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Determination of the set of Metzler matrices for given stable polynomials

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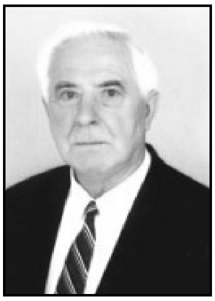

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Abstract

The problem of determination of the set of Metzler matrices for given stable polynomials is formulated and partly solved. For stable polynomial of the second degree there exists a set of Metzler matrices if and only if the polynomial has only real negatives zeros. If the stable polynomial has only real negative zeros then the set of corresponding Metzler matrices is given by the set of lower or upper triangular matrices with diagonal entries equal to the negative real zeros and any nonnegative off-diagonal entries. Sufficient condition are establish for the existence of the set of Metzler matrices for stable polynomials with a real negative zeros and the complex conjugate zeros).

Keywords: determination, existence, Metzler matrix, stable polynomial.

Wyznaczanie zbioru macierzy Metzlera dla danych stabilnych wielomianów

Streszczenie

W artykule sformułowani i częściowo rozwiązano problem wyznaczania zbioru macierzy Metzlera dla danych stabilnych wielomianów. Wykazano, ze dla stabilnych wielomianów stopnia drugiego istnieje zbiór macierzy Metzlera wtedy i tylko wtedy, gdy wielomian ten ma tylko ujemne pierwiastki rzeczywiste. Jeżeli stabilny wielomian dowolnego stopnia ma tylko pierwiastki rzeczywiste, to odpowiadający jemu zbiór macierzy Metzlera jest dany zbiorem macierzy dolno lub górno-trójkątnych z elementami na głównej przekątnej równych ujemnym zerom tego wielomianu oraz nieujemnymi elementami poza główną przekątną. Warunkiem koniecznym na to, aby dla danego stabilnego wielomianu istniał zbiór macierzy Metzlera jest posiadanie przez ten wielomian co najmniej dwóch zer przeczystych. Podano warunki dostateczne na istnienie zbioru macierzy Metzlera dla danych stabilnych wielomianów z ujemnymi zerami rzeczywistymi i zespolonymi parami sprzężonymi.

Słowa kluczowe: wyznaczanie, istnienie, macierz Metzlera, wielomian stabilny.

1. Introduction

Determination of the state space equations for given transfer matrix is a classical problem, called realization problem, which has been addressed in many papers and books [1, 3, 16, 17-19]. An overview on the positive realization problem is given in [1-3]. The realization problem for positive continuous-time and discrete-time linear systems has been considered in [4-9, 12] and the positive realization problem for discrete-time systems with delays in [8-10]. The fractional positive linear systems has been addressed in [14, 15, 18]. The realization problem for fractional linear systems has been analyzed in [11] and for positive 2D hybrid systems in [13]. A method based on the similarity transformation of the standard realization to the discrete positive one has been proposed in [12]. Conditions for the existence of positive stable realization with system Metzler matrix for proper transfer function has been established in [6].

It is well-known [1, 2, 16] that to find a realization for a given transfer function first we have to find a state matrix for the given denominator of the transfer function.

In this paper the problem of determination of the set of Metzler matrices for given stable polynomials is formulated and partly solved. Necessary and sufficient conditions or only sufficient conditions will be established for the existence of the set of Metzler matrices for given stable polynomials. Two procedures for finding the sets of Metzler matrices will be proposed.

The paper is organized as follows. In section 2 the problem for polynomials of the second degree is formulated and solved. For stable polynomials of the third degree the problem is formulated and partly solved in section 3. An extension of these consideration for *n*-degree polynomials is presented in section 4. Concluding remarks are given in section 5.

2. Set of Metzler matrices corresponding to second degree stable polynomials

In this section the following problem will be addressed. Given a stable polynomial of the form

$$p(s) = s^2 + a_1 s + a_0 \tag{1}$$

with positive coefficients $a_0>0$, $a_1>0$ find a Metzler matrix of the form

$$A_M = \begin{bmatrix} -a_{11} & a_{12} \\ a_{21} & -a_{22} \end{bmatrix}, \quad a_{ij} \geq 0, \; i,j = 1,2 \tag{2}$$

such that

$$\det[I_2 s - A_M] = p_2(s). \tag{3}$$

It will be shown that for a given stable polynomial (1) there exists a Metzler matrix (2) if and only if the polynomial has real negative zeros s_1, s_2, i.e.

$$p_2(s) = (s - s_1)(s - s_2) = s^2 + a_1 s + a_0. \tag{4}$$

Lemma 1. The Metzler matrix (2) for any its entries $a_{ij} \geq 0$ $i,j = 1,2$ has only real eigenvalues.

