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Reduction of descriptor 2D continuous-discrete linear systems to standard systems

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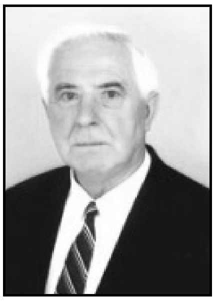

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Abstract

A method of reduction of descriptor 2D continuous-discrete linear systems with regular pencil to equivalent standard linear systems is proposed. The method is based on the shuffle algorithm. A polynomial matrix version of the method is also given. The inverse problem of reduction of standard systems to descriptor systems is considered as well.

Keywords: descriptor, 2D continuous-discrete, linear, system, regular pencil, reduction, shuffle algorithm.

Redukcja singularnych układów 2W ciągło-dyskretnych liniowych do układów standardowych

Streszczenie

W pracy podano metodę redukcji singularnych układów 2W ciągło-dyskretnych o pękach regularnych do równoważnych układów standardowych. Metoda ta opiera się na algorytmie przesuwania. Podano również wersję wielomianową tej metody opartą na działaniach elementarnych na macierzach wielomianowych. Pokazano, że odwracając procedurę metody wielomianowej można przekształcić układ standardowy do układu singularnego.

Słowa kluczowe: singularne układy, ciągło-dyskretne, pęk regularny, redukcja, algorytm przesuwania.

1. Introduction

Descriptor linear systems with regular pencils have been considered in many papers and books [4-12, 16-19]. The descriptor linear systems with singular pencils has been addressed in [10, 11]. The Drazin inverse of matrix to analysis of linear algebraic-differential equations has been applied in [4-7, 9, 11]. The positive descriptor linear systems with regular pencils have been analyzed in [1-3, 19]. The shuffle algorithm has been applied to checking the positivity of descriptor linear systems with regular pencil in [12].

A new class of descriptor 2D continuous-discrete linear systems has been proposed in [14]. The reachability and minimum energy control of positive 2D continuous-discrete linear systems have been investigated in [15].

In this paper a method for reduction of descriptor 2D continuous-discrete linear systems with regular pencils to equivalent standard linear systems will be proposed. A polynomial inversion of the method will be also given.

The paper is organized as follows. In section 2 a method of the reduction of descriptor 2D continuous-discrete linear systems with regular pencils to equivalent standard linear systems is proposed. A polynomial matrix method of the reduction is presented in section 3. An inverse problem of the reduction of 2D standard linear systems to descriptor systems is discussed in section 4. Concluding remarks are given in section 5.

In the paper the following elementary row operations will be used [11]:
- Multiplication of the i-th row by a real number c (or operators s, z). This operation will be denoted by $L[i\times c]$.
- Addition to the i-th row of the j-th row multiplied by a real number c (or operators s, z). This operation will be denoted by $L[i+j\times c]$.
- Interchange of the i-th and j-th rows. This operation will be denoted by $L[i, j]$.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, Z_+ - the set of nonnegative integers, I_n - the $n\times n$ identity matrix.

2. Reduction of descriptor systems to equivalent standard systems

Consider the descriptor 2D continuous-discrete linear system

$$E\dot{x}(t,i+1) = A_0 x(t,i) + A_1\dot{x}(t,i) + A_2 x(t,i+1) + Bu(t,i) \quad (1)$$

where $t\in\Re_+ = [0,+\infty]$, $i\in Z_+ = \{0,1,...\}$, $x(t,i)\in\Re^n$, $u(t,i)\in\Re^m$ are the state and input vectors and $E, A_0, A_1, A_2 \in\Re^{n\times n}$, $B\in\Re^{n\times m}$. It is assumed that det$E$=0, and

$$\det[Esz - A_0 - A_1 s - A_2 z] \neq 0 \text{ for some } s,z\in C, \quad (2)$$

where C is the field of complex numbers.

The systems (1) satisfying (2) will be called the descriptor system with regular pencil otherwise with singular pencil. It will be shown that the descriptor system with regular pencil can be reduced by the use of the shuffle algorithm [17, 18, 11] to the equivalent standard 2D continuous-discrete linear systems

$$\begin{aligned}\dot{x}(t,i+1) &= A_{00}x(t,i) + A_{01}x(t,i+1) + ... + A_{0,p_0}x(t,i+p_0)\\ &+ A_{10}\dot{x}(t,i) + A_{12}\dot{x}(t,i+2) + ... + A_{1,p_1}\dot{x}(t,i+p_1) + ...\\ &+ A_{q,0}x^{(q)}(t,i) + A_{q,1}x^{(q)}(t,i+1) + ... + A_{q,p_q}x^{(q)}(t,i+p_q)\\ &+ B_{00}u(t,i) + B_{01}u(t,i+1) + ... + B_{0,p_0}u(t,i+p_0) + B_{10}\dot{u}(t,i)\\ &+ B_{11}\dot{u}(t,i+1) + ... + B_{1,p_1}\dot{u}(t,i+p_1) + ... + B_{q,0}u^{(q)}(t,i)\\ &+ B_{q,1}u^{(q)}(t,i+1) + ... + B_{q,p_q}u^{(q)}(t,i+p_q)\end{aligned} \quad (3)$$

