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Normalized finite fractional discrete-time derivative – a new concept and its application to OBF modelling

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Abstract

This paper presents new results of modelling of linear open-loop stable systems by means of discrete-time finite fractional orthonormal basis functions, in particular the Laguerre functions. New stability conditions are offered and useful modification of the finite fractional derivative, called the normalized finite fractional derivative, is introduced. Simulation examples illustrate the usefulness of the new modelling methodology.

Keywords: identification, fractional systems, othonormal basis function.

Znormalizowane skończone równanie różnicowe niecałkowitego rzędu – nowa koncepcja i jej aplikacja w modelowaniu opartym na funkcjach bazy ortonormalnej

Streszczenie

W artykule przedstawiono nową koncepcję modelowania stabilnych systemów dynamicznych z zastosowaniem funkcji bazy ortonormalnej i równań różnicowych niecałkowitego rzędu. Przypomniano klasyczne równanie różnicowe niecałkowitego rzędu (Grunwalda-Letnikowa). Następnie wprowadzono tzw. skończone równanie różnicowe niecałkowitego rzędu oraz zaproponowano jego modyfikację nazwaną znormalizowanym skończonym równaniem różnicowym niecałkowitego rzędu. Ponadto przedstawiono opis modeli bazujących na funkcjach bazy ortonormalnej opartych zarówno na skończonym równaniu różnicowym niecałkowitego rzędu, jak również znormalizowanym skończonym równaniu różnicowym niecałkowitego rzędu i przedstawiono warunki stabilności tych modeli. Przykłady symulacyjne potwierdzają wysoką skuteczność prezentowanej metodologii w sensie niskich błędów predykcji generowanych przez wprowadzone modele. Ponadto w oparciu o przykłady symulacyjne zaprezentowano pewne zasady doboru parametrów α i K wchodzących w skład modeli.

Słowa kluczowe: identyfikacja, równania różnicowe niecałkowitego rzędu, funkcje bazy ortonormalnej.

1. Introduction

Fractional-order dynamic models have recently attracted a high research interest. Their specific properties can make them more adequate for modelling selected industrial systems [10, 13]. A number of discrete-time fractional-order systems have been modelled both via a transfer function [6-9] and state space models [4, 5].

On the other hand, control theoreticians and practicians involved in modelling and control have given a remarkable attention to modelling by means of Orthonormal Basis Functions (OBF) [3]. The orthonormality property of the employed all-pass filters can lead, by construction, to a well-posed estimation problem. Additionally, the output-error structure of OBF models contributes both to higher accuracy and robustness of a parameter estimator, as compared to those for e.g. ARX-like models [3].

Discrete-time fractional OBF modelling is a new research area and there are only a few papers on this topic available [11, 12]. On the other hand, continuous-time fractional OBF models have been studied in Refs. [1, 2].

2. Classical OBF-based modelling

It is well known that the classical OBF-based model output can be presented in the form

$$\hat{y}(t) = \sum_{i=1}^{M} c_i L_i(q) u(t) \tag{1}$$

including a series of orthonormal transfer functions $L_i(z)$ and the weighting parameters c_i, i=1,...,M, characterizing the model dynamics [3]. In case of discrete Laguerre models to be exploited hereinafter, the orthonormal transfer functions

$$L_i(z) = \frac{k}{z-p}\left[\frac{1-pz}{z-p}\right]^{i-1} \quad i=1,...,M \tag{2}$$

where $k=\sqrt{1-p^2}$, consist of a first-order low-pass factor and (i-1) th-order *all-pass* filters. The unknown parameters c_i, i=1,…,M, are easily estimated using e.g. Adaptive Least Squares (ALS) or Least Mean Squares (LMS) algorithms formalized in a linear regression fashion.

3. Fractional OBF-based modelling

The main problem is to describe filter (2) using a fractional derivative. In order to cope with the problem we reformulate Laguerre functions (2) in the following form [11]

$$L_i(q) = G_L(q^{-1})(G_R(q^{-1}) - p)^{i-1} \tag{3}$$

where

$$G_L(q^{-1}) = \frac{kq^{-1}}{1 - pq^{-1}}, \quad G_R(q^{-1}) = \frac{k^2 q^{-1}}{1 - pq^{-1}}, \tag{4}$$

where p is the Laguerre pole and q^{-1} is the backward shift operator. A block diagram of the model output calculation process for the fractional Laguerre model is presented in Fig. 1.

