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Stability and robust stability conditions for a general model of scalar continuous-discrete linear systems

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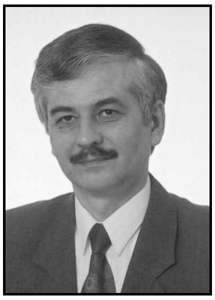

Mikołaj Busłowicz received the MSc, PhD and DSc degrees in Electrical Engineering in 1974, 1977 and 1988, respectively, all from the Faculty of Electrical Engineering of the Warsaw University of Technology. Since 1978 he has worked at the Białystok University of Technology. Since 2002 he has been professor and since 2005 full professor at the Białystok University of Technology. His main research interests include analysis and synthesis of time delay systems, positive systems, fractional systems and continuous-discrete systems.

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Abstract

The problems of asymptotic stability and robust stability of the general model of scalar linear dynamic continuous-discrete systems, standard and positive, are considered. Simple analytic conditions for asymptotic stability and for robust stability are given. These conditions are expressed in terms of coefficients of the model. The considerations are illustrated by numerical examples.

Keywords: continuous-discrete system, positive system, scalar system, stability, robust stability.

Warunki stabilności oraz odpornej stabilności modelu ogólnego skalarnych liniowych układów ciągło-dyskretnych

Streszczenie

W pracy rozpatrzono problemy stabilności oraz odpornej stabilności modelu ogólnego (1) skalarnych liniowych układów ciągło-dyskretnych, standardowych oraz dodatnich. Bazując na podanym w twierdzeniu 3 kryterium stabilności analizowanej klasy układów, wyprowadzono proste analityczne warunki asymptotycznej stabilności oraz odpornej stabilności. Warunki asymptotycznej stabilności oraz odpornej stabilności standardowego układu ciągło-dyskretnego podano w twierdzeniu 4 oraz w twierdzeniu 6, odpowiednio. Natomiast warunki asymptotycznej stabilności oraz odpornej stabilności dodatniego układu ciągło-dyskretnego podano w twierdzeniach 5 i 8, odpowiednio. Wszystkie warunki są wyrażone w terminach współczynników modelu (1) (lub wartości krańcowych przedziałów (13), z których te współczynniki mogą przyjmować swoje wartości). Rozważania zostały zilustrowane przykładami liczbowymi.

Słowa kluczowe: układ ciągło-dyskretny, dodatni, skalarny, stabilność, odporna stabilność.

1. Introduction

In continuous-discrete systems both continuous-time and discrete-time components are relevant and interacting and these components cannot be separated. Such systems are also called 2D hybrid systems or hybrid systems, see [1 - 5], for example.

The models and basic properties of positive continuous-discrete linear systems are given in [6]. A new general model of positive continuous-discrete linear systems is introduced in the paper [1].

The realisation problem of positive continuous-discrete systems is considered in [4, 5, 6]. The problems of stability and robust stability of continuous-discrete linear systems are investigated in [7 - 13].

The main purpose of this paper is to present simple analytical conditions for stability and robust stability for a general model of scalar continuous-discrete linear systems, standard and positive.

The following notation will be used: $\Re$ - the set of real numbers, Z_+ - the set of non-negative integers, $\Re_+ = [0,\infty]$.

2. The main result

Consider the state equation of the general model of a scalar continuous-discrete linear system (for $i \in Z_+$ and $t \in \Re_+$)

$$\dot{x}(t,i+1) = a_0 x(t,i) + a_1 \dot{x}(t,i) + a_2 x(t,i+1) + bu(t,i), \tag{1}$$

where $\dot{x}(t,i) = \partial x(t,i)/\partial t$, $x(t,i) \in \Re$, $u(t,i) \in \Re$ and a_0, a_1, a_2, b are real constant coefficients.

The boundary conditions for equation (1) have the forms

$$x(0,i) = x(i),\ \ i \in Z_+ \ \text{and}\ x(t,0) = x(t),\ \ \dot{x}(t,0) = \dot{x}(t),\ \ t \in \Re_+. \tag{2}$$

The model (1) will be called the standard general scalar model.

Definition 1. The general scalar model (1) is called positive (internally) if $x(t,i) \geq 0$ for all boundary conditions $x(i) \geq 0$, $i \in Z_+$ and $x(t) \geq 0$, $\dot{x}(t) \geq 0$, $t \in \Re_+$, and all inputs $u(t,i) \geq 0$, $t \in R_+$, $i \in Z_+$.

From [6] and definition 1 we have the following theorem.