Proof. The characteristic polynomial

$$\begin{aligned} p_2(s) = \det[I_2 s - A_M] &= \begin{vmatrix} s + a_{11} & -a_{12} \\ -a_{21} & s + a_{22} \end{vmatrix} \\ &= s^2 + (a_{11} + a_{22})s + a_{11}a_{22} - a_{12}a_{21} \end{aligned} \tag{5}$$

has only real zeros, since

$$\begin{aligned} (a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21}) &= a_{11}^2 + a_{22}^2 - 2a_{11}a_{22} + 4a_{12}a_{21} \\ &= (a_{11} - a_{22})^2 + 4a_{12}a_{21} \geq 0 \quad \text{for} \quad a_{ij} \geq 0, \; i,j = 1,2. \end{aligned} \tag{6}$$

□

Theorem 1. For a given stable polynomial (1) there exists a set of Metzler matrices of the form (2) if and only if the polynomial (1) has real negative zeros ($a_1^2 > 4a_0$). The set of Metzler matrices has the form (2) with the diagonal matrices

$$a_{11} = \frac{1}{2}\left(a_1 \pm \sqrt{a_1^2 - 4(a_0 + a_{12}a_{21})}\right),$$

$$a_{22} = \frac{1}{2}\left(a_1 \mp \sqrt{a_1^2 - 4(a_0 + a_{12}a_{21})}\right) \tag{7}$$

and off-diagonal entries $a_{12}\geq 0$, $a_{21}\geq 0$ satisfying the conditions

$$a_1^2 - 4(a_0 + a_{12}a_{21}) \geq 0. \tag{8}$$

Proof. From (7) it follows that $a_{i2}>0$, $i=1,2$ if and only the conditions (8) is satisfied and the polynomial (5) is equal to the polynomial (1) since

$$a_{11} + a_{22} = a_1,\ \ a_{11}a_{22} - a_{12}a_{21} = a_0 \tag{9}$$

and by Lemma 1 it has only real negative eigenvalues. □

Remark 1. For $a_{12}=a_{21}=0$ the Metzler matrix (2) is diagonal of the form

$$A_m = \begin{bmatrix} -s_1 & 0 \\ 0 & -s_2 \end{bmatrix} \tag{10}$$

where

$$s_1 = \frac{1}{2}\left(a_1 + \sqrt{a_1^2 - 4a_0}\right),\ \ s_2 = \frac{1}{2}\left(a_1 - \sqrt{a_1^2 - 4a_0}\right), \tag{11}$$

are the zeros of the polynomial (1).

Example 1. Let the stable polynomial (1) have the form

$$p_2(s) = (s+2)(s+3) = s^2 + 5s + 6. \tag{12}$$

In this case the set of corresponding Metzler matrix has the form (2) with the diagonal entries

$$\begin{aligned} a_{11} &= \frac{1}{2}\left(5 \pm \sqrt{5^2 - 4(6 + a_{12}a_{21})}\right) = \frac{1}{2}\left(5 \pm \sqrt{1 - 4a_{12}a_{21}}\right) \\ a_{22} &= \frac{1}{2}\left(5 \mp \sqrt{5^2 - 4(6 + a_{12}a_{21})}\right) = \frac{1}{2}\left(5 \mp \sqrt{1 - 4a_{12}a_{21}}\right) \end{aligned} \tag{13}$$

for $1-4a_{12}a_{21}\geq 0$. From (13) for $a_{12}=a_{21}=0$ we obtain $a_{11}=2$, $a_{22}=3$ or $a_{11}=3$, $a_{22}=2$.