Where $x^{(h)}(t,i) = \dfrac{d^h x(t,i)}{dt^h}$, h = 1,2,… and $A_{k,j}\in\Re^{n\times n}$, $B_{k,j}\in\Re^{n\times m}$, $k = 0,1,\dots,q; j = 0,1,\dots,p_q$.

The equation (2.1) and (2.3) are called equivalent, if and only if, they have the same solution $x(t,i)$, $t\in\Re_+$, $i\in Z_+$ for suitable admissible inputs and consistent initial conditions.

Theorem 1. The descriptor system (1) satisfying (2) is equivalent to the standard system (3).

Proof. Performing elementary row operations on the array (or equivalently on (1))

$$E \quad A_0 \quad A_1 \quad A_2 \quad B \tag{4}$$

we obtain

$$\begin{matrix} E^1 & A_0^1 & A_1^1 & A_2^1 & B^1 \\ 0 & A_0^2 & A_1^2 & A_2^2 & B^2 \end{matrix} \tag{5}$$

where $E^1 \in \Re^{n_1 \times n}$ has full row rank. From (5) we obtain

$$E^1\dot{x}(t,i+1) = A_0^1 x(t,i) + A_1^1\dot{x}(t,i) + A_2^1 x(t,i+1) + B^1 u(t,i) \tag{6a}$$

$$0 = A_0^2 x(t,i) + A_1^2\dot{x}(t,i) + A_2^2 x(t,i+1) + B^2 u(t,i) \tag{6b}$$

Let U_{ad} be the set of admissible inputs $u(t,i)$ and X_c^0 be the set of consistent initial conditions $x(0,i) = x_{0,i}$, $\dot{x}(0,i) = x_{1,i}$, $i \in Z_+$, $x(t,0) = x_{t,0}$, $t \in \Re_+$, $x_{0,i}, x_{1,i}, x_{t,0} \in X_c^0$ of the equ. (1). Note that the admissible inputs and the consistent initial conditions should satisfy the equation (6b) for $t = 0$ and $i \in Z_+$. Among the matrices A_0^2, A_1^2, A_2^2 we choose the matrix A_k^2, $k \in \{0,1,2\}$ such that

$$\operatorname{rank}\begin{bmatrix} E^1 \\ A_k^2 \end{bmatrix} > \operatorname{rank} E^1,\ k \in \{0,1,2\}. \tag{7}$$

Case 1.

If (7) holds for $k = 1$ then in the equation (6b) we substitute i by $i + 1$, and

$$A_1^2\dot{x}(t,i+1) = -A_0^2 x(t,i+1) - A_2^2 x(t,i+2) - B^2 u(t,i+1). \tag{8}$$

The equations (6a) and (8) can be written in the form

$$\begin{aligned}\begin{bmatrix} E^1 \\ A_1^2 \end{bmatrix}\dot{x}(t,i+1) &= \begin{bmatrix} A_0^1 \\ 0 \end{bmatrix}x(t,i) + \begin{bmatrix} A_1^1 \\ 0 \end{bmatrix}\dot{x}(t,i) + \begin{bmatrix} A_2^1 \\ -A_0^2 \end{bmatrix}x(t,i+1) \\ &+ \begin{bmatrix} 0 \\ -A_2^2 \end{bmatrix}x(t,i+2) + \begin{bmatrix} B^1 \\ 0 \end{bmatrix}u(t,i) + \begin{bmatrix} 0 \\ -B^2 \end{bmatrix}u(t,i+1).\end{aligned} \tag{9a}$$

Note that the array

$$\begin{matrix} E^1 & A_0^1 & A_1^1 & A_2^1 & 0 & B^1 & 0 \\ A_1^2 & 0 & 0 & -A_0^2 & -A_2^2 & 0 & -B^2 \end{matrix} \tag{9b}$$

(or equivalently (9a)) can be obtained from (5) by the shuffle of A_1^2.

Case 2.

