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Reachability and minimum energy control of positive discrete-time linear systems with multiple delays in state and control

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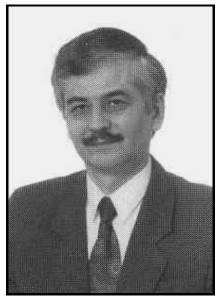

Uzyskał dyplom mgr inż. elektryka w 1974 roku na Wydziale Elektrycznym Politechniki Warszawskiej. Na tym samym wydziale w 1977 roku uzyskał stopień doktora nauk technicznych, a w 1988 roku doktora habilitowanego. Tytuł profesora uzyskał w 2002 roku. Autor trzech monografii oraz ponad 140 artykułów. Zainteresowania naukowe koncentrują się wokół problematyki analizy i syntezy układów regulacji automatycznej z opóźnieniami czasowymi, układów dodatnich oraz układów o niepewnych parametrach.

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Abstract

A notion of positive linear discrete-time systems with multiple delays in state and control is introduced. The necessary and sufficient conditions for positivity, reachability and minimum energy control are given. Considerations are illustrated by example.

Keywords: linear positive systems, discrete-time systems, time-delay, reachability, minimum energy control.

Osiągalność i sterowanie z minimalną energią liniowych dodatnich układów dyskretnych z wieloma opóźnieniami stanu i sterowania

Streszczenie

W pracy podano warunki, przy spełnieniu których liniowy dyskretny układ z wieloma opóźnieniami zmiennych stanu i sterowania jest układem dodatnim. Podano też warunki konieczne i wystarczające osiągalności oraz sterowania z minimalną energią. Rozważania zilustrowano przykładem.

Słowa kluczowe: liniowy układ dodatni, dyskretny, opóźnienie, osiągalność, sterowanie z minimalną energią.

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values for non-negative initial states and non-negative controls. Industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models are examples of positive systems. A variety of models having positive linear systems behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc. Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of art in the positive systems theory is given in the monographs [4, 5].

Recently some results known for standard linear positive systems have been extended for positive systems with time-delays. The conditions for reachability and minimum energy control of positive discrete-time systems with delay in state were given in [2]. The problem of reachability and controllability of linear positive discrete-time systems with delays in control or in state was discussed in [8]. An overview of some recent developments in theory of positive discrete-time linear systems with delays in state was presented in [3] and [6].

The aim of this paper is to give the notion of the internally positive linear discrete-time systems with multiple delays in state and control and necessary and sufficient conditions for the internal positivity, reachability and minimum energy control.

To the best of the authors' knowledge, the reachability and minimum energy control problems for positive discrete-time systems with multiple delays in state and control have not been studied yet.

2. Preliminaries

Let $\Re^{n\times m}$ be the set of $n\times m$ matrices with entries from the field of real numbers and $\Re^n = \Re^{n\times 1}$. The set of $n\times m$ matrices with real non-negative entries will be denoted by $\Re_+^{n\times m}$ and $\Re_+^n = \Re_+^{n\times 1}$. The set of non-negative integers will be denoted by Z_+.

Consider the discrete-time linear system with delays described in the state space by the equations

$$x_{i+1} = \sum_{k=0}^{h} A_k x_{i-k} + \sum_{j=0}^{q} B_j u_{i-j}, \quad i \in Z_+, \tag{1a}$$

$$y_i = Cx_i + Du_i, \tag{1b}$$

where h and q are positive integers, $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state, input and output vectors, respectively, $A_k \in \Re^{n\times n}$ $(k = 0,1,...,h)$, $B_j \in \Re^{n\times m}$ $(j = 0,1,...,q)$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$.

The initial conditions for (1a) are given by

$$x_{-i} \in \Re^n \quad (i = 0,1,...,h), \quad u_{-j} \in \Re^m \quad (j = 1,2,...,q). \tag{2}$$

The general form of the solution of state equation (1a) is as follows [1]

$$\begin{aligned} x_i = {} & \Phi(i)x_0 + \sum_{j=-h}^{-1} \sum_{k=1}^{h+j+1} \Phi(i-k)A_{k-1-j}x_j \\ & + \sum_{j=-q}^{-1} \sum_{k=1}^{q+j+1} \Phi(i-k)B_{k-1-j}u_j \\ & + \sum_{j=0}^{i-1} \sum_{k=0}^{q} \Phi(i-1-k-j)B_k u_j, \end{aligned} \tag{3}$$

where

$$\Phi(i) = Z^{-1}\{(zI_n - \sum_{k=0}^{h} A_k z^{-k})^{-1} z\} \tag{4}$$

is the state-transition matrix and Z^{-1} denotes the inverse z-transform.