3. Set of Metzler matrices corresponding to third degree stable polynomials

In this section for the given stable polynomial

$$p_3(s) = s^3 + a_2 s^2 + a_1 s + a_0,\ \ a_i > 0,\ \ i = 0,1,2 \tag{14}$$

a set of Metzler matrices of the form

$$A_M = \begin{bmatrix} -a_{11} & a_{12} & a_{13} \\ a_{21} & -a_{22} & a_{23} \\ a_{31} & a_{32} & -a_{33} \end{bmatrix},\ \ a_{ij} \geq 0,\ \ i,j = 1,2,3 \tag{15}$$

will be found such that

$$\det[I_3 s - A_M] = p_3(s). \tag{16}$$

The following two cases will be considered:
Case 1. The polynomial (14) has only real negative zeros $s_1=-\alpha_1$, $s_2=-\alpha_2$, $s_3=-\alpha_3$.
Case 2. The polynomial (14) has one real negative zeros $s_1=-\alpha_1$, and two complex conjugate zeros $s_2=-\alpha+j\beta$, $s_3=-\alpha-j\beta$.

Case 1.
Theorem 2. If

$$p_3(s) = (s+\alpha_1)(s+\alpha_2)(s+\alpha_3) = s^3 + a_2 s^2 + a_1 s + a_0 \tag{17}$$

then the desired set of Metzler matrices satisfying (16) is given by the set of lower triangular matrices

$$A_M = \begin{bmatrix} -a_{11} & 0 & 0 \\ a_{21} & -a_{22} & 0 \\ a_{31} & a_{32} & -a_{33} \end{bmatrix},\ \ a_{ij} \geq 0,\ \ i,j = 1,2,3 \tag{18a}$$

or the set of upper triangular matrices

$$A_M = \begin{bmatrix} -a_{11} & a_{12} & a_{13} \\ 0 & -a_{22} & a_{23} \\ 0 & 0 & -a_{33} \end{bmatrix},\ \ a_{ij} \geq 0,\ \ i,j = 1,2,3 \tag{18b}$$

with diagonal entries $-a_{ii}$, $i=1,2,3$ equal to the negative zeros $-\alpha_1$, $-\alpha_2$, $-\alpha_3$, and nonnegative off-diagonal entries.

Proof. Let the matrix A_M have the form (18). Then

$$\begin{aligned} \det[I_3 s - A_M] &= (s+a_{11})(s+a_{22})(s+a_{33}) \\ &= (s+\alpha_1)(s+\alpha_2)(s+\alpha_3) = p_3(s) \end{aligned} \tag{19}$$

if the diagonal entries $-a_{ij}$ $i,j=1,2,3$, are equal for any choice to the negative zeros $-\alpha_1$, $-\alpha_2$, $-\alpha_3$ and any nonnegative off-diagonal entries. □

Lemma 2. If the stable polynomial (14) has only real negative zeros $-\alpha_1$, $-\alpha_2$, $-\alpha_3$ (not necessarily distinct) then the coefficients a_2 and a_1 of (17) satisfies the condition

$$a_2^2 \geq 3a_1. \tag{20}$$

Proof. Taking into account that

$$\begin{aligned} a_2 &= -(s_1 + s_2 + s_3) = \alpha_1 + \alpha_2 + \alpha_3, \\ a_1 &= s_1(s_2 + s_3) + s_2 s_3 = \alpha_1(\alpha_2 + \alpha_3) + \alpha_2\alpha_3 \end{aligned}$$

we obtain

$$\begin{aligned} a_2^2 &= (\alpha_1 + \alpha_2 + \alpha_3)^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 \\ &+ 2[\alpha_1(\alpha_2 + \alpha_3) + \alpha_2\alpha_3] \geq 3[\alpha_1(\alpha_2 + \alpha_3) + \alpha_2\alpha_3] = 3a_1 \end{aligned}$$

since by Lemma A.1 (see Appendix A)

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 \geq \alpha_1(\alpha_2 + \alpha_3)\alpha_2\alpha_3.\ \ □$$

Example 2. Consider the polynomial

$$p_3(s) = (s+2)^2(s+3) = s^3 + 7s^2 + 16s + 12 \tag{21}$$

with real negative zeros $\alpha_1 = \alpha_2 = -2,\ \alpha_3 = -3$.
The coefficients of (21) satisfies the condition (20) since $a_2^2 = 7^2 = 49$ and $3a_1 = 3\cdot 16 = 48$.