If (7) holds for $k = 2$ then differentiating (6b) with respect to time t we obtain

$$A_2^2\dot{x}(t,i+1) = -A_0^2\dot{x}(t,i) - A_1^2\ddot{x}(t,i) - B^2\dot{u}(t,i). \tag{10}$$

The equations (9a) and (9b) can be written in the form

$$\begin{aligned}\begin{bmatrix} E^1 \\ A_2^2 \end{bmatrix}\dot{x}(t,i+1) &= \begin{bmatrix} A_0^1 \\ 0 \end{bmatrix}x(t,i) + \begin{bmatrix} A_1^1 \\ -A_0^2 \end{bmatrix}\dot{x}(t,i) + \begin{bmatrix} A_2^1 \\ 0 \end{bmatrix}x(t,i+1) \\ &+ \begin{bmatrix} 0 \\ -A_1^2 \end{bmatrix}\ddot{x}(t,i) + \begin{bmatrix} B^1 \\ 0 \end{bmatrix}u(t,i) + \begin{bmatrix} 0 \\ -B^2 \end{bmatrix}\dot{u}(t,i).\end{aligned} \tag{11a}$$

Note that the array

$$\begin{matrix} E^1 & A_0^1 & A_1^1 & A_2^1 & 0 & B^1 & 0 \\ A_2^2 & 0 & -A_0^2 & 0 & -A_1^2 & 0 & -B^2 \end{matrix} \tag{11b}$$

(or equivalently (11a)) can be obtained from (5) by the shuffle of A_2^2.

Case 3.

If (7) holds for $k = 0$ then differentiating (6b) with respect to time t and substitute i by $i + 1$ we obtain

$$A_0^2\dot{x}(t,i+1) = -A_1^2\ddot{x}(t,i+1) - A_2^2\dot{x}(t,i+2) - B^2\dot{u}(t,i+1). \tag{12}$$

The equations (6a) and (12) can be written in the form

$$\begin{aligned}\begin{bmatrix} E^1 \\ A_0^2 \end{bmatrix}\dot{x}(t,i+1) &= \begin{bmatrix} A_0^1 \\ 0 \end{bmatrix}x(t,i) + \begin{bmatrix} A_1^1 \\ 0 \end{bmatrix}\dot{x}(t,i) + \begin{bmatrix} A_2^1 \\ 0 \end{bmatrix}x(t,i+1) \\ &+ \begin{bmatrix} 0 \\ -A_2^2 \end{bmatrix}\dot{x}(t,i+2) + \begin{bmatrix} 0 \\ -A_1^2 \end{bmatrix}\ddot{x}(t,i+1) + \begin{bmatrix} B^1 \\ 0 \end{bmatrix}u(t,i) \\ &+ \begin{bmatrix} 0 \\ -B^2 \end{bmatrix}\dot{u}(t,i+1).\end{aligned} \tag{13a}$$

Note that the array

$$\begin{matrix} E^1 & A_0^1 & A_1^1 & A_2^1 & 0 & 0 & B^1 & 0 \\ A_0^2 & 0 & 0 & 0 & -A_2^2 & -A_1^2 & 0 & -B^2 \end{matrix} \tag{13b}$$

(or equivalently (13a)) can be obtained from (5) by the shuffle of A_0^2.

If $\det\begin{bmatrix} E^1 \\ A_1^2 \end{bmatrix} \neq 0$ then premultiplying the equation (9a) by the inverse matrix $\begin{bmatrix} E^1 \\ A_1^2 \end{bmatrix}^{-1}$ we obtain

$$\begin{aligned}\dot{x}(t,i+1) &= A_{00}x(t,i) + A_{01}x(t,i+1) + A_{02}x(t,i+2) \\ &+ A_{10}\dot{x}(t,i) + B_{00}u(t,i) + B_{01}u(t,i+1)\end{aligned} \tag{14a}$$

where

$$\begin{gathered} A_{00} = \begin{bmatrix} E^1 \\ A_1^2 \end{bmatrix}^{-1}\begin{bmatrix} A_0^1 \\ 0 \end{bmatrix},\ A_{10} = \begin{bmatrix} E^1 \\ A_1^2 \end{bmatrix}^{-1}\begin{bmatrix} A_1^1 \\ 0 \end{bmatrix}, \\ A_{01} = \begin{bmatrix} E^1 \\ A_1^2 \end{bmatrix}^{-1}\begin{bmatrix} A_2^1 \\ -A_0^2 \end{bmatrix},\ A_{02} = \begin{bmatrix} E^1 \\ A_1^2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ -A_2^2 \end{bmatrix}, \\ B_{00} = \begin{bmatrix} E^1 \\ A_1^2 \end{bmatrix}^{-1}\begin{bmatrix} B^1 \\ 0 \end{bmatrix},\ B_{01} = \begin{bmatrix} E^1 \\ A_1^2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ -B^2 \end{bmatrix}. \end{gathered} \tag{14b}$$