Theorem 1. The scalar general model (1) is positive (internally) if and only if

$$a_0 \geq 0,\ a_1 \geq 0,\ a_2 \in \Re,\ b \geq 0 \ \text{and}\ a = a_0 + a_1 a_2 \geq 0. \tag{3}$$

The characteristic function of equation (1) (polynomial in two independent variables *s* and *z*) has the form

$$w(s,z) = sz - a_0 - sa_1 - za_2. \tag{4}$$

Definition 2. The general scalar model (1) is called asymptotically stable (or Hurwitz-Schur stable) if for $u(t,i) \equiv 0$ and bounded boundary conditions (2) the condition $x(t,i) \to 0$ holds for $t, i \to \infty$.

From [8, 9] we have the following theorem.

Theorem 2. The general scalar model (1) is asymptotically stable if and only if

$$w(s,z) \neq 0,\ \ \mathrm{Re}\, s \geq 0,\ \ |z| \geq 1. \tag{5}$$

Polynomial (4) satisfying the condition (5) is called continuous-discrete stable (C-D stable) or Hurwitz-Schur stable.

Theorem 3. The general scalar model (1) is asymptotically stable if and only if $s(j\omega) < 0$ for all $\omega \in [0, 2\pi]$, where

$$s(j\omega) = \frac{a_0 + a_2 \exp(j\omega)}{\exp(j\omega) - a_1}. \tag{6}$$

Proof. In [8] it was shown that the model of continuous-discrete linear system with the characteristic polynomial $w(s,z)$ is asymptotically stable if and only if

$$w(s, \exp(j\omega)) \neq 0,\ \ \mathrm{Re}\, s \geq 0,\ \ \forall \omega \in [0, 2\pi]. \tag{7}$$

The condition of Theorem 3 follows directly from (7) for the polynomial (4).

From (6) for $\omega = 0$ and $\omega = \pi$ we have, respectively,

$$s_0 = s(j0) = \frac{a_0 + a_2}{1 - a_1}, \quad s_\pi = s(j\pi) = \frac{a_2 - a_0}{1 + a_1}. \tag{8}$$

From (6) and (8) it follows that the function $s(j\omega)$ is discontinuous in the points $\omega = 0$ and $\omega = \pi$ for $a_1 = 1$ and $a_1 = -1$, respectively. Therefore, for excluding this discontinuity, we will assume that $a_1 \neq \pm 1$ and we consider the following values of the coefficient a_1: $a_1 > 1$, $-1 < a_1 < 1$ and $a_1 < -1$.

Let $s(j\omega) = u(\omega) + jv(\omega)$, $u(\omega) = \operatorname{Re} s(j\omega)$, $v(\omega) = \operatorname{Im} s(j\omega)$. It is easy to check that $[u(\omega) - s_c]^2 + v^2(\omega) = r^2$, where

$$s_c = 0.5(s_0 + s_\pi) = \frac{a_2 + a_0 a_1}{1 - a_1^2}; \quad r = |s_0 - s_c| = \left| \frac{-a_0 - a_1 a_2}{1 - a_1^2} \right|. \tag{9}$$

This means that the plot of $s(j\omega)$, $\omega \in [0, 2\pi]$, where $s(j\omega)$ is defined by (6), is a circle with the center s_c and radius r. Hence, the condition $s(j\omega) < 0$, $\omega \in [0, 2\pi]$, holds if and only if

$$\min\left\{ \frac{a_0 + a_2}{1 - a_1}, \frac{a_2 - a_0}{1 + a_1} \right\} < 0. \tag{10}$$

Theorem 4. The standard scalar model (1) is asymptotically stable if and only if one of the following conditions holds:

$$a_1 > 1, \quad -a_0 < a_2 < a_0, \tag{11a}$$

$$-1 < a_1 < 1, \quad a_2 < a_0, \quad a_2 < -a_0. \tag{11b}$$

$$a_1 < -1, \quad a_0 < a_2 < -a_0, \tag{11c}$$

Moreover, this system is unstable if $a_2 > -a_0$ and $a_2 > a_0$.

Proof. The proof follows directly from (10) for $a_1 \neq \pm 1$.

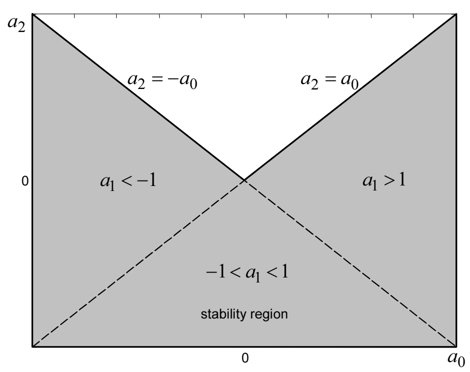

Fig. 1. Stability region for standard model (1)
Rys. 1. Obszar stabilności standardowego modelu (1)

From Theorem 4 we have the stability region in the plane (a_0, a_2) shown in Fig. 1.

