i = 1,\ldots,n$$

for some $r_i > 0$, $i = 1,\ldots,n$.

By suitable choice of $r_i > 0$, $i = 1,\ldots,n$ the Grešgorin's theorem can be extended [14] and we may solve the stabilization problems in such cases when for $r_1 = r_2 = \ldots = r_n = 1$ do not exist solutions to the problems.

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