If $\det\begin{bmatrix} E^1 \\ A_2^2 \end{bmatrix} \neq 0$ then premultiplying the equation (11a) by the inverse matrix $\begin{bmatrix} E^1 \\ A_2^2 \end{bmatrix}^{-1}$ we obtain

$$\begin{aligned}\dot{x}(t,i+1) &= A_{00}x(t,i) + A_{01}x(t,i+1) + A_{10}\dot{x}(t,i) \\ &+ A_{20}\ddot{x}(t,i) + B_{00}u(t,i) + B_{10}\dot{u}(t,i)\end{aligned} \tag{15a}$$

where

$$\begin{aligned} A_{00} &= \begin{bmatrix} E^1 \\ A_2^2 \end{bmatrix}^{-1}\begin{bmatrix} A_0^1 \\ 0 \end{bmatrix},\ A_{10} = \begin{bmatrix} E^1 \\ A_2^2 \end{bmatrix}^{-1}\begin{bmatrix} A_1^1 \\ -A_0^2 \end{bmatrix}, \\ A_{01} &= \begin{bmatrix} E^1 \\ A_2^2 \end{bmatrix}^{-1}\begin{bmatrix} A_2^1 \\ 0 \end{bmatrix},\ A_{20} = \begin{bmatrix} E^1 \\ A_2^2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ -A_1^2 \end{bmatrix}, \\ B_{00} &= \begin{bmatrix} E^1 \\ A_2^2 \end{bmatrix}^{-1}\begin{bmatrix} B^1 \\ 0 \end{bmatrix},\ B_{10} = \begin{bmatrix} E^1 \\ A_2^2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ -B^2 \end{bmatrix}. \end{aligned} \tag{15b}$$

If $\det\begin{bmatrix} E^1 \\ A_0^2 \end{bmatrix} \neq 0$ then premultiplying the equation (13a) by the inverse matrix $\begin{bmatrix} E^1 \\ A_0^2 \end{bmatrix}^{-1}$ we obtain

$$\begin{aligned}\dot{x}(t,i+1) &= A_{00}x(t,i) + A_{01}x(t,i+1) + A_{10}\dot{x}(t,i) \\ &+ A_{12}\dot{x}(t,i+2) + A_{21}\ddot{x}(t,i+1) + B_{00}u(t,i) + B_{11}\dot{u}(t,i+1)\end{aligned} \tag{16a}$$

where

$$\begin{aligned} A_{00} &= \begin{bmatrix} E^1 \\ A_0^2 \end{bmatrix}^{-1}\begin{bmatrix} A_0^1 \\ 0 \end{bmatrix},\ A_{10} = \begin{bmatrix} E^1 \\ A_0^2 \end{bmatrix}^{-1}\begin{bmatrix} A_1^1 \\ 0 \end{bmatrix}, \\ A_{01} &= \begin{bmatrix} E^1 \\ A_0^2 \end{bmatrix}^{-1}\begin{bmatrix} A_2^1 \\ 0 \end{bmatrix},\ A_{21} = \begin{bmatrix} E^1 \\ A_0^2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ -A_1^2 \end{bmatrix}, \\ A_{12} &= \begin{bmatrix} E^1 \\ A_0^2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ -A_2^2 \end{bmatrix},\ B_{00} = \begin{bmatrix} E^1 \\ A_0^2 \end{bmatrix}^{-1}\begin{bmatrix} B^1 \\ 0 \end{bmatrix}, \\ B_{11} &= \begin{bmatrix} E^1 \\ A_0^2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ -B^2 \end{bmatrix}. \end{aligned} \tag{16b}$$

In Case 1 if $\det\begin{bmatrix} E^1 \\ A_1^2 \end{bmatrix} = 0$ then performing elementary row operations on the array (9b) we obtain

$$\begin{matrix} E^2 & A_0^3 & A_1^3 & A_2^3 & A_{02}^3 & B_{00}^3 & B_{01}^3 \\ 0 & A_0^4 & A_1^4 & A_2^4 & A_{02}^4 & B_{00}^4 & B_{01}^4 \end{matrix} \tag{17}$$

where $\operatorname{rank} E^2 \geq \operatorname{rank} E^1$. Let $\det\begin{bmatrix} E^2 \\ A_1^4 \end{bmatrix} \neq 0$ then performing on (17) the shuffle we obtain

$$\begin{aligned}\begin{bmatrix} E^1 \\ A_1^4 \end{bmatrix}\dot{x}(t,i+1) &= \begin{bmatrix} A_0^3 \\ 0 \end{bmatrix}x(t,i) + \begin{bmatrix} A_1^3 \\ 0 \end{bmatrix}\dot{x}(t,i) + \begin{bmatrix} A_2^3 \\ -A_0^4 \end{bmatrix}x(t,i+1) \\ &+ \begin{bmatrix} A_{02}^3 \\ -A_2^4 \end{bmatrix}x(t,i+2) + \begin{bmatrix} 0 \\ -A_{02}^4 \end{bmatrix}x(t,i+3) + \begin{bmatrix} B_{00}^3 \\ 0 \end{bmatrix}u(t,i) \\ &+ \begin{bmatrix} B_{01}^3 \\ -B_{00}^4 \end{bmatrix}u(t,i+1) + \begin{bmatrix} 0 \\ -B_{01}^4 \end{bmatrix}u(t,i+2)\end{aligned} \tag{18}$$