Now we consider the positive scalar general model (1). In this case the conditions (3) hold. Taking into account assumption $a_1 \neq \pm 1$ we will consider the following values of the coefficient a_1: $a_1 > 1$ and $0 \le a_1 < 1$.

From the above, the conditions (3) and Theorem 4 we obtain the following theorem.

Theorem 5. The scalar general model (1) is positive and asymptotically stable if and only if one of the following conditions holds:

$$a_1 > 1, \quad a_0 \ge 0, \quad -a_0 / a_1 \le a_2 < a_0, \tag{12a}$$

$$0 \le a_1 < 1, \quad a_0 \ge 0, \quad -a_0 / a_1 \le a_2 < -a_0. \tag{12b}$$

Stability regions in the plane (a_0, a_2) for the positive model (1) are shown in Figs. 2 and 3.

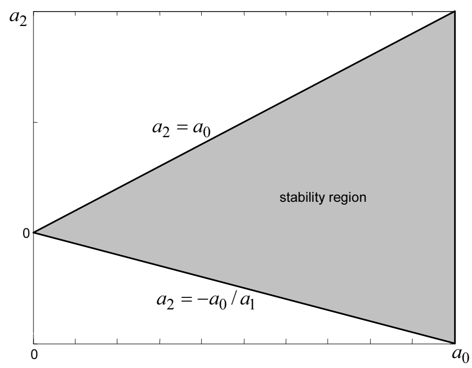

Fig. 2. Stability region described by (12a) for positive model (1)
Rys. 2. Obszar stabilności opisany przez (12a) dodatniego modelu (1)

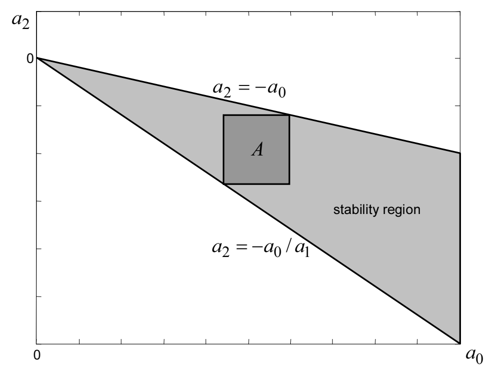

Fig. 3. Stability region described by (12b) for positive model (1)
Rys. 3. Obszar stabilności opisany przez (12b) dodatniego modelu (1)

Now we consider the scalar continuous-discrete linear system with uncertain parameters. In this case values of coefficients in the model (1) are not precisely known. We will assume that the coefficients of (1) are interval numbers, i.e.

$$a_i \in A_i = [a_i^-, a_i^+], \quad a_i^- \le a_i^+, \quad i = 1,2,3, \tag{13}$$

where a_i^- and a_i^+ $(i = 1,2,3)$ are given real numbers.

The model (1) with interval coefficients (13) is robustly stable if and only if it is asymptotically stable for all $a_i \in A_i$, $i = 1,2,3$.

From Theorem 4 it follows that for the standard uncertain system (1), (13) we must consider the following cases:

1) $A_1 \subset (1, \infty) \;\Leftrightarrow\; a_1^- > 1$,

2) $A_1 \subset (-1, 1) \;\Leftrightarrow\; a_1^- > -1$ and $a_1^+ < 1$,

3) $A_1 \subset (-\infty, -1) \;\Leftrightarrow\; a_1^+ < -1$.

Let $A = A_0 \times A_2$ ($\times$ denotes the Cartesian product) be the set of values of uncertain coefficients a_0 and a_2. This set is a rectangle in the plane (a_0, a_2) with the sides parallel to the axes and with the vertices V_i, $i = 1,2,3,4$. Values of coefficients a_0 and a_2 in the vertices are as follows:

$V_1: \; a_0 = a_0^-, \; a_2 = a_2^-, \quad V_2: \; a_0 = a_0^-, \; a_2 = a_2^+,$

$V_3: \; a_0 = a_0^+, \; a_2 = a_2^+, \quad V_4: \; a_0 = a_0^+, \; a_2 = a_2^-.$

From above and Theorem 4 it follows that the standard scalar uncertain model (1), (13) is robustly stable if and only if the rectangle $A = A_0 \times A_2$ lies in a suitable stability sub-region shown in Fig. 1, corresponding to the appropriate values of a_1. Analytical conditions for them are formulated in the following theorem.