The state-transition matrix $\Phi(i)$ satisfies the equation

$$\Phi(i+1) = A_0\Phi(i) + A_1\Phi(i-1) + ... + A_h\Phi(i-h) \quad (5)$$

with the initial conditions

$$\Phi(0) = I_n, \quad \Phi(i) = 0 \text{ for } i < 0. \quad (6)$$

Definition 1. The system (1) is called (internally) positive if $x_i \in \Re_+^n$ and $y_i \in \Re_+^p$ ($i \in Z_+$) for every $x_{-i} \in \Re_+^n$, $u_{-j} \in \Re_+^m$, $i = 0,1,...,h$, $j = 1,2,...,q$, and all inputs $u_i \in \Re_+^m$, $i \in Z_+$.

Theorem 1. The system (1) is internally positive if and only if

$$A_k \in \Re_+^{n\times n} (k = 0,1,...,h), B_j \in \Re_+^{n\times m} (j = 0,1,...,q), \\ C \in \Re_+^{p\times n}, D \in \Re_+^{p\times m}. \quad (7)$$

Proof. Defining

$$\tilde{x}_i = \begin{bmatrix} x_i \\ x_{i-1} \\ \vdots \\ x_{i-h+1} \\ x_{i-h} \end{bmatrix} \in \Re_+^{\tilde{n}}, \quad \tilde{u}_i = \begin{bmatrix} u_i \\ u_{i-1} \\ \vdots \\ u_{i-q+1} \\ u_{i-q} \end{bmatrix} \in \Re_+^{\tilde{m}}, \quad (8)$$

$$A = \begin{bmatrix} A_0 & A_1 & \cdots & A_{h-1} & A_h \\ I_n & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & I_n & 0 \end{bmatrix}, \quad (9a)$$

$$\tilde{B} = \begin{bmatrix} B_0 & B_1 & \cdots & B_{q-1} & B_q \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad (9b)$$

$$\tilde{C} = [C \ \ 0 \ \ \cdots \ \ 0], \quad \tilde{D} = [D \ \ 0 \ \ \cdots \ \ 0], \quad (9c)$$

equations (1) can be written in the form

$$\tilde{x}_{i+1} = A\tilde{x}_i + \tilde{B}\tilde{u}_i, \quad i \in Z_+, \quad (10a)$$

$$y_i = \tilde{C}\tilde{x}_i + \tilde{D}\tilde{u}_i, \quad (10b)$$

where $\tilde{n} = (h+1)n$, $\tilde{m} = (q+1)m$ and

$$\tilde{x}_0 = \begin{bmatrix} x_0 \\ x_{-1} \\ \vdots \\ x_{-h+1} \\ x_{-h} \end{bmatrix} \in \Re_+^{\tilde{n}}, \quad \tilde{u}_0 = \begin{bmatrix} u_0 \\ u_{-1} \\ \vdots \\ u_{-q+1} \\ u_{-q} \end{bmatrix} \in \Re_+^{\tilde{m}}. \quad (11)$$

System (10) is called (internally) positive if $\tilde{x}_i \in \Re_+^{\tilde{n}}$ and $y_i \in \Re_+^p$ ($i \in Z_+$) for every $\tilde{x}_0 \in \Re_+^{\tilde{n}}$ and any input sequence $\tilde{u}_i \in \Re_+^{\tilde{m}}$, $i \in Z_+$.

In [4] and [5] it was shown that system (10) is positive if and only if

$$A \in \Re_+^{\tilde{n}\times\tilde{n}}, \tilde{B} \in \Re_+^{\tilde{n}\times m}, \tilde{C} \in \Re_+^{p\times\tilde{n}}, \tilde{D} \in \Re_+^{p\times m}. \quad (12)$$

Hence, system (1) is positive if and only if the matrices A, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ satisfy conditions (12) that are equivalent to (7). ■

3. Reachability

For simplicity it will be assumed that $h = q$ in (1a). The case $h \neq q$ is similarly analyzed.

Definition 2. A state $x_f \in \Re_+^n$ is called reachable in *N* steps if there exists a sequence of inputs $u_i \in \Re_+^m$, $i = 0,1,..., N-1$, that transfers the system (1) from zero initial conditions (2) to the state x_f.

Definition 3. If every state $x_f \in \Re_+^n$ is reachable in *N* steps then the system is called reachable in *N* steps.

