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A rationalized algorithm for complex-valued inner product calculation

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Abstract

This paper presents a rationalized algorithm for calculating a complex-valued inner product. The main idea of algorithm synthesis uses the well-known opportunity to calculate the product of two complex numbers with three multiplications and five additions of real numbers. Thus, the proposed algorithmic solution reduces the number of real multiplications and additions compared to the schoolbook implementation, and takes advantage of parallelization of calculation offered by field-programmable gate arrays (FPGAs).

Keywords: complex-valued inner product, arithmetic complexity reduction.

Zracjonalizowany algorytm wyznaczania zespolonego iloczynu skalarnego

Streszczenie

W artykule został przedstawiony równoległy algorytm wyznaczania iloczynu skalarnego dwóch wektorów, których elementami są liczbami zespolonymi. Proponowany algorytm wyróżnia się w stosunku do całkowicie równoległej implementacji metody naiwnej zredukowaną złożonością multiplikatywną. Jeśli metoda naiwna wymaga wykonania 4*N* mnożeń (układów mnożących podczas implementacji sprzętowej) oraz 2(2*N*-1) dodawań (sumatorów) liczb rzeczywistych to proponowany algorytm wymaga tylko 3*N* mnożeń oraz 6*N*-1 dodawań. W pracy została przedstawiona zracjonalizowana wektorowo-macierzowa procedura obliczeniowa wyznaczania takich iloczynów a także zdefiniowane konstrukcje macierzowe, wchodzące w skład owej procedury. Przy implementacji sprzętowej proponowany algorytm posiada niewątpliwe walory w stosunku do implementacji naiwnego sposobu zrównoleglenia obliczeń wymagającego więcej bloków mnożących. A ponieważ blok mnożący pochłania znacznie więcej zasobów sprzętowych platformy implementacyjnej niż sumator, to redukcja liczby tych bloków przy projektowaniu jednostek obliczeniowych jest sprawą niezwykle aktualną. W przypadku implementacji jednostki do obliczania iloczynu skalarnego w strukturze FPGA proponowane rozwiązanie pozwala zaoszczędzić pewną część umieszczonej w układzie puli bloków mnożących lub też elementów logicznych.

Słowa kluczowe: zespolony iloczyn skalarny, redukcja złożoności obliczeniowej.

1. Introduction

Let $\mathbf{X}_{N\times 1} = [x_0, x_1, ..., x_{N-1}]^{\mathrm{T}}$ and $\mathbf{Y}_{N\times 1} = [y_0, y_1, ..., y_{N-1}]$ be two N-element complex-valued vectors, where $x_i = a_i + jb_i$ and $y_i = c_i + jd_i$. Then the inner product of $\mathbf{X}_{N\times 1}$ and $\mathbf{Y}_{N\times 1}$ is a complex number $z = e + jf$ defined as:

$$z = \sum_{i=0}^{N-1} x_i y_i \,, \; i = 0,1,...,N-1 \qquad (1)$$

In this expression a_i, b_i, c_i, d_i, e and f are real numbers and j is the imaginary unit, satisfying $j^2 = -1$.

It is well known that complex multiplication requires four real multiplications and two real additions, because:

$$(a + jb)(c + jd) = ac - bd + j(ad + bc) \,.$$

Consequently, we can observe that the schoolbook computation of (1) requires N complex multiplications ($4N$ real multiplications) and $N-1$ complex additions ($2(2N-1)$ real additions). However, it is possible to perform the complex multiplication with three multiplication and five additions, because [1, 2]:

$$(a + jb)(c + jd) = ac - bd + j[(a+b)(c+d) - ac - bd] \qquad (2)$$

Taking into account this fact, expression (1) can be calculated in a creative manner using $3N$ multiplications and $7N-2$ additions of real numbers. The proposed algorithm has the same number of real multiplications. The number of real additions, however, is less. Let us consider the synthesis algorithm in detail.

