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Positive realization of fractional continuous-time linear systems with delays

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Abstract

The positive realization problem for single-input single-output (SISO) fractional continuous-time linear systems with delays in state vector and input is formulated and a method for finding a positive realization of a given proper transfer function is proposed. Sufficient conditions for the existence of a positive realization of this class of linear systems are established. A procedure for computation of a positive realization is proposed and illustrated by a numerical example.

Keywords: fractional, positive, delay, realization, existence, computation.

Realizacje dodatnie ciągłych układów niecałkowitego rzędu z opóźnieniami

Streszczenie

Przedstawiono rozwiązanie zadania realizacji dodatniej dla ciągłych układów niecałkowitego rzędu z opóźnieniem w wektorze stanu i wymuszeniu o jednym wejściu i jednym wyjściu (SISO). Zaproponowano metodę wyznaczania realizacji dodatniej na podstawie znanej transmitancji operatorowej dla tej klasy układów. Podano warunki wystarczające istnienia realizacji dodatniej omawianej klasy układów. Przedstawiono procedurę obliczania realizacji dodatniej, która jest zilustrowana przykładem numerycznym. Praca jest rozszerzeniem wcześniejszych rozważań [8, 9] dla układów bez opóźnień. W rozdziale 2 zawarto podstawowe definicje i twierdzenia dotyczące rozwiązania układu i dodatniości rozpatrywanej klasy układów oraz sformułowanie zadania realizacji dodatniej. Rozdział 3 przedstawia rozwiązanie tego zadania wraz z warunkami wystarczającymi istnienia rozwiązania i przykładem. Podsumowanie jest w rozdziale 4.

Słowa kluczowe: niecałkowity rząd, dodatnie, opóźniania, realizacja istnienie, wyznaczanie.

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of art in positive systems theory is given in the monographs [2, 8]. The realization problem for positive discrete-time and continuous-time systems without and with delays was considered in [1, 2, 4, 7]. The first definition of the fractional derivative was introduced by Liouville and Riemann at the end of the 19th century [12, 13]. This idea has been used by engineers for modeling different processes [3, 10, 14]. Mathematical fundamentals of fractional calculus are given in the monographs [11-13]. The fractional order controllers have been developed in [14, 15]. A class of positive fractional discrete-time linear system with and without delays has been introduced in [5, 9]. The realization problem for positive fractional systems with and without delays was considered in [6, 9, 15, 16].

The main purpose of this paper is to present a method for computation of a positive realization of SISO fractional continuous-time linear systems with delays in state vector and input for given proper transfer function. Sufficient conditions for the existence of a positive realization of this class of systems will be established and a procedure for computation of a positive realization will be proposed.

The paper is organized as follows. In section 2 basic definition and theorem concerning positive fractional systems with delays are recalled. Also in this section using the Laplace transform the transfer matrix (function) of the fractional linear systems with delays is derived and the positive realization problem is formulated. Main result is given in section 3 where solution to the realization problem for given transfer function of the fractional continuous-time linear systems with delays is given. In the same section the sufficient conditions for the positive realization are derived and the procedure for computation of the positive realization is proposed. Concluding remarks are given in section 4.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n - the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n - the $n\times n$ identity matrix, $\mathcal{L}[f(t)]$ - Laplace transform of the continuous-time function $f(t)$.

2. Preliminaries and problem formulation

Consider a fractional continuous-time linear system with q delays in state vector and input described by the equations

$$ {}_0D_t^\alpha x(t) = \sum_{k=0}^{q} A_k x(t-kd) + \sum_{l=0}^{q} B_l u(t-ld), \quad 0<\alpha\le 1 \tag{1a}$$

$$ y(t) = Cx(t) + Du(t) \tag{1b}$$

where $x(t)\in\Re^N$ is state vector, $u(t)\in\Re^m$ is input vector, $y(t)\in\Re^p$ is output vector, and $A_k\in\Re^{N\times N}$, $B_l\in\Re^{N\times m}$, $k, l = 0,1,\dots q$; $C\in\Re^{p\times N}$, $D\in\Re^{p\times m}$, d is a delay.