Definition 4. If for every state $x_f \in \Re_+^n$ there exists a natural number *N* such that the state x_f is reachable in *N* steps then the system is called reachable.

Recall that the set $X \subset \Re^n$ is called the cone if the following implication holds: if $x \in X$ then $\alpha x \in X$ for every $\alpha \in \Re_+$. The cone *X* is called convex if for any $x_1, x_2 \in X$ every point of the line segment $x = (1-\lambda)x_1 + \lambda x_2 \in X$ for $0 \le \lambda \le 1$. The cone *X* is called solid if its interior contains the sphere $K(x,r)$ with the center at the point $x \in X$ and radius *r*.

Theorem 2. The set of reachable states of positive system (1) is a positive convex cone. This cone is solid if and only if there exists an $N \in Z_+$ such that the rank of the reachability matrix

$$R_N = [\Psi(N-1), \Psi(N-2), \cdots, \Psi(1), \Psi(0)], \quad (13)$$

is equal to *n*, where

$$\Psi(i) = \sum_{k=0}^{h} \Phi(i-k)B_k, \quad (14)$$

and $\Phi(i)$ is the state-transition matrix.

Proof. For $x_{-i} = 0$ $(i = 0,1,...,h)$, $u_{-j} = 0$ $(j = 1,2,...,q)$ and $i = N > 0$ solution (3) of (1a) has the form

$$x_N = \sum_{j=0}^{N-1}\sum_{k=0}^{h} \Phi(N-1-k-j)B_k u_j = R_N u_0^N, \quad (15)$$

where R_N has form (13) with $\Psi(i)$ defined by (14) and

$$u_0^N = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}. \quad (16)$$

If $\text{rank} R_N = n$ then from (15) it follows that if u_0^N steers system (1) from zero initial conditions to x_N, then αu_0^N steers this system from zero initial conditions to αx_N for every $\alpha \in \Re$. Therefore, the set of states which are reachable in *N* steps is a cone.

Let X_N denotes the positive cone of reachable states of positive system (1).

If $\bar{x}_N = R_N \bar{u}_0^N \in X_N$ and $\tilde{x}_N = R_N \tilde{u}_0^N \in X_N$, then

$$(1-\lambda)\bar{x}_N + \lambda\tilde{x}_N = (1-\lambda)R_N\bar{u}_0^N + \lambda R_N\tilde{u}_0^N$$
$$= R_N[(1-\lambda)\bar{u}_0^N + \lambda\tilde{u}_0^N] = R_N v_0^N \in X_N,$$

where $v_0^N = (1-\lambda)\bar{u}_0^N + \lambda\tilde{u}_0^N$. Hence, the cone X_N is convex.

Let $K(0,\varepsilon)$ be the sphere with the center $x = 0$ and radius ε. From the assumption $\operatorname{rank} R_N = n$ it follows that the system is reachable if the input is unbounded. In this case there exists an input Δu_0^N that steers the state of system (1) to an arbitrary point inside the sphere. From the linearity of the system and superposition principle it follows that the input $u_0^N + \Delta u_0^N$ can steer the system to an arbitrary point inside the sphere $K(x,\varepsilon)$, where u_0^N is the input that steers system (1) to x. The input Δu_0^N can be chosen so that all entries of $u_0^N + \Delta u_0^N$ are non-negative and $K(x,\varepsilon) \subset X_N$. Hence, the cone X_N is solid. On the other hand, if X_N contains the sphere $K(x,\varepsilon)$ then there exists an input Δu_0^N that steers system (1) to an arbitrary point inside the sphere $K(0,\varepsilon)$ only if rank $R_N = n$. ■

The cone X_N of the reachable states of positive system (1) usually increases with N, i.e. $X_{N_1} \subset X_{N_2}$ for $N_2 > N_1$. The following theorem gives the conditions under which this cone in invariant with respect to N.

Theorem 3. The cone X_N of reachable states of positive system (1) is invariant for $N > \tilde{n} = (h+1)n$ if and only if $\operatorname{rank} R_N = n$ and the coefficients of the characteristic polynomial

$$\det(z^{h+1}I_n - \sum_{k=0}^{h} A_k z^{h-k})$$
$$= \det(zI_{\tilde{n}} - A) = z^{\tilde{n}} + a_{\tilde{n}-1}z^{\tilde{n}-1} + \ldots + a_1 z + a_0 \tag{17}$$

are non-positive, i.e. $a_k \le 0$ for $k = 0,1,\ldots,\tilde{n}-1$.