2. Development of algorithm

First, we represent the vector $\mathbf{X}_{N\times 1} = [x_0, x_1, ..., x_{N-1}]^{\mathrm{T}}$ in the form:

$$\mathbf{X}_{2N\times 1} = [a_0, b_0, a_1, b_1, ..., a_{N-1}, b_{N-1}]^{\mathrm{T}}$$

and vector $\mathbf{Y}_{N\times 1} = [y_0, y_1, ..., y_{N-1}]$ in the form:

$$\mathbf{Y}_{2N\times 1} = [c_0, d_0, c_1, d_1, ..., c_{N-1}, d_{N-1}]^{\mathrm{T}} \,.$$

Next, we introduce some auxiliary matrices:

$$\mathbf{A}^{(0)}_{3N\times 2N} = \left[\begin{array}{c} \mathbf{I}_{2N} \\ \hdashline \mathbf{C}_{N\times 2N} \end{array} \right],$$

where

$$\mathbf{C}_{N\times 2N} = \mathbf{I}_N \otimes \bar{\mathbf{1}}_{1\times 2} \,, \; \bar{\mathbf{1}}_{1\times 2} = [-1 \mid 1],$$

$$\mathbf{A}^{(1)}_{3N\times 2N} = (\mathbf{1}_{1\times N} \otimes \mathbf{I}_2) \oplus \mathbf{1}_{1\times N} \,, \; \mathbf{A}^{(2)}_{2\times 3} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

$\mathbf{I}_N$ - is an identity $N \times N$ matrix and signs „$\otimes$” and „$\oplus$” denote the tensor product and direct sum of two matrices, respectively [3].

Using the above matrices the rationalized computational procedure for calculating the scalar product can be written as follows:

$$\mathbf{Z}_{2\times1} = \mathbf{A}^{(2)}_{2\times3}\mathbf{A}^{(1)}_{3\times3N}\mathbf{D}_{3N}\mathbf{A}^{(0)}_{3N\times2N}\mathbf{X}_{2N\times1} \tag{3}$$

where $\mathbf{Z}_{2\times1} = [e \mid f]^{\mathrm{T}}$, and $\mathbf{D}_{3N} = diag(s_0, s_1, ..., s_{3N-1})$ - is a diagonal matrix, with elements $\{s_k\}$, $(k = 0,1,...,3N-1)$ defined as components of vector $\mathbf{S}_{3N\times1} = [s_0, s_1, ..., s_{3N-1}]^{\mathrm{T}}$.

In turn, the vector $\mathbf{S}_{3N\times1}$ can be calculated using the following procedure:

$$\mathbf{S}_{3N\times1} = \mathbf{P}_{3N}\widetilde{\mathbf{H}}_{3N}\mathbf{P}_{3N\times2N}\mathbf{Y}_{2N\times1},$$

where

$$\widetilde{\mathbf{H}}_{3N} = (\mathbf{I}_N \otimes \mathbf{H}_2) \oplus \mathbf{I}_N, \quad \mathbf{P}_{3N} = (\mathbf{I}_N \otimes \mathbf{J}_2) \oplus \mathbf{I}_N,$$

$$\mathbf{P}_{3N\times2N} = \left[\begin{array}{c} \mathbf{I}_{2N} \\ \hdashline \overline{\mathbf{P}}_{N\times2N} \end{array}\right], \quad \overline{\mathbf{P}}_{N\times2N} = \mathop{\blacksquare}_{\alpha=0}^{N-1} \mathbf{e}^{(2\alpha+1)}_{1\times2N},$$

$\mathbf{e}^{(2\alpha+1)}_{1\times2N}$ - is a $(2\alpha+1)^{th}$ row of the $2N$-th order identity matrix, $\mathbf{J}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ - is a 2 by 2 exchange matrix [2], and $\blacksquare$ denotes horizontal concatenation sign [4].