The following Caputo definition of the fractional derivative will be used [11, 13]

$$ {}_0D_t^\alpha f(t) = \frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau \tag{2}$$

where $n-1<\alpha\le n\in N=\{1,2,...\}$ and $\alpha\in\Re$ is the order of fractional derivative, $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt$ is the gamma function and $f^{(n)}(\tau) = \dfrac{d^n f(\tau)}{d\tau_n}$.

Theorem 1. The solution of equation (1a) is given by

$$x(t)=\Phi_0(t)x_0+\int_0^t \Phi(t-\tau)\Phi_d(\tau-d)d\tau,\quad x(0)=x_0 \tag{3}$$

where

$$\Phi_0(t)=E_\alpha(At^\alpha)=\sum_{k=0}^{\infty}\frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}, \tag{4a}$$

$$\Phi(t)=\sum_{k=0}^{\infty}\frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}, \tag{4b}$$

$$\Phi_d(t-d)=\sum_{k=1}^{q}A_k x(t-kd)+\sum_{l=0}^{q}B_l u(t-ld) \tag{4c}$$

and $E_\alpha(At^\alpha)$ is the Mittage-Leffler matrix function.

Proof. Using the Laplace transform ($\mathcal{L}$) to (1a) and taking into account that

$$\mathcal{L}[{}_0D_t^\alpha x(t)]=s^\alpha X(s)-s^{\alpha-1}x_0,\quad \mathcal{L}[x(t)]=X(s),\\ \mathcal{L}[x(t-d)]=e^{-sd}X(s) \tag{5}$$

we obtain the equation

$$s^\alpha X(s)-s^{\alpha-1}x_0=A_0X(s)+\sum_{k=1}^{q}A_k e^{-sdk}X(s)\\ +\sum_{l=0}^{q}B_l e^{-sdl}U(s),\quad 0<\alpha\le 1 \tag{6}$$

which can be written in the form

$$X(s)=[I_N s^\alpha-A_0]^{-1}\left(s^{\alpha-1}x_0+\sum_{k=1}^{q}A_k e^{-sdk}X(s)+\sum_{l=0}^{q}B_l e^{-sdl}U(s)\right) \tag{7}$$

where $U(s)=\mathcal{L}[u(t)]$.

It is easy to check [9] that

$$[I_N s^\alpha-A_0]^{-1}=\sum_{r=0}^{\infty}A_0^r s^{-(r+1)\alpha} \tag{8}$$

since

$$[I_N s^\alpha-A_0]\left(\sum_{r=0}^{\infty}A_0^r s^{-(r+1)\alpha}\right)=I_N. \tag{9}$$

Substitution of (8) into (7) yields

$$X(s)=\sum_{r=0}^{\infty}A_0^r s^{-(r\alpha+1)}x_0\\ +\sum_{r=0}^{\infty}A_0^r s^{-(r+1)\alpha}\left(\sum_{k=1}^{q}A_k e^{-sdk}X(s)+\sum_{l=0}^{q}B_l e^{-sdl}U(s)\right) \tag{10}$$

Using the inverse Laplace transform ($\mathcal{L}^{-1}$) to (2.10) and the convolution theorem we obtain

$$x(t)=\mathcal{L}^{-1}[X(s)]=\sum_{r=0}^{\infty}A_0^r\mathcal{L}^{-1}[s^{-(r\alpha+1)}]x_0\\ +\sum_{r=0}^{\infty}A_0^r\mathcal{L}^{-1}\left[s^{-(k+1)\alpha}\left(\sum_{k=1}^{q}A_k e^{-sdk}X(s)+\sum_{l=0}^{q}B_l e^{-sdl}U(s)\right)\right]\\ =\Phi_0(t)x_0+\int_0^t\Phi(t-\tau)\Phi_d(\tau-d)d\tau \tag{11}$$

where

$$\Phi_0(t)=\sum_{r=0}^{\infty}A_0^r\mathcal{L}^{-1}[s^{-(r\alpha+1)}]=\sum_{r=0}^{\infty}\frac{A_0^r t^{r\alpha}}{\Gamma(r\alpha+1)},$$

$$\Phi(t)=\sum_{r=0}^{\infty}A_0^r\mathcal{L}^{-1}[s^{-(r+1)\alpha}]=\sum_{r=0}^{\infty}\frac{A^r t^{(r+1)\alpha-1}}{\Gamma[(r+1)\alpha]},$$

$$\Phi_d(t-d)=\mathcal{L}^{-1}\left[\sum_{k=1}^{q}A_k e^{-sdk}X(s)+\sum_{l=0}^{q}B_l e^{-sdl}U(s)\right]\\ =\sum_{k=1}^{q}A_k x(t-kd)+\sum_{l=0}^{q}B_l u(t-ld).\ \square$$