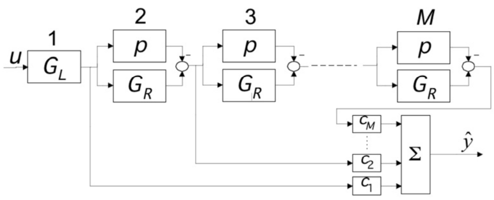

Fig. 1. Block diagram of the fractional Laguerre model
Rys.1. Schemat blokowy modelu Laguerre'a opisanego równaniem różnicowym niecałkowitego rzędu

Now, filters $G_L(q^{-1})$ and $G_R(q^{-1})$ are to be described by fractional derivative.

3.1. Finite Fractional Derivative

Recall, the familiar Grünwald-Letnikov derivative [4, 5]

$$\Delta^{\alpha} y(t) = \sum_{j=0}^{t} (-1)^j C_j(\alpha) y(t) q^{-j} = y(t) + \sum_{j=1}^{t} (-1)^j C_j(\alpha) y(t) q^{-j} \tag{5}$$

where

$$C_j(\alpha) = \binom{\alpha}{j} = \begin{cases} 1 & j = 0 \\ \dfrac{\alpha(\alpha-1)....(\alpha-j+1)}{j!} & j > 0 \end{cases} \tag{6}$$

In [11], a truncated or finite fractional derivative (FFD) has in analogy to FIR been introduced for practical, feasibility reasons, with the convergence to zero of the series $C_j(\alpha)$ enabling assuming $C_j(\alpha) \approx 0$ for some $j>K$, where K is the number of backward output samples used to calculate the fractional derivative. Finally, the outputs from the FFD versions of the $G_L(q^{-1})$ and $G_R(q^{-1})$ filters can be obtained [11]

$$G_L\text{: } y_L(t) = (p-1) y_L(t) q^{-1} + k u(t) q^{-1} - \sum_{j=1}^{K} (-1)^j C_j(\alpha) y_L(t) q^{-j}$$

$$G_R\text{: } y_R(t) = (p-1) y_R(t) q^{-1} + k^2 u_R(t) q^{-1} - \sum_{j=1}^{K} (-1)^j C_j(\alpha) y_R(t) q^{-j} \tag{7}$$

3.2. Normalized Finite Fractional Derivative

An important problem encountered in an FFD-based model is the incorrect gain of the model, with its discrepancy with respect to the corresponding FD one being dependent on K. This can be illustrated in the step response for the difference $\Delta^{\alpha}(\mathbf{1}(t))$ with α=0.5 and K=20, in which case $\lim_{t\to\infty}(\Delta^{\alpha}\mathbf{1}(t)) \approx 0.12$, whereas the limit is zero for FD. This may cause a (considerable) difference between the steady-state outputs of a plant and its FFD model. Also this may affect a stability condition for FFD.

In order to provide $\lim_{t\to\infty}(\Delta^{\alpha}\mathbf{1}(t)) = 0$, it is sufficient to incorporate a „normalization" factor N into the FFD to obtain the normalized FFD, or NFFD

$$\Delta^{\alpha} y(t) = y(t) + \frac{1}{N} \sum_{j=1}^{K} (-1)^j C_j(\alpha) y(t) q^{-j} \tag{8}$$

where

$$N = -\sum_{j=1}^{K} (-1)^j C_j(\alpha) \tag{9}$$

The outputs from the NFFD versions of the $G_L(q^{-1})$ and $G_R(q^{-1})$ are as follows [12]

$$G_L\text{: } y_L(t) = (p-1) y_L(t) q^{-1} + k u(t) q^{-1} - \frac{1}{N} \sum_{j=1}^{K} (-1)^j C_j(\alpha) y_L(t) q^{-j}$$

$$G_R\text{: } y_R(t) = (p-1) y_R(t) q^{-1} + k^2 u_R(t) q^{-1} - \frac{1}{N} \sum_{j=1}^{K} (-1)^j C_j(\alpha) y_R(t) q^{-j} \tag{10}$$

4. Stability issues for fractional Laguerre filters

Based on a number of simulation runs, we can offer stability conditions for FFD-OBF and NFFD-OBF models.

Conjecture 1.

The FFD Laguerre model (3) and (7) is asymptotically stable if

$$p - \sum_{j=1}^{K} (-1)^j C_j(\alpha) < 2 \quad \text{for } \alpha \in (0,2) \tag{11}$$

Conjecture 2.

The NFFD Laguerre model (3) and (10) is asymptotically stable if

$$p < 1 \quad \text{for } \alpha \in (0,2) \tag{12}$$

Conjectures 1 and 2 can be easily proved for $\alpha \in (0,1)$ [12]. Proving the conjectures for $\alpha \in (1,2)$ is a subject of the current research work.

5. Modelling application

Example 1.