Theorem 6. The standard uncertain model (1), (13) is robustly stable if and only if one of the following conditions holds:

$$a_1^- > 1 \text{ and } a_2^+ < a_0^-,\ a_2^- > -a_0^-, \tag{14a}$$

$$a_1^- > -1,\ a_1^+ < 1 \text{ and } a_2^+ < -a_0^+,\ a_2^+ < a_0^-, \tag{14b}$$

$$a_1^+ < -1 \text{ and } a_2^+ < -a_0^+,\ a_2^- > a_0^+. \tag{14c}$$

Now we consider the positive uncertain model (1), (13). By generalization of Theorem 1 we obtain the following.

Theorem 7. The general uncertain model (1), (13) is positive if and only if

$$a_0^- \geq 0,\ a_1^- \geq 0 \text{ and } a_2^- \geq -a_0^- / a_1^+,\ a_2^+ = \infty. \tag{15}$$

In the case of the positive uncertain model (1), (13) the rectangle $A = A_0 \times A_2$ must lie in the region shown in Fig. 2 for $A_1 \subset (1, \infty)$ and in the region shown in Fig. 3 for $A_1 \subset [0, 1)$. An example of the set $A = A_0 \times A_2$ location in the stability region is shown in Fig. 3. From Fig. 3 it follows that in this case the positive model (1), (13) is robustly stable if and only if the model (1) with the coefficients a_0 and a_2 corresponding to the vertices V_1 and V_3 of the set A is asymptotically stable.

From Theorems 5 and 7 we have the following theorem.

Theorem 8. The general uncertain model (1), (13) is positive and robustly stable if and only if one of the following conditions holds:

$$a_1^- > 1 \text{ and } a_0^- \geq 0,\ a_2^+ < a_0^-,\ a_2^- \geq -a_0^- / a_1^+, \tag{16a}$$

$$0 \leq a_1^- \leq a_1^+ < 1 \text{ and } a_2^+ < -a_0^+,\ a_2^- \geq -a_0^- / a_1^+. \tag{16b}$$

3. Illustrative examples

Example 1. Consider the general model (1) with the coefficients $a_0 = 1,\ a_1 = 2,\ a_2 \in \Re$.

From Theorems 4, 1 and 5 we have that the model is:

- asymptotically stable if and only if $-1 < a_2 < 1$,
- the positive system if and only if $a_2 \geq -0.5$,
- positive and asymptotically stable if and only if $-0.5 \leq a_2 < 1$.

Example 2. Consider the general uncertain model (1) with the coefficients $a_0 \in A_0 = [-1, 2],\ a_1 \in A_1 = [-0.8, 0.5],\ a_2 \in \Re$.

From condition (14b) of Theorem 6 it follows that the model is robustly stable if and only if $a_2^+ < -2$.

Example 3. Consider the general model (1) with the coefficients $a_0 \in A_0 = [2, 4],\ a_1 \in A_1 = [0.1, 0.5]$ and $a_2 \in \Re$.

From Theorem 7 it follows that the model is positive if and only if $a_2 \in [-4, \infty)$. Moreover, from Theorem 8 we have that this model is positive and robustly stable if and only if $a_2 \in [-4, -2)$.

4. Concluding remarks

Simple analytical conditions for stability and robust stability of the general model of scalar continuous-discrete linear systems, standard and positive, are given. These conditions are expressed in terms of the model coefficients.

In particular it has been shown that:

- the general standard model (1) is asymptotically stable if and only if plot of the function (6) lies in the open left half-plane of the complex plane for all $\omega \in [0, 2\pi]$ (Theorem 3),
- the general standard model (1) is asymptotically stable if and only if one of the conditions (11) holds (Theorem 4),
- the general model (1) is positive and asymptotically stable if and only if one of the conditions (12) holds (Theorem 5),
- the general uncertain standard model (1), (13) is robustly stable if and only if one of the conditions (14) holds (Theorem 6),
- the general uncertain model (1), (13) is positive if and only if the conditions (15) holds (Theorem 7),
- the general uncertain model (1), (13) is positive and robustly stable if and only if one of the conditions (16) holds (Theorem 8).

The work was supported by the Ministry of Science and High Education of Poland under grants No. N N514 1939 33 and S/WE/1/06.

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otrzymano / received: 02.11.2009
przyjęto do druku / accepted: 04.01.2010 *artykuł recenzowany*