and after premultiplication of (18) by the inverse matrix $\begin{bmatrix} E^2 \\ A_1^4 \end{bmatrix}^{-1}$ we obtain

$$\begin{aligned}\dot{x}(t,i+1) &= A_{00}x(t,i) + A_{10}\dot{x}(t,i) + A_{01}x(t,i+1) \\ &+ A_{02}x(t,i+2) + A_{03}x(t,i+3) + B_{00}u(t,i) \\ &+ B_{01}u(t,i+1) + B_{02}u(t,i+2)\end{aligned} \tag{19a}$$

where

$$\begin{aligned} A_{00} &= \begin{bmatrix} E^1 \\ A_1^4 \end{bmatrix}^{-1}\begin{bmatrix} A_0^3 \\ 0 \end{bmatrix},\ A_{10} = \begin{bmatrix} E^1 \\ A_1^4 \end{bmatrix}^{-1}\begin{bmatrix} A_1^3 \\ 0 \end{bmatrix}, \\ A_{01} &= \begin{bmatrix} E^1 \\ A_1^4 \end{bmatrix}^{-1}\begin{bmatrix} A_2^3 \\ -A_0^4 \end{bmatrix},\ A_{02} = \begin{bmatrix} E^1 \\ A_1^4 \end{bmatrix}^{-1}\begin{bmatrix} A_{02}^3 \\ -A_2^4 \end{bmatrix}, \\ A_{03} &= \begin{bmatrix} E^1 \\ A_1^4 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ -A_{02}^4 \end{bmatrix},\ B_{00} = \begin{bmatrix} E^1 \\ A_1^4 \end{bmatrix}^{-1}\begin{bmatrix} B_{00}^2 \\ 0 \end{bmatrix}, \\ B_{01} &= \begin{bmatrix} E^1 \\ A_1^4 \end{bmatrix}^{-1}\begin{bmatrix} B_{01}^3 \\ -B_{00}^4 \end{bmatrix},\ B_{02} = \begin{bmatrix} E^1 \\ A_1^4 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ -B_{01}^4 \end{bmatrix}. \end{aligned} \tag{19b}$$

In Case 2 and Case 3 we proceed in a similar way. If $\det\begin{bmatrix} E^2 \\ A_1^4 \end{bmatrix} = 0$ then we repeat the previous step of the procedure. After finite number of steps l we obtain nonsingular matrix $\begin{bmatrix} E^l \\ A_k^v \end{bmatrix}$, $k \in \{0,1,2\}$ and the desired equation (3) since the elementary row operations and shuffles do not change the rank of the matrix $[Esz - A_0 - A_1 s - A_2 z]$ which by assumption (2) is nonsingular [17, 18, 11]. This completes the proof. □

Example 1. Consider the descriptor system (1) with the matrices

$$\begin{aligned} E &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A_0 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ A_1 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}. \end{aligned} \tag{20}$$

The matrices (20) satisfy the condition

$$\begin{aligned}\det[Esz - A_0 - A_1 s - A_2 z] &= \begin{vmatrix} sz & -s-1 & 0 \\ -1 & -z & 0 \\ -s-z & 0 & -1 \end{vmatrix} \\ &= sz^2 + s + 1 \neq 0\end{aligned} \tag{21}$$

for some $s, z \in C$. Using the shuffle algorithm reduce the system (1) with (20) to the standard 2D continuous-discrete system

$$\dot{x}(t,i+1) = A_{00}x(t,i) + A_{01}x(t,i+1) + A_{10}\dot{x}(t,i) + A_{21}\ddot{x}(t,i+1) + B_{10}\dot{u}(t,i) + B_{11}\dot{u}(t,i+1) \tag{22a}$$

where

$$A_{00} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A_{10} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ A_{01} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix},\ A_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},\ B_{10} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix},\ B_{11} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{22b}$$