Theorem 3. The Metzler matrices

$$A_{M1} = \begin{bmatrix} -a_{11} & 0 & a_{13} \\ 0 & -a_{22} & 0 \\ 1 & 0 & -a_{33} \end{bmatrix},\ \ A_{M2} = \begin{bmatrix} -a_{11} & 0 & 1 \\ 0 & -a_{22} & 0 \\ a_{31} & 0 & -a_{33} \end{bmatrix} \tag{22}$$

with nonnegative entries a_{11}, a_{22}, a_{33}, a_{13}, (a_{31}) have only real eigenvalues.

Proof. Using (22) we obtain

$$\det[I_3 s - A_{M1}] = \begin{vmatrix} s+a_{11} & 0 & -a_{13} \\ 0 & s+a_{22} & 0 \\ -1 & 0 & s+a_{33} \end{vmatrix}$$
$$= (s+a_{22})[(s+a_{11})(s+a_{33}) - a_{13}] \tag{23}$$
$$= (s+a_{22})[s^2 + (a_{11}+a_{33})s + a_{11}a_{33} - a_{13}].$$

The polynomial

$$s^2 + (a_{11}+a_{33})s + a_{11}a_{33} - a_{13} \tag{24}$$

has only real zeros since

$$(a_{11}+a_{33})^2 - 4(a_{11}a_{33} - a_{13}) = (a_{11}-a_{33})^2 - 4a_{13} > 0$$

for all nonnegative a_{11}, a_{22}, a_{33}, and a_{13}. The proof for A_{M2} is similar. □

Case 2.

Consider the polynomial (14) with one real negative zero $s_1 = -\alpha_1$ and two complex conjugate zeros $s_2 = -\alpha + j\beta$, $s_3 = -\alpha - j\beta$, $\alpha > 0$, $\beta > 0$

$$p_3(s) = (s+\alpha_1)(s+\alpha-j\beta)(s+\alpha+j\beta)$$
$$= s^3 + (\alpha_1+2\alpha)s^2 + [2\alpha_1\alpha + \alpha^2 + \beta^2]s + \alpha_1(\alpha^2+\beta^2) \tag{25a}$$
$$= s^3 + a_2 s^2 + a_1 s + a_0$$

where

$$a_2 = \alpha_1 + 2\alpha,\ a_1 = 2\alpha_1\alpha + \alpha^2 + \beta^2,\ a_0 = \alpha_1(\alpha^2+\beta^2) \tag{25b}$$

From (3.12b) we have

$$a_2^2 = (\alpha_1+2\alpha)^2 = \alpha_1^2 + 4\alpha_1\alpha + 4\alpha^2,$$

$$3a_1 = 3(2\alpha_1\alpha + \alpha^2 + \beta^2) = 6\alpha_1\alpha + 3(\alpha^2+\beta^2)$$

and $a_2^2 > 3a_1$ if and only if $\alpha_1^2 + \alpha^2 > 2\alpha_1\alpha + 3\beta^2$ and

$$(\alpha_1 - \alpha)^2 > 3\beta^2 \tag{26}$$

Therefore, we have the following corollary

Corollary 2. The conditions

$$a_2^2 > 3a_1 \tag{27}$$

is independent separately of α_1 and α and it depends only of the difference $\alpha_1 - \alpha$.

From (26) and Fig. 1 it follows the conditions $a_2^2 > 3a_1$ is satisfied for both cases a) and b) but only case a) corresponds to the Metzler matrix since for stable Metzler matrix the real zero is always located on right hand side of the complex conjugate zeros [1].

Theorem 4. For a given stable polynomial (25) with one real negative zero $s_1 = -\alpha_1$ and two complex conjugate zeros $s_2 = -\alpha + j\beta$, $s_3 = -\alpha - j\beta$ ($\alpha > 0$, $\beta > 0$) there exists a set of Metzler matrices (15) such that (16) holds if

$$a_2^2 > 3a_1 \text{ and } \alpha > \alpha_1 \tag{28}$$

Proof. Using (3.2) and (3.3) we obtain

$$p(s) = \begin{vmatrix} s+a_{11} & -a_{12} & -a_{13} \\ -a_{21} & s+a_{22} & -a_{23} \\ -a_{31} & -a_{32} & s+a_{33} \end{vmatrix} = s^3 + a_2 s^2 + a_1 s + a_0, \tag{29a}$$

where

$$a_2 = a_{11} + a_{22} + a_{33},$$
$$a_1 = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{12}a_{21} - a_{23}a_{32} - a_{13}a_{31}, \tag{29b}$$
$$a_0 = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{22}a_{13}a_{31}$$
$$- a_{33}a_{12}a_{21} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32}.$$