As an example, consider the case of calculating the inner product of complex-valued vectors of length 4.

$$\mathbf{Z}_{2\times1} = \mathbf{A}^{(2)}_{2\times3}\mathbf{A}^{(1)}_{3\times12}\mathbf{D}_{12}\mathbf{A}^{(0)}_{12\times8}\mathbf{X}_{8\times1} \tag{4}$$

$$\mathbf{D}_{12} = diag(s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}),$$

$$\mathbf{S}_{12\times1} = \mathbf{P}_{12}\widetilde{\mathbf{H}}_{12}\mathbf{P}_{12\times8}\mathbf{Y}_{8\times1} \tag{5}$$

$$\mathbf{X}_{4\times1} = [x_0, x_1, x_2, x_3]^{\mathrm{T}},$$

$$\mathbf{X}_{8\times1} = [a_0, b_0, a_1, b_1, a_2, b_2, a_3, b_3]^{\mathrm{T}},$$

$$\mathbf{Y}_{8\times1} = [c_0, d_0, c_1, d_1, c_2, d_2, c_3, d_3]^{\mathrm{T}},$$

$$\mathbf{S}_{12\times1} = [s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}]^{\mathrm{T}},$$

$$\mathbf{A}^{(0)}_{12\times8} = \left[\begin{array}{cccc:cccc} 1 &&&&&&& \\ & 1 &&&& \mathbf{0}_4 && \\ && 1 &&&&& \\ &&& 1 &&&& \\ \hdashline &&&& 1 &&& \\ & \mathbf{0}_4 &&&& 1 && \\ &&&&&& 1 & \\ &&&&&&& 1 \\ \hdashline 1 & -1 & \multicolumn{2}{:c:}{\mathbf{0}_{1\times2}} & \multicolumn{4}{c}{\multirow{2}{*}{$\mathbf{0}_{2\times4}$}} \\ \multicolumn{2}{c:}{\mathbf{0}_{1\times2}} & 1 & -1 &&&& \\ \hdashline \multicolumn{4}{c:}{\multirow{2}{*}{$\mathbf{0}_{2\times4}$}} & 1 & -1 & \multicolumn{2}{:c}{\mathbf{0}_{1\times2}} \\ &&&& \multicolumn{2}{c:}{\mathbf{0}_{1\times2}} & 1 & -1 \end{array}\right],$$

$$\mathbf{P}_{12\times8} = \left[\begin{array}{cccc:cccc} 1 &&&&&&& \\ & 1 &&&& \mathbf{0}_4 && \\ && 1 &&&&& \\ &&& 1 &&&& \\ \hdashline &&&& 1 &&& \\ & \mathbf{0}_4 &&&& 1 && \\ &&&&&& 1 & \\ &&&&&&& 1 \\ \hdashline 0 & 1 & \multicolumn{2}{:c:}{\mathbf{0}_{1\times2}} & \multicolumn{4}{c}{\multirow{2}{*}{$\mathbf{0}_{2\times4}$}} \\ \multicolumn{2}{c:}{\mathbf{0}_{1\times2}} & 0 & 1 &&&& \\ \hdashline \multicolumn{4}{c:}{\multirow{2}{*}{$\mathbf{0}_{2\times4}$}} & 0 & 1 & \multicolumn{2}{:c}{\mathbf{0}_{1\times2}} \\ &&&& \multicolumn{2}{c:}{\mathbf{0}_{1\times2}} & 0 & 1 \end{array}\right],$$

$$\mathbf{A}^{(1)}_{3\times12} = \left[\begin{array}{cc:cc:cc:cc:cccc} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \multicolumn{4}{c}{\multirow{2}{*}{$\mathbf{0}_{2\times4}$}} \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 &&&& \\ \hdashline \multicolumn{8}{c:}{\mathbf{0}_{1\times8}} & 1 & 1 & 1 & 1 \end{array}\right],$$