Using (20) and (4) we obtain the array

$$\begin{array}{ccccc} E & A_0 & A_1 & A_2 & B \end{array} = \begin{array}{ccccc} E^1 & A^1_{00} & A^1_{10} & A^1_{01} & B^1_{00} \\ 0 & A^2_{00} & A^2_{10} & A^2_{01} & B^2_{00} \end{array}$$

$$\begin{array}{ccccccccccccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \end{array} \tag{23}$$

with $\operatorname{rank} E^1 = 1$. From (23) we have

$$E^1\dot{x}(t,i+1) = A^1_{00}x(t,i) + A^1_{10}\dot{x}(t,i) + A^1_{01}x(t,i+1) + B^1_{00}u(t,i) \tag{24a}$$

$$0 = A^2_{00}x(t,i) + A^2_{10}\dot{x}(t,i) + A^2_{01}x(t,i+1) + B^2_{00}u(t,i) \tag{24b}$$

where

$$E = [1\ \ 0\ \ 0],\ A^1_{00} = [0\ \ 1\ \ 0],\ A^1_{10} = [0\ \ 1\ \ 0],\ A^1_{01} = [0\ \ 0\ \ 0],\ B^1_{00} = [0],$$
$$A^2_{00} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ A^2_{10} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ A^2_{01} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ B^2_{00} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{24c}$$

Differentiation of (24b) with respect to time t yields

$$0 = A^2_{00}\dot{x}(t,i) + A^2_{10}\ddot{x}(t,i) + A^2_{01}\dot{x}(t,i+1) + B^2_{00}\dot{u}(t,i). \tag{25}$$

The equations (24a) and (25) can be written in the form

$$\begin{bmatrix} E^1 \\ A^2_{01} \end{bmatrix}\dot{x}(t,i+1) = \begin{bmatrix} A^1_{00} \\ 0 \end{bmatrix}x(t,i) + \begin{bmatrix} A^1_{10} \\ -A^2_{00} \end{bmatrix}\dot{x}(t,i) + \begin{bmatrix} A^1_{01} \\ 0 \end{bmatrix}x(t,i+1) + \begin{bmatrix} 0 \\ -A^2_{10} \end{bmatrix}\ddot{x}(t,i) + \begin{bmatrix} B^1_{00} \\ 0 \end{bmatrix}u(t,i) + \begin{bmatrix} 0 \\ -B^2_{00} \end{bmatrix}\dot{u}(t,i). \tag{26}$$

The array

$$\begin{array}{ccccccc} E^1 & A^1_{00} & A^1_{10} & A^1_{01} & 0 & B^1_{00} & 0 \\ A^2_{01} & 0 & -A^2_{00} & 0 & -A^2_{10} & 0 & -B^2_{00} \end{array}$$

$$= \begin{array}{ccccccccccccccccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{array} \tag{27}$$

can be obtained from (23) by the shuffle of $-A^2_{01}$.

Remark 2. To simplify the notation in the array (27) the zero matrices $\begin{bmatrix} A^1_{01} \\ 0 \end{bmatrix}$, $\begin{bmatrix} B^1_{00} \\ 0 \end{bmatrix}$ will be omitted.

Performing the elementary row operation $L[3 + 1 \times (-1)]$ on (27) we obtain

$$\begin{array}{ccccc} E^2 & A^3_{00} & A^3_{10} & A^3_{20} & B^3_{10} \\ 0 & A^4_{00} & A^4_{10} & A^4_{20} & B^4_{10} \end{array}$$

$$= \begin{array}{ccccccccccccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 1 \end{array} \tag{28}$$

Performing the shuffle of $-A^4_{10}$ on (28) we get

$$\begin{array}{cccccccc} E^2 & A^3_{00} & A^3_{10} & A^3_{01} & A^3_{20} & A^3_{21} & B^3_{10} & B^3_{11} \\ A^4_{10} & 0 & 0 & -A^4_{00} & 0 & -A^4_{20} & 0 & -B^4_{10} \end{array}$$

$$= \begin{array}{cccccccccccccccccccccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{array} \tag{29}$$

and after performing the elementary row operations $L[3 + 2 \times 1]$, $L[3 \times (-1)]$ and omitting the zero matrix $\begin{bmatrix} A^3_{20} \\ 0 \end{bmatrix}$ we have

$$\begin{array}{ccccccc} I_3 & A_{00} & A_{10} & A_{01} & A_{21} & B_{10} & B_{11} \end{array}$$

$$= \begin{array}{ccccccccccccccccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 1 \end{array} \tag{30}$$

and the desired matrices (22b).

3. Polynomial matrix method

In this section it will be shown that the reduction of the descriptor system (1) to the standard system (3) is equivalent to the transformation of the polynomial matrix

$$[Esz - A_0 - A_1 s - A_2 z \quad B] \tag{31}$$

(corresponding to the system (1)) to the polynomial matrix

$$\begin{aligned} [I_n sz &- A_{00} - A_{01}z - \ldots - A_{0,p_0}z^{p_0} - A_{10}s - A_{12}sz^2 - \ldots \\ &- A_{1,p_1}sz^{p_1} - A_{q,0}s^q - A_{q,1}s^q z - \ldots - A_{q,p_q}s^q z^{p_q} \\ B_{00} &+ B_{01}z + \ldots + B_{0,p_0}z^{p_0} + B_{10}s + B_{11}sz + \ldots + B_{1,p_1}sz^{p_1} \\ &+ \ldots + B_{q,0}s^q + B_{q,1}s^q z + \ldots + B_{q,p_q}s^q z^{p_q}] \end{aligned} \tag{32}$$