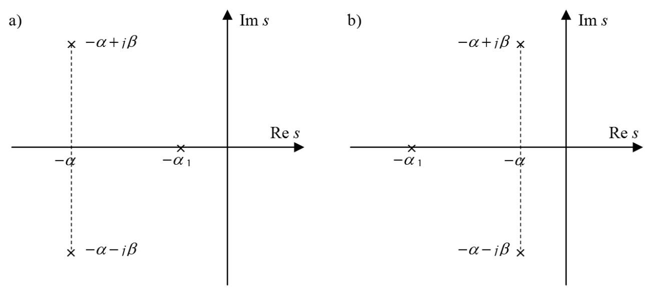

Fig. 1. Zeros placement
Rys. 1. Rozmieszczenie zer

From (29b) we have

$$a_2^2 = (a_{11}+a_{22}+a_{33})^2 = a_{11}^2 + a_{22}^2 + a_{33}^2 + 2(a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33})$$

$$3a_1 = 3(a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{12}a_{21} - a_{23}a_{32} - a_{13}a_{31})$$

and $a_2^2 > 3a_1$ if and only if

$$a_{11}^2 + a_{22}^2 + a_{33}^2 + 3[(a_{12}a_{21} + a_{23}a_{32} + a_{13}a_{31})$$
$$> a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33}. \tag{30}$$

By Lemma A.1 we have

$$a_{11}^2 + a_{22}^2 + a_{33}^2 > a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} \tag{31}$$

for $a_{ii} \neq a_{jj}$ for at least one pair (i,j) $i,j \in \{1,2,3\}$.

Therefore if (28) holds then it is possible to choose the entries of (15) so that the condition (30) is satisfied. □

The set of Metzler matrices (15) for a given stable polynomial (25) can be computed as follows. Choose the diagonal entries a_{11}, a_{22}, a_{33} so that

$$a_2 = a_{11} + a_{22}a_{33}. \tag{32}$$

Knowing the diagonal entries and a_1, a_0 find

$$\bar{a}_1 = a_{12}a_{21} + a_{23}a_{32} + a_{13}a_{31} = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_1 \tag{33}$$

$$\bar{a}_0 = a_{11}a_{23}a_{32} + a_{22}a_{13}a_{31} + a_{33}a_{12}a_{21} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$= a_{11}a_{22}a_{33} - a_0. \tag{34}$$

Choose four of the unknown nonnegative entries of (32), for example

$$a_{21} = a_{13} = 0,\ a_{12} = a_{23} = 1. \tag{35}$$

In this case from (33) and (34) we have

$$a_{32} = \overline{a}_1 = a_{11}(a_{22} + a_{33}) + a_{22}a_{33} - a_1, \tag{36a}$$

$$a_{31} = \overline{a}_0 - a_{11}\overline{a}_1 \tag{36b}$$

and the matrix (15) has the form

$$A_M = \begin{bmatrix} -a_{11} & 1 & 0 \\ 0 & -a_{22} & 1 \\ a_{31} & a_{32} & -a_{33} \end{bmatrix}. \tag{37}$$

To find the matrix (37) the following procedure can be used.

Procedure 1.
Step 1. Knowing a_2 choose a_{11}, a_{22}, a_{33} so that (32) is met.
Step 2. Knowing $\overline{a}_1$, $\overline{a}_0$ choose $a_{21} = a_{13} = 0$ and $a_{12} = a_{23} = 1$ and using (36) find a_{32} and a_{31}.
Step 3. Find the desired matrix (37).