$$\widetilde{\mathbf{H}}_{12} = \left[\begin{array}{cc:cc:cc:cc:cccc} 1 & 1 & \multicolumn{2}{c:}{\multirow{2}{*}{$\mathbf{0}_2$}} & \multicolumn{4}{c:}{\multirow{4}{*}{$\mathbf{0}_4$}} & \multicolumn{4}{c}{\multirow{8}{*}{$\mathbf{0}_{8\times4}$}} \\ 1 & -1 &&&&&&&&&& \\ \hdashline \multicolumn{2}{c:}{\multirow{2}{*}{$\mathbf{0}_2$}} & 1 & 1 &&&&&&&& \\ && 1 & -1 &&&&&&&& \\ \hdashline \multicolumn{4}{c:}{\multirow{4}{*}{$\mathbf{0}_4$}} & 1 & 1 & \multicolumn{2}{c:}{\multirow{2}{*}{$\mathbf{0}_2$}} &&&& \\ &&&& 1 & -1 &&&&&& \\ \hdashline &&&& \multicolumn{2}{c:}{\multirow{2}{*}{$\mathbf{0}_2$}} & 1 & 1 &&&& \\ &&&&&& 1 & -1 &&&& \\ \hdashline \multicolumn{8}{c:}{\multirow{4}{*}{$\mathbf{0}_{4\times8}$}} & 1 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathbf{0}_2$}} \\ &&&&&&&& 0 & 1 && \\ &&&&&&&& \multicolumn{2}{c}{\multirow{2}{*}{$\mathbf{0}_2$}} & 1 & 0 \\ &&&&&&&&&& 0 & 1 \end{array}\right],$$

$$\mathbf{P}_{12} = \left[\begin{array}{cc:cc:cc:cc:cccc} 0 & 1 & \multicolumn{2}{c:}{\multirow{2}{*}{$\mathbf{0}_2$}} & \multicolumn{4}{c:}{\multirow{4}{*}{$\mathbf{0}_4$}} & \multicolumn{4}{c}{\multirow{8}{*}{$\mathbf{0}_{8\times4}$}} \\ 1 & 0 &&&&&&&&&& \\ \hdashline \multicolumn{2}{c:}{\multirow{2}{*}{$\mathbf{0}_2$}} & 0 & 1 &&&&&&&& \\ && 1 & 0 &&&&&&&& \\ \hdashline \multicolumn{4}{c:}{\multirow{4}{*}{$\mathbf{0}_4$}} & 0 & 1 & \multicolumn{2}{c:}{\multirow{2}{*}{$\mathbf{0}_2$}} &&&& \\ &&&& 1 & 0 &&&&&& \\ \hdashline &&&& \multicolumn{2}{c:}{\multirow{2}{*}{$\mathbf{0}_2$}} & 0 & 1 &&&& \\ &&&&&& 1 & 0 &&&& \\ \hdashline \multicolumn{8}{c:}{\multirow{4}{*}{$\mathbf{0}_{4\times8}$}} & 1 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathbf{0}_2$}} \\ &&&&&&&& 0 & 1 && \\ &&&&&&&& \multicolumn{2}{c}{\multirow{2}{*}{$\mathbf{0}_2$}} & 1 & 0 \\ &&&&&&&&&& 0 & 1 \end{array}\right]$$

The graph-structural model for realization of the proposed algorithm is illustrated in Fig. 1. In turn, Fig. 2 shows a graph-structural model for computing elements of the matrix $\mathbf{D}_{3N}$ in accordance with the procedure (4). In this paper, the graph-structural models are oriented from left to right. Note that the circles in these figures show the operation of multiplication by a real number (variable) inscribed inside a circle. Straight lines in the figures denote the operation of data transfer. The data are summarized at points where lines converge (the dash-dotted lines indicate the subtraction operation). We use the usual lines without arrows specifically so as not to clutter the picture.