(corresponding to the system (3)) by the use of elementary row operations. Performing elementary row operations on the matrix (31) such that E^1 has full row rank we obtain

$$\begin{bmatrix} E^1 sz - A_0^1 - A_1^1 s - A_2^1 z & B^1 \\ -A_0^2 - A_1^2 s - A_2^2 z & B^2 \end{bmatrix}. \quad (33)$$

The execution of the shuffle A_1^2 is equivalent to the multiplication of the second block matrix of (33) by $-z$

$$\begin{bmatrix} E^1 sz - A_0^1 - A_1^1 s - A_2^1 z & B^1 \\ A_0^2 z + A_1^2 sz + A_2^2 z^2 & -B^2 z \end{bmatrix}. \quad (34a)$$

The execution of the shuffle A_2^2 is equivalent to the multiplication of the second block matrix of (33) by $-s$

$$\begin{bmatrix} E^1 sz - A_0^1 - A_1^1 s - A_2^1 z & B^1 \\ A_0^2 s + A_1^2 s^2 + A_2^2 sz & -B^2 s \end{bmatrix}. \quad (34b)$$

The execution of the shuffle A_0^2 is equivalent to the multiplication of the second block matrix of (33) by $-sz$

$$\begin{bmatrix} E^1 sz - A_0^1 - A_1^1 s - A_2^1 z & B^1 \\ A_0^2 sz + A_1^2 s^2 z + A_2^2 sz^2 & -B^2 sz \end{bmatrix}. \quad (34c)$$

If $\det\begin{bmatrix} E^1 \\ A_1^2 \end{bmatrix} \neq 0$ then performing elementary row operations on (34a) we obtain

$$[I_n sz - A_{00} - A_{10}s - A_{01}z - A_{02}z^2 \quad B_{00} + B_{01}z]. \quad (35a)$$

If $\det\begin{bmatrix} E^1 \\ A_2^2 \end{bmatrix} \neq 0$ then performing elementary row operations on (34b) we obtain

$$[I_n sz - A_{00} - A_{10}s - A_{20}s^2 - A_{01}z \quad B_{00} + B_{01}z]. \quad (35b)$$

If $\det\begin{bmatrix} E^1 \\ A_0^2 \end{bmatrix} \neq 0$ then performing elementary row operations on (34c) we obtain

$$[I_n sz - A_{00} - A_{10}s - A_{01}z - A_{12}sz^2 - A_{21}s^2 z \quad B_{00} + B_{11}sz]. \quad (35c)$$

If $\det\begin{bmatrix} E^1 \\ A_1^2 \end{bmatrix} = 0$ then performing elementary row operations on the matrix (34a) such that E^2 has full row rank we obtain

$$\begin{bmatrix} E^2 sz - A_0^3 - A_1^3 s - A_2^3 z - A_{02}^3 z^2 & B_{00}^3 + B_{01}^3 z \\ -A_0^4 - A_1^4 s - A_2^4 z - A_{02}^4 z^2 & B_{00}^4 + B_{01}^4 z \end{bmatrix}. \quad (36)$$

In a similar way we precede if $\det\begin{bmatrix} E^1 \\ A_0^2 \end{bmatrix} = 0$ or $\det\begin{bmatrix} E^1 \\ A_2^2 \end{bmatrix} = 0$.

Continuing this procedure after finite number of steps we obtain the desired polynomial matrix (32). The details of the method will be demonstrated on the following numerical example.

Example 2. Using the polynomial matrix method reduce the descriptor system (1) with the matrices (20) to the standard system (22a) with the matrices (22b). In this case the polynomial matrix (31) corresponding to (20) has the form

$$[Esz - A_0 - A_1 s - A_2 z \quad B] = \begin{bmatrix} sz & -s-1 & 0 & 0 \\ -1 & -z & 0 & 1 \\ -s-z & 0 & -1 & -1 \end{bmatrix}. \quad (37)$$

Differentiation of (24b) with respect to time and the execution of the shuffle $-A_{01}^2$ is equivalent to multiplication of the second and third rows of (37) by $-s$. Therefore

$$\begin{bmatrix} E^1 sz - A_{00}^1 - A_{10}^1 s - A_{01}^1 z & B_{00}^1 \\ A_{00}^2 s + A_{10}^2 s^2 + A_{01}^2 sz & -B_{00}^2 s \end{bmatrix} = \begin{bmatrix} sz & -s-1 & 0 & 0 \\ s & sz & 0 & -s \\ sz + s^2 & 0 & s & s \end{bmatrix}. \quad (38a)$$