Definition 1. The fractional continuous-time system with delays in state vector and input (2.1) is called the internally positive fractional system if and only if $x(t)\in\Re_+^N$ and $y(t)\in\Re_+^p$ for $t\ge 0$ for any initial conditions $x_0(t)\in\Re_+^N$, $t\in[-kd,0]$, $k=0,1,\ldots,q$ and all inputs $u(t)\in\Re_+^m$, $t\ge -ld$ $l=0,1,\ldots,q$.

Theorem 2. The fractional continuous-time system with delays in state vector and input (2.1) is internally positive if and only if the matrix A_0 is a Metzler matrix and

$$A_k\in\Re_+^{N\times N}, k=1,2,\ldots,q\quad B_l\in\Re_+^{N\times m}, l=0,1,\ldots,q\\ C\in\Re_+^{p\times N},\quad D\in\Re_+^{p\times m}. \tag{12}$$

Proof. Sufficiency. It is well known [9] that $\Phi_0(t)\in\Re_+^{N\times N}$ and $\Phi(t)\in\Re_+^{N\times N}$ for small $t>0$ only if A_0 is a Metzler matrix. By Theorem 2.1 the solution of the equation (2.1a) has the form (2.3) and $x(t)\in\Re_+^N$, $t\ge 0$ holds if $\Phi(t)\in\Re_+^{N\times N}$, $\Phi_0(t)\in\Re_+^{N\times N}$ (this holds when A_0 is a Metzler matrix) and $\Phi_d(t-d)\in\Re_+^{N\times m}$ (this holds when $A_k\in\Re_+^{N\times N}$, $k=1,2,\ldots,q$ and $B_l\in\Re_+^{N\times m}$, $l=0,1,\ldots,q$ - see (2.4c)) since $x_0(t)\in\Re_+^N$, $t\in[-kd,0]$, $k=0,1,\ldots,q$ and $u(t)\in\Re_+^m$, $t\ge -ld$ $l=0,1,\ldots,q$.

Necessity. Let $u(t)=0$, $t\ge -ld$ $l=0,1,\ldots,q$ and $x_0(0)=e_i$ (the i-th column of the identity matrix I_N) and $x_0(-kd)=0$, $k=1,\ldots,q$. The trajectory of the system does not leave the orthant $\Re_+^N$ only if ${}_0D_t^\alpha x(0)=A_0e_i\ge 0$, what implies $a_{i,j}\ge 0$ for $i\ne j$. The matrix A_0 has to be a Metzler matrix. For the same reason, for $x_0(0)=0$ and $x_0(-kd)>0$, $k=1,\ldots,q$ we have

$${}_0D_t^\alpha x(0)=\sum_{k=1}^{q}A_k x_0(-kd)+\sum_{l=0}^{q}B_l u(-ld)\ge 0$$

what implies $A_k\in\Re_+^{N\times N}$, $k=1,\ldots,q$ and $B_l\in\Re_+^{N\times m}$, $l=0,1,\ldots,q$ since $u(-ld)\in\Re_+^m$ may be arbitrary. From (2.1b) for $u(t)=0, t\ge 0$ we have $y(0)=Cx_0\ge 0$ and $C\in\Re_+^{p\times N}$, since $x_0\in\Re_+^N$ may be arbitrary. In a similar way, assuming $x_0=0$ we obtain $y(0) = Du(0) \geq 0$ and $D \in \Re_+^{p\times m}$, since $u(0) \in \Re_+^m$ may be arbitrary. □