Modelling of an oscillatory, "regular" (non-FD) system, with the transfer function

$$G(z) = \frac{z^4}{(z^2 - 1.85z + 0.9)(z - 0.8)(z^2 - 1.2z + 0.9)}$$

An NFFD-based Laguerre model is analyzed, with p=0.84 and M=12 for various K and α. Fig. 2 presents the results of modelling in terms of plots of MSPE vs. α and K.

It can be seen from Fig.2 that there is the single global minimum of MSPE for α=1.31 and K=12. Increasing K over 12 leads to decreased modelling accuracies. Simlar MSPE plots for an NFFD-based Laguerre model with M=12 and K=12 vs. α and dominant Laguerre pole p show that the optimum α=1.32 is independent of p.

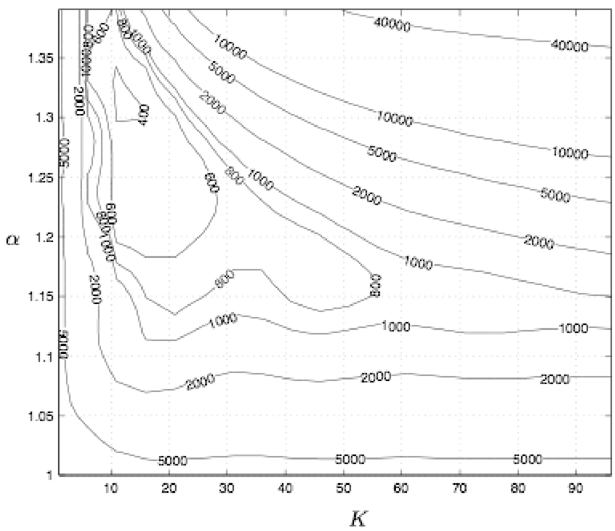

Fig. 2. Plots of MSPE for NFFD-based Laguerre model vs. α and K
Rys. 2. Wykres MSPE modelu Laguerre'a opartego na NFFD w funkcji α i K

Table 2 presents MSPE for the optimum NFFD-based Laguerre model versus FFD-based Laguerre model and two "regular" (non-FD) OBF models, namely Laguerre and Kautz models.

Tab. 1. Mean square prediction error for the analyzed models
Tab. 1. Średni błąd predykcji analizowanych modeli

Models	MSPE
NFFD-based Laguerre model	367.4
FFD-based Laguerre model	412.0
"Regular" Laguerre model	5281
"Regular" Kautz model	487.0

Clearly, the modelling accuracy is highest for the NFFD-based Laguerre model, which is rather surprising as it is the Kautz filters that are dedicated to model oscillatory systems. Also, NFFD-based Laguerre model provides better results than the FFD-based Laguerre one, which additionally confirms the usefulness of the introduced normalization.

Example 2.

Modelling of an overdamped "regular" system, with the transfer function

$$G(z) = \frac{0.001z^4}{(z^2 - 1.8z + 0.9)(z - 0.8)(z - 0.95)(z - 0.99)}$$

MSPE plots for the NFFD-based Laguerre model with M=12 and p=0.96 vs. α and K show that there is the single global minimum of MSPE for α=0.78 and K=41. Table 2 presents MSPE for the optimum NFFD-based Laguerre model versus "regular" Laguerre and "regular" Kautz models.

Tab. 2. Mean square prediction error for the analyzed models
Tab. 2. Średni błąd predykcji analizowanych modeli

Models	MSPE
NFFD-based Laguerre model	2.87
"Regular" Laguerre model	47.01
"Regular" Kautz model	47.01

Also in this case, the results obtained for the NFFD-based Laguerre model are best. The same results for the two "regular" models are caused by the same optimal poles p in both cases.

Based on the above results and those presented in Refs. [11, 12], we can formulate general hints concerning the selection of α and K:

- Optimum selection of α is dependent on the nature of a specific system; for oscillatory systems we select $\alpha \in (1,2)$, whereas for non-oscillatory ones $\alpha \in (0,1)$.
- In case of modelling the regular (non-FD) systems, we always have an optimum value of K, after which increased values of K result in increased MSPE. For fast-dynamic systems the optimum values of K are low (below 20) and for slow-dynamics systems the optimum values of K are high (even hundreds).

6. Conclusions

This paper has offered new tools for modelling discrete-time systems by means of fractional derivative-based orthonormal basis functions, in particular Laguerre functions.

Firstly, finite fractional derivative systems have been recalled and a new effective modification, called Normalized FFD or NFFD, has been introduced. The normalization removes the gain modeling inaccuracy, while preserving the classical FD stability conditions for an NFFD-based Laguerre model. Secondly, the stability conditions for FFD-based and NFFD-based Laguerre models have been given. Simulation experiments confirm the adequacy of the offered modeling methodology. Also, indications towards selection of α and K parameter, for the NFFD-based Laguerre model have been offered.

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