Performing the elementary row operation $L[3+1\times(-1)]$ on (38a) we obtain

$$\begin{bmatrix} E^2 sz - A_{00}^3 - A_{10}^3 s - A_{20}^3 s^2 & B_{10}^3 s \\ -A_{00}^4 - A_{10}^4 s - A_{20}^4 s^2 & -B_{10}^4 s \end{bmatrix} = \begin{bmatrix} sz & -s-1 & 0 & 0 \\ s & sz & 0 & -s \\ s^2 & 1+s & s & s \end{bmatrix}. \quad (38b)$$

The execution of the shuffle A_{10}^4 is equivalent to multiplication of the third row of (38b) by $-z$

$$\begin{bmatrix} E^2 sz - A_{00}^3 - A_{10}^3 s - A_{20}^3 s^2 & B_{10}^3 s \\ A_{00}^4 z + A_{10}^4 sz + A_{20}^4 s^2 z & B_{10}^4 sz \end{bmatrix} = \begin{bmatrix} sz & -s-1 & 0 & 0 \\ s & sz & 0 & -s \\ -s^2 z & -sz-z & -sz & -sz \end{bmatrix} \quad (39a)$$

and after performing the elementary row operations $L[3+2\times 1]$, $L[3\times(-1)]$ we obtain the desired result

$$[I_3 sz - A_{00} - A_{10}s - A_{01}z - A_{21}s^2 z \quad B_{10}s + B_{11}sz] = \begin{bmatrix} sz & -s-1 & 0 & 0 \\ s & sz & 0 & s \\ -s+s^2 z & z & sz & s+sz \end{bmatrix}. \quad (39b)$$

The polynomial matrix (39b) is equivalent to (30).

4. Reduction of the standard system to equivalent descriptor system by polynomial matrix method

By inverting the procedure of polynomial matrix method it is possible to transform the standard system (3) to the descriptor system (1). In the inverted procedure:
elementary row operation $L[i+j\times c]$ should be replaced by $L[i+j\times(-c)]$,
multiplication by $-s$ (shuffle of A_1^2) should be replaced by the multiplication by $-s^{-1}$,
multiplication by $-z$ (shuffle of A_2^2) should be replaced by the multiplication by $-z^{-1}$,
multiplication by $-sz$ (shuffle of A_0^2) should be replaced by the multiplication by $-s^{-1}z^{-1}$.
The details of the procedure will be demonstrated on the following numerical example.

Example 3. Consider the standard 2D continuous-discrete linear system (22a) with the matrices (22b). Transform the standard system to the descriptor system (1) with matrices (20).
Using the matrices (22b) we obtain

$$[I_3sz - A_{00} - A_{10}s - A_{01}z - A_{21}s^2z \quad B_{10}s + B_{11}sz] = \begin{bmatrix} sz & -s-1 & 0 & 0 \\ s & sz & 0 & -s \\ -s+s^2z & z & sz & s+sz \end{bmatrix}. \tag{40}$$

Performing the elementary row operations $L[3\times(-1)]$ and $L[3+2\times(-1)]$ on (40) we obtain

$$\begin{bmatrix} sz & -s-1 & 0 & 0 \\ s & sz & 0 & -s \\ -s^2z & -z-sz & -sz & -sz \end{bmatrix}. \tag{41}$$

Multiplication of the third row of (41) by $-z^{-1}$ yields

$$\begin{bmatrix} sz & -s-1 & 0 & 0 \\ s & sz & 0 & -s \\ s^2 & 1+s & s & s \end{bmatrix} \tag{42}$$

and after performing the elementary row operation $L[3+1\times 1]$ on (42) we obtain

$$\begin{bmatrix} sz & -s-1 & 0 & 0 \\ s & sz & 0 & -s \\ sz+s^2 & 0 & s & s \end{bmatrix}. \tag{43}$$

Multiplication of the second and third rows of (43) by $-s^{-1}$ we obtain

$$\begin{bmatrix} sz & -s-1 & 0 & 0 \\ -1 & -z & 0 & 1 \\ -s-z & 0 & -1 & -1 \end{bmatrix} = [Esz - A_0 - A_1s - A_2z \quad B] \tag{44}$$

and the desired matrices (20).

5. Concluding remarks

A method of reduction of descriptor 2D continuous-discrete linear systems with regular pencil to equivalent standard linear systems has been proposed. The method is based on the application of the shuffle algorithm to the descriptor system. A polynomial matrix version of the reduction method has been also proposed. The inverse problem of reduction of standard systems by the use of the inverse polynomial matrix method has been also considered. The proposed methods have been illustrated by numerical examples. The proposed method can be extended to descriptor linear systems with singular pencils and to fractional descriptor 2D continuous-discrete linear systems

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