Similarly as in proof of Theorem 1 using the Laplace transform with zero initial conditions to (1) we obtain the equations

$$X(s) = [I_N s^\alpha - \sum_{k=0}^{q} A_k e^{-sdk}]^{-1} \sum_{k=0}^{q} B_k e^{-sdk} U(s), \quad Y(s) = CX(s) + DU(s) \tag{13}$$

based on which we obtain the transfer matrix of the fractional continuous-time linear system with delays (1)

$$T(s^\alpha, w) = C[I_N s^\alpha - \sum_{k=0}^{q} A_k e^{-sdk}]^{-1} \sum_{k=0}^{q} B_l e^{-sdk} + D = C[I_N \lambda - \sum_{k=0}^{q} A_k w^{-k}]^{-1} \sum_{k=0}^{q} B_k w^{-k} + D, \; s^\alpha = \lambda, \; e^{sd} = w. \tag{14}$$

The transfer matrix is called proper if

$$\lim_{\lambda\to\infty} T(\lambda) = K \in \Re^{p\times m} \tag{15}$$

and it is called strictly proper if $K = 0$.

Definition 2. Matrices (12) are called a positive realization of transfer matrix $T(\lambda) \in \Re^{p\times m}(\lambda)$ if they satisfy the equality (14). The realization is called minimal if the dimension of A_k, $k = 0,1,\ldots,q$ is minimal among all realizations of $T(\lambda)$.

The problem under considerations can be stated as follows. Given a proper rational transfer matrix $T(\lambda, w) \in \Re^{p\times m}(\lambda, w)$ of the continuous-time system with the fractional order α, find its positive realization (12), where $\Re^{p\times m}(\lambda, w)$ is the set of $p\times m$ rational matrices in λ and w.

Similar as in case of discrete-time fractional systems with delays [15] the transfer matrix of the system (1) is the function of the operators λ and w^{-k}, $k = 0,1,\ldots,q$. For single-input single-output (SISO) systems the proper transfer function has the following form

$$T(\lambda, w) = \frac{\sum_{r=0}^{N-1} n_r(w)\lambda^r}{\lambda^n - \sum_{r=0}^{N-1} d_r(w)\lambda^r} + D = \frac{N(\lambda, w)}{d(\lambda, w)} + D \tag{16}$$

and

$$n_r(w) = b_r^0 + b_r^1 w^{-1} + \ldots + b_r^q w^{-q}, \quad d_r(w) = a_r^0 + a_r^1 w^{-1} + \ldots + a_r^q w^{-q}, \quad r = 0,1,\ldots,N-1. \tag{17}$$

for known α.

3. Problem solution for SISO systems

Transfer function (14) can be written in the following form

$$T(\lambda, w) = \frac{C(H_{ad}(\lambda, w))[B_0 + B_1 w^{-1} + \ldots + B_q w^{-q}]}{\det H(\lambda, w)} + D \tag{18}$$

where

$$\begin{aligned} H(\lambda, w) &= [I_N \lambda - A_0 - A_1 w^{-1} - \ldots - A_q w^{-q}], \\ N(\lambda, w) &= C(H_{ad}(\lambda, w))[B_0 + B_1 w^{-1} + \ldots + B_q w^{-q}] \\ &= n_{N-1}(w)\lambda^{N-1} + \ldots + n_1(w)\lambda + n_0(w), \\ d(\lambda, w) &= \det H(\lambda, w) = \lambda^N - d_{N-1}(w)\lambda^{N-1} - \ldots - d_1(w)\lambda - d_0(w) \end{aligned} \tag{19}$$

the $H_{ad}(\lambda, w)$ is the adjont matrix. From (18) we have

$$D = \lim_{\lambda\to\infty} T(\lambda, w) \tag{20}$$

since $\lim_{\lambda\to\infty} \left[I_N \lambda - \sum_{k=0}^{q} A_k w^{-k} \right]^{-1} = 0$. The strictly proper transfer function is given by the equation

$$T_{sp}(\lambda, w) = T(\lambda, w) - D \tag{21}$$

and the realization problem is down to finding the matrices A, B and C.

Theorem 3. There exists a positive realization of the proper transfer function $T(\lambda,w)$ of the fractional continuous-time linear system with delays (2.1) for $0 < \alpha < 1$ if the following conditions are satisfied:

1) $T(\infty, w) = \lim_{\lambda\to\infty} T(\lambda, w) \in \Re_+^{p\times m}$, which is equivalent to $D \in \Re_+^{p\times m}$.