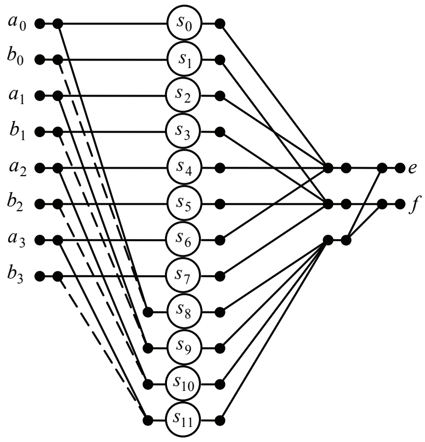

Fig. 1. Graph-structural model of complex-valued inner product computation according to procedure (4)

Rys. 1. Model grafostrukturalny opisujący organizację zracjonalizowanego procesu obliczenia iloczynu skalarnego zgodnie z proponowaną metodą według procedury (4)

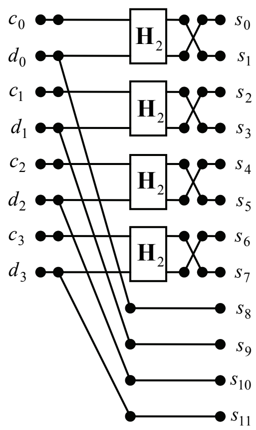

Fig. 2. The graph-structural model of computation process organization for vector $\mathbf{S}_{12\times1}$ element calculations according to (5).

Rys. 2. Grafostrukturalny model organizacji obliczeń elementów wektora $\mathbf{S}_{12\times1}$ zgodnie z procedurą (5)

3. Discussion of computational complexity

The described algorithm for computing the inner product of complex-valued vectors requires 3N multiplications and additions of 6N-1 of real numbers. Compared to the schoolbook way of computing, it gives 25% reduction in multiplicative complexity due to increase in the additive by 50%. Tab. 1 shows the number of multiplications and additions of real numbers for three different methods: the schoolbook method according to formula (1), the creative method according to formula (2), and the proposed algorithm according to procedure (3). As can be seen, the developed algorithm has the same multiplicative complexity as the creative method (2), but has a lower additive complexity. This means that the proposed algorithm has some advantages and deserves the right to exist.

4. Concluding remarks

The paper presents an affective algorithm for complex-valued inner product computation. The idea of constructing the algorithm uses the opportunity to calculate the product of complex numbers with three multiplications and five additions of real numbers. However, in comparison with the direct use of the summation of products of complex numbers calculated by the creative way in accordance with (2), the proposed algorithm has a lower additive complexity. This allows the effective use of parallelization of computational process of complex-valued matrix multiplication [5-7] and results in reduction of the computation time.

Tab. 1. Estimates for the computational complexity of the discussed methods for calculating the complex-valued inner product of two vectors

Tab. 1. Ocena złożoności obliczeniowej dla trzech omawianych sposobów wyznaczania iloczynu skalarnego wektorów, których elementy są liczbami zespolonymi

N	Multiplicative complexity			Additive complexity		
	(1)	(2)	(3)	(1)	(2)	(3)
2	8	6	6	6	12	11
3	12	9	9	10	19	17
4	16	12	12	14	26	23
5	20	15	15	18	33	29
6	24	18	18	22	40	35
7	28	21	21	26	47	41
8	32	24	24	30	54	47
9	36	27	27	34	61	53
10	40	30	30	38	68	59
11	44	33	33	42	75	65
12	48	36	36	46	82	71
13	52	39	39	50	89	77
14	56	42	42	54	96	83
15	60	45	45	58	103	89
16	64	48	48	62	110	95
17	68	51	51	66	117	101
18	72	54	54	70	124	107
19	76	57	57	74	131	113
20	80	60	60	78	138	119
21	84	63	63	82	145	125
22	88	66	66	86	152	131
23	92	69	69	90	159	137
24	96	72	72	94	166	143
25	100	75	75	98	173	149
26	104	78	78	102	180	155
27	108	82	82	104	187	161
27	112	85	85	108	194	167
29	116	88	88	112	201	173
30	120	92	92	116	208	179
31	124	95	95	120	215	185

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