Example 3.2. Find a set of Metzler matrices (37) for the polynomial

$$p_3(s) = s^3 + 20.1s^2 + 103s + 10.1. \tag{38}$$

The polynomial (38) has one real zero $s_1 = -\alpha = -0.1$ and two complex conjugate zeros $s_2 = -10+j$, $s_3 = -10-j$ ($\alpha = 10$, $\beta = 1$) The conditions (26) are satisfied since $a_2^2 = 20.1^2 > 3a_1 = 3 \cdot 103$ and $\alpha = 10 > \alpha_1 = -0.1$.
Using Procedure 1 we obtain the following
Step 1. In this case we choose for example $a_{11}=7$, $a_{22}=8$, $a_{33}=5.1$ so that $a_{11}+a_{22}+a_{33}=a_2=20.1$
Step 2. Assuming (35) from (36) we obtain

$$\begin{aligned} a_{32} = \overline{a}_1 &= a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_1 \\ &= 7 \cdot 8 + 7 \cdot 5.1 + 8 \cdot 5.1 - 103 = 29.5, \end{aligned} \tag{39a}$$

$$\begin{aligned} a_{31} = \overline{a}_0 - a_{11}\overline{a}_1 &= a_{11}a_{22}a_{33} - a_1 - a_{11}\overline{a}_1 \\ &= 7 \cdot 8 \cdot 5.1 - 103 - 7 \cdot 29.5 = 70. \end{aligned} \tag{39b}$$

Step 3. The desired Metzler matrix has the form

$$A_M = \begin{bmatrix} -7 & 1 & 0 \\ 0 & -8 & 1 \\ 70 & 29.5 & -5.1 \end{bmatrix}. \tag{40}$$

Remark 1. The set of Metzler matrices of the form (37) corresponding to the polynomial (14) is not unique.
It is easy to check that if we choose $a_{12}=10$, $a_{23}=1$, $a_{31}=7$, $a_{32}=29.5$ then the matrix A_M takes the form

$$A_M = \begin{bmatrix} -7 & 10 & 0 \\ 0 & -8 & 1 \\ 7 & 29.5 & -5.1 \end{bmatrix}. \tag{41}$$

and its characteristic polynomial

$$p_3(s) = \begin{vmatrix} s+7 & -10 & 0 \\ 0 & s+8 & -1 \\ -7 & -29.5 & s+5.1 \end{vmatrix} = s^3 + 20.1s^2 + 103s + 10.1 \tag{42}$$

is equal to the given are (38).

If we assume $a_{21}=a_{32}=0$ and $a_{12}=a_{31}=1$ than from (33) and (34) we obtain

$$a_{13} = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_1, \tag{43a}$$

$$a_{23} = a_{11}a_{22}a_{33} - a_0 - a_{22}a_{13}. \tag{43b}$$

Knowing a_{11}, a_{22}, a_{33}, a_1 and a_0 we can find from (43a) a_{13} and next from (43b) a_{23}.
Therefore, we have also the following procedure for finding of the set of Metzler matrices of the form

$$A_M = \begin{bmatrix} -a_{11} & 1 & a_{13} \\ 0 & -a_{22} & a_{23} \\ 1 & 0 & -a_{33} \end{bmatrix} \tag{44}$$

corresponding to the given stable polynomial (14).

Procedure 2.
Step 1. Knowing a_2 choose a_{11}, a_{22}, a_{33} so that (32) is met.
Step 2. Knowing a_1, a_0 and using (43a) find a_{13} and from (43b) a_{23}
Step 3. Find the desired matrix (44).

Example 4. Using Procedure 2 find the Metzler matrix (44) for the given stable polynomial (38).
Step 1. Is the same as in example 3.2 and we choose $a_{11}=7$, $a_{22}=8$, $a_{33}=5.1$.
Step 2. Using (43a) and (43b)we obtain

$$a_{13} = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_1 = 132.5 - 103 = 29.5$$

and

$$a_{23} = a_{11}a_{22}a_{33} - a_0 - a_{22}a_{13} = 285.6 - 101 - 246.1 = 39.5.$$

Step 3. The desired Metzler matrix corresponding to the polynomial (38) has the form

$$A_M = \begin{bmatrix} -7 & 1 & 29.5 \\ 0 & -8 & 39.5 \\ 1 & 0 & -5.1 \end{bmatrix}. \tag{45}$$

The following example shows the importance of the assumption $\alpha > \alpha_1$ of Theorem 3.