2) Coefficients of the polynomial $d(\lambda,w)$ are nonnegative $a_r^k \geq 0$, $k = 1,\ldots q$; $r = 0,1,\ldots N-2$ and the coefficient a_{N-1}^0 can be arbitrary.

3) Coefficients of $N(\lambda,w)$ are nonnegative $b_r^k \geq 0$, $k = 0,1,\ldots q$; $r = 0,1,\ldots N-1$.

The realization has the following form

$$A_0 = \begin{bmatrix} 0 & \ldots & 0 & a_0^0 \\ 1 & \ldots & 0 & a_1^0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 1 & a_{N-1}^0 \end{bmatrix}, \quad A_r = \begin{bmatrix} 0 & \ldots & 0 & a_0^k \\ 0 & \ldots & 0 & a_1^k \\ \vdots & \ldots & \vdots & \vdots \\ 0 & \ldots & 0 & a_{N-1}^k \end{bmatrix}, k = 1,\ldots q; \quad B_k = \begin{bmatrix} b_0^k \\ \vdots \\ b_{N-1}^k \end{bmatrix}, k = 0,1,\ldots q; \quad C = [0 \; \ldots \; 0 \; 1]. \tag{22}$$

Proof. According to (18) the strictly proper transfer function have the form

$$T_{sp}(\lambda, w) = \frac{N(\lambda, w)}{d(\lambda, w)}. \tag{23}$$

Solving denominator $d(\lambda,w)$ of the transfer function (23) according to n-th column and using the matrices (22) we obtain the following polynomial

$$d(\lambda,w)=\det[I_N\lambda-A_0-A_1w^{-1}-...-A_qw^{-q}]$$

$$=\det\begin{bmatrix}\lambda & 0 & ... & 0 & -(\sum_{k=0}^{q}a_0^kw^{-k})\\ -1 & \lambda & ... & 0 & -(\sum_{k=0}^{q}a_1^kw^{-k})\\ \vdots & \vdots & ... & \vdots & \vdots\\ 0 & 0 & ... & -1 & \lambda-(\sum_{k=0}^{q}a_{N-1}^kw^{-k})\end{bmatrix} \quad (24)$$

$$=\lambda^N-(a_{N-1}^0+a_{N-1}^1w^{-1}+...+a_{N-1}^qw^{-q})\lambda^{N-1}$$
$$-...-(a_1^0+a_1^1w^{-1}+...+a_1^qw^{-q})\lambda \quad -(a_0^0+a_0^1w^{-1}+...+a_0^qw^{-q})$$
$$=\lambda^N-d_{N-1}(w)\lambda^{N-1}-...-d_1(w)\lambda-d_0(w)$$

which is equal to (19).

It is well known [8] that if A_r, r=0,1,…q have the canonical form (22) then

$$C[I_N\lambda-A_0-A_1w^{-1}-...-A_qw^{-q}]_{ad}=[1 \quad \lambda \quad ... \quad \lambda^{N-1}]. \quad (25)$$

Now solving the numerator $N(\lambda,w)$ of the transfer function (23) we obtain the polynomial

$$N(\lambda,w)=[1 \quad \lambda \quad ... \quad \lambda^{N-1}]\begin{bmatrix}b_0^0+b_0^1w^{-1}+...+b_0^qw^{-q}\\ b_1^0+b_1^1w^{-1}+...+b_1^qw^{-q}\\ \vdots\\ b_{N-1}^0+b_{N-1}^1w^{-1}+...+b_{N-1}^qw^{-q}\end{bmatrix} \quad (26)$$

$$=(b_{N-1}^0+b_{N-1}^1w^{-1}+...+b_{N-1}^qw^{-q})\lambda^{N-1}$$
$$-...-(b_1^0+b_1^1w^{-1}+...+b_1^qw^{-q})\lambda-(b_0^0+b_0^1w^{-1}+...+b_0^qw^{-q})$$
$$=n_{N-1}(w)\lambda^{N-1}+n_{N-2}(w)\lambda^{N-2}+...+n_1(w)\lambda+n_0(w)$$

which is equal to (19). This shows that the matrices (3.5) are realization of the strictly proper transfer function (21), moreover if the conditions of Theorem 3 are satisfied then the matrices (22) are positive realization. □

Based on Theorem 3 the following procedure for finding the realization of $T(\lambda,w)$ is proposed.

Procedure 1.

Step 1. Using (20) compute the matrix D and strictly proper transfer function.

Step 2. Knowing coefficients a_r^k, k = 0,1,…q; r = 0,1,…N – 1 of the polynomial $d(\lambda,w)$ and the matrices (22) find $A_k, k=0,1,...q$.

Step 3. Knowing coefficients b_r^k, k = 0,1,…q; r = 0,1,…N – 1 of the $N(\lambda,w)$ and the matrices (22) find $B_k, k=0,1,...q$.

Example 1. Find the positive realization of the continuous-time linear systems with delay and fractional order α = 0.5 given by the transfer function

$$T(\lambda,w)=\frac{\lambda^3-(w^{-1}-1)\lambda^2-(w^{-1}-1)\lambda-(w^{-1}+1)}{\lambda^3-(w^{-1}+1)\lambda^2-(w^{-1}+2)\lambda-(2w^{-1}+1)}. \quad (27)$$

In his case n = 3, p = 1, q = 1. Based on the procedure we obtain the following.

Step 1. Using (20) we obtain the matrix D

$$D=\lim_{\lambda\to\infty}T(\lambda,w)=[1] \quad (28)$$

and the strictly proper transfer function

$$T_{sp}(\lambda,w)=T(\lambda,w)-D$$
$$=\frac{2\lambda^2+3\lambda+3w^{-1}+2}{\lambda^3-(w^{-1}+1)\lambda^2-(w^{-1}+2)\lambda-(2w^{-1}+1)}. \quad (29)$$

Step 2. Denominator have the form

$$d(\lambda,w)=\lambda^3-(w^{-1}+1)\lambda^2-(w^{-1}+2)\lambda-(2w^{-1}+1)$$
$$=\lambda^3-(a_2^1w^{-1}+a_2^0)\lambda^2-(a_1^1w^{-1}+a_1^0)\lambda-(a_0^1w^{-1}+a_0^0) \quad (30)$$

and the matrices A_0, A_1 have the form

$$A_0=\begin{bmatrix}0&0&1\\1&0&2\\0&1&1\end{bmatrix}, A_1=\begin{bmatrix}0&0&2\\0&0&1\\0&0&1\end{bmatrix}. \quad (31)$$

Step 3. Numerator have the form

$$N(\lambda,w)=2\lambda^2+3\lambda+3w^{-1}+2$$
$$=(b_2^1w^{-1}+b_2^0)\lambda^2+(bw^{-1}+b_1^0)\lambda+(b_0^1w^{-1}+b_0^0) \quad (32)$$

the matrices B_0, B_1 have the form

$$B_0=\begin{bmatrix}b_0^0\\b_1^0\\b_2^0\end{bmatrix}=\begin{bmatrix}2\\3\\2\end{bmatrix}, \quad B_1=\begin{bmatrix}b_0^1\\b_1^1\\b_2^1\end{bmatrix}=\begin{bmatrix}3\\0\\0\end{bmatrix} \quad (33)$$

and

$$C=[0 \quad 0 \quad 1]. \quad (34)$$

The realization of the fractional continuous-time linear system with delays is given by (28), (31), (33) and (34). These matrices satisfy the Theorem 1 and they are the positive realization of the transfer function (27).

4. Concluding remarks

A method for computation of a positive realization of a given proper transfer function of fractional continuous-time linear systems with delays in state and input has been proposed. Sufficient conditions for the existence of a positive realization of this class of fractional systems have been established. A procedure for computation of a positive realization has been proposed. The effectiveness of the procedure has been illustrated by a numerical example. Extension of these considerations for multi-input multi-output fractional systems with delays is possible. In general case the proposed procedure does not provide a minimal realization of a given transfer matrix. An open problem is formulation of the necessary and sufficient conditions for the existence of positive minimal realizations for fractional systems.

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