Example 5. Using Procedure 2 find the Metzler matrix (44) for the given stable polynomial (38).

$$p_3(s) = s^3 + 9s^2 + 25s + 17 \tag{46}$$

and

$$p_3(s) = s^3 + 6s^2 + 10s + 8. \tag{47}$$

The polynomial (46) has the zeros $s_1 = -1$, $s_2 = -4+j$, $s_3 = -4-j$ and the polynomial (47) the zeros $s_1 = -4$, $s_2 = -1+j$, $s_3 = -1-j$.
The first polynomial (46) satisfies the both conditions (28) but the second one does not satisfy the condition $\alpha > \alpha_1$ since $\alpha=4$, $\alpha_1=1$.
The first condition $a_2^2 > 3a_1$ of (28) is met for both polynomials.
Using Procedure 2 for the first polynomial (47) we obtain the following
Step 1. For $a_2=9$ we choose $a_{11}=2$, $a_{22}=3$ and $a_{33}=4$.
Step 2. Using (43a) and (43b)we obtain

$$a_{13} = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_1 = 26 - 25 = 1$$

and

$$a_{23} = a_{11}a_{22}a_{33} - a_0 - a_{22}a_{13} = 24 - 17 - 3 = 4.$$

Step 3. The desired Metzler matrix corresponding to the polynomial (46) has the form

$$A_M = \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 4 \\ 1 & 0 & -4 \end{bmatrix}. \tag{48}$$

Using Procedure 2 for the second polynomial (47) we obtain the following

Step 1. For a_2=6 we choose a_{11}=a_{22}=a_{33}=2.

Step 2. Using (43a) and (43b)we obtain

$$a_{13} = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_1 = 12 - 10 = 2$$

and

$$a_{23} = a_{11}a_{22}a_{33} - a_0 - a_{22}a_{13} = 8 - 8 - 4 = -4 < 0$$

and the corresponding Metzler matrix (44) does not exists to the polynomial (47). It can be easily shown that the Metzler matrix does not exist for any choice of the diagonal entries a_{11}, a_{22}, a_{33} satisfying the condition (32) for a_2=6.

4. Extension for general case

If the given stable polynomial $p_n(s)$ has only real negative zeros then an extension of Theorem 3.1 is immediately.

Theorem 5. If the stable polynomial

$$p_n(s) = s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0 \tag{49}$$

has only real negative zeros s_1, s_2,…s_n (not necessary distinct)

$$p(s) = (s + s_1)(s + s_2)...(s + s_n), \ \ s_k < 0, \ \ k = 1,...,n \tag{50}$$

than there exists always the set triangular of Metzler matrices

$$M_A = \begin{bmatrix} s_1 & a_{12} & ... & a_{1n} \\ 0 & s_2 & ... & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & s_n \end{bmatrix} \tag{51a}$$

$$M'_A = \begin{bmatrix} s_1 & 0 & ... & 0 \\ a_{21} & s_2 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & ... & s_n \end{bmatrix}, \ (a_{ij} \geq 0, \ i, j = 1,2,...,n) \tag{51b}$$

satisfying the condition

$$\det[I_3 s - M_A] = p(s) \ \text{ or } \ \det[I_3 s - M'_A] = p(s) \tag{52}$$

Proof. Using (51a) and (50) we obtain

$$\det[I_n s - M_A] = \begin{vmatrix} s - s_1 & -a_{12} & ... & -a_{1n} \\ 0 & s - s_2 & ... & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & s - s_n \end{vmatrix} \tag{53}$$
$$= (s + s_1)(s + s_2)...(s + s_n) = p(s)$$

for any a_{ij}≥0, i,j=1,2,…n. The proof for (51b) is similar. □

If the polynomial (49) has real and complex conjugate zeros than an extension of the consideration of section 3 is not so easy but possible as shown the following example.

Example 6. Find a set of Metzler matrices for the stable polynomial

$$p_4(s) = s^4 + 10s^3 + 34s^2 + 42s + 17. \tag{54}$$

Note that the polynomial (54) can be decomposed as the product of the following two stable polynomials

$$p_1(s) = s + 1 \ \text{ and } \ p_3(s) = s^3 + 9s^2 + 25s + 17 \tag{55}$$

$$\text{i.e. } \ p_4(s) = p_1(s)p_3(s). \tag{56}$$

In Example 5 it has been shown that the Metzler matrix corresponding $p_3(s)$ has the form (48). Therefore the desired Metzler matrix corresponding to the polynomial (54) is

$$A_M = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & -3 & 4 \\ 0 & 1 & 0 & -4 \end{bmatrix} \tag{57}$$

since

$$\det[I_4 s - A_M] = \begin{vmatrix} s+1 & 0 & 0 & 0 \\ 0 & s+2 & 1 & 1 \\ 0 & 0 & s+3 & 4 \\ 0 & 1 & 0 & s+4 \end{vmatrix} \tag{58}$$
$$= (s+1)\det\begin{bmatrix} s+2 & -1 & -1 \\ 0 & s+3 & -4 \\ -1 & 0 & s+4 \end{bmatrix} = p_1(s)p_3(s).$$

The determination of the set of Metzler matrices corresponding to stable polynomial (54) is not a trivial task and it will be considered in general case in a next paper.

5. Concluding remarks

The problem of finding the set of Metzler matrices for given stable polynomials has been formulated and partly solved. For stable polynomials of the second degree has been solved completely. It has been shown that there exists the set of Metzler matrices of the form (2) for the given stable polynomial (1) if and only if the polynomial has only real negative zeros (Theorem 1). If the stable polynomial (14) has only real negative zeros then the set of corresponding Metzler matrices is given by the set of lower or upper triangular matrices (18) with diagonal entries equal to the negative zeros and nonnegative off-diagonal entries (Theorem 2). Sufficient conditions for the existence of the set of Metzler matrices corresponding to stable polynomial with a real negative zeros and two complex conjugate zeros have been established (Theorem 3). Two procedures have been proposed for finding the Metzler matrices corresponding to stable polynomial (25) with one real negative zeros and two complex conjugate zeros. Theorem 2 has been extended to the stable n-degree polynomials (49) when it has only real negative zeros (Theorem 5). An open problem is an extension of these considerations for general case of stable n-degree polynomials and establishing necessary and sufficient conditions for the existence of a solution to the problem.

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6. Appendix

Lemma A1. If at least for one pair of indices (i, j) of the real numbers $a_1, a_2, \dots a_n$

$$a_i \neq a_j \text{ for } i \neq j \quad (i,j) \in \{1,2,...,n\} \tag{A.1}$$

then

$$a_1^2 + a_2^2 + ... + a_n^2 > a_1 a_2 + a_1 a_3 + ... + a_1 a_n + a_2 a_3 + ... + a_{n-1} a_n \tag{A.2}$$

Proof. Taking into account that

$$(a_i - a_j)^2 = a_i^2 + a_j^2 - 2a_{ij} > 0 \text{ for } a_i \neq a_j \quad (i,j) \in \{1,2,...,n\} \tag{A.3}$$

we obtain

$$\begin{aligned} & a_1^2 + a_2^2 > 2a_1 a_2 \quad \text{for} \quad a_1 \neq a_2 \\ & a_1^2 + a_3^2 > 2a_1 a_3 \quad \text{for} \quad a_1 \neq a_3 \\ & \dots\dots\dots\dots\dots\dots \\ & a_1^2 + a_n^2 > 2a_1 a_n \quad \text{for} \quad a_1 \neq a_n \\ & \dots\dots\dots\dots\dots\dots \\ & a_{n-1}^2 + a_n^2 > 2a_{n-1} a_n \quad \text{for} \quad a_{n-1} \neq a_n \end{aligned} \tag{A.4}$$

Summing up (A.4) we obtain

$$(n-1)(a_1^2 + a_2^2 + ... + a_n^2) > (n-1)(a_1 a_2 + a_1 a_3 + ... + a_{n-1} a_n) \tag{A.5}$$

for at least one pair (i, j) $a_i \neq a_j$
Dividing (A5) by $n - 1$ we obtain (A.2).

Lemma A2. Let

$$\frac{dp_3(s)}{ds} = 3s^2 + 2a_2 s + a_1 \tag{A.6}$$

be the first derivative of the polynomial (3.1).
The polynomial (A.6) has real zeros if and only if

$$a_2^2 \geq 3a_1. \tag{A.7}$$

Proof. The polynomial has real zeros if and only if $(2a_2)^2 \geq 4 \cdot 3a_1$ and this is equivalent to the condition (A.7).

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