Proof. In the same way as in [2] it can be proved that

$$\Phi(\tilde{n}+j) = -a_{\tilde{n}-1}\Phi(\tilde{n}+j-1) - \ldots$$
$$- a_1\Phi(j+1) - a_0\Phi(j), \qquad j \in Z_+. \tag{18}$$

Hence, $\Phi(\tilde{n}+j)$ for any $j \in Z_+$ is a linear non-negative combination of $\Phi(j+k)$ $(k = 0,1,\ldots,\tilde{n}-1)$ if and only if $a_k \le 0$, $k = 0,1,\ldots,\tilde{n}-1$.

From (14) for $i = \tilde{n}+j$ we have

$$\Psi(\tilde{n}+j) = \sum_{k=0}^{h}\Phi(\tilde{n}+j-k)B_k, \quad j \in Z_+. \tag{19}$$

Because $B_k \in \Re_+^{n\times m}$ for $k = 0,1,\ldots,h$, the matrix $\Psi(\tilde{n}+j)$ for any $j \in Z_+$ is a linear non-negative combination of $\Phi(\tilde{n}+j-k)$, $k = 0,1,\ldots,h$. Hence, if $\operatorname{rank} R_N = n$, then $X_{\tilde{n}+j} = X_{\tilde{n}}$ for all $j \in Z_+$ if and only if all the coefficients a_k $(k = 0,1,\ldots,\tilde{n}-1)$ of polynomial (17) are non-positive. ■

By definition 3 positive system (1) is reachable if and only if the reachability cone is equal to $\Re_+^n$.

Denote by $\operatorname{Im}_+ R_N$ the positive image of the matrix $R_N \in \Re_+^{n\times Nm}$, i.e.

$$\operatorname{Im}_+ R_N = \{y \in \Re_+^n : y = R_N u, u \in \Re_+^{Nm}\}. \tag{20}$$

Theorem 4. Positive system (1) is reachable if and only if there exists an $N \in Z_+$ such that $\operatorname{rank} R_N = n$ and

1) $\operatorname{Im}_+ R_N = \Re_+^n$, where R_N is defined by (13),
2) n linearly independent columns can be chosen from R_N so that the matrix $\bar{R}_N$ constructed from them is a monomial matrix (every row and every column has only one positive entry and the remaining entries are equal to zero),
3) n linearly independent columns can be chosen from R_N so that the matrix $\bar{R}_N$ constructed from them has the inverse $\bar{R}_N^{-1}$ with non-negative entries, i.e. $\bar{R}_N^{-1} \in \Re_+^{n\times n}$.

Proof. If $x_N = x_f$ in (15) then

$$x_f = R_N u_0^N. \tag{21}$$

From (21) it follows that for every $x_f \in \Re_+^n$ there exists $u_0^N \in \Re_+^{Nm}$ if and only if the condition 1) is satisfied. If 1) is satisfied then n linearly independent columns (being a base of $\Re_+^n$) can be chosen from R_N if and only if in every row and every column only one entry is positive and all the remaining entries are zero. The matrix constructed from these columns is a monomial matrix. The inverse matrix of a positive matrix is positive if and only if it is a monomial matrix [5]. Therefore, conditions 2) and 3) are equivalent. ■

From the above it follows that if the conditions of Theorem 3 hold then the cone of reachable states of positive system (1) is invariant for $N \ge \tilde{n} = (h+1)n$. This means that if this system is not reachable in $N = \tilde{n}$ steps, then it is not reachable in $N > \tilde{n}$ steps (it is not reachable).

In certain cases the cone of reachable states may be invariant for $N < \tilde{n}$. This follows from the fact that if $m > 1$ then condition $\operatorname{rank} R_N = n$ can be satisfied for $N < \tilde{n}$. In such a case, if the conditions of Theorem 4 hold, then positive system (1) is reachable in $N < \tilde{n}$ steps (see example below).

Theorem 5. Positive system (1) is reachable if there exists an $N \in Z_+$ such that the rank of the reachability matrix R_N of form (13) is equal to n and

$$R_N^{\mathrm{T}}[R_N R_N^{\mathrm{T}}]^{-1} \in \Re_+^{Nm\times n}. \tag{22}$$

Moreover, if (22) holds then the sequence of controls $u_i \in \Re_+^m$, $i = 0,1,\ldots,N-1$, that transfer the system (1) from zero initial conditions (2) to the desired final state $x_f \in \Re_+^n$, can be computed from

$$u_0^N = R_N^{\mathrm{T}}[R_N R_N^{\mathrm{T}}]^{-1}x_f = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}. \tag{23}$$

Proof. If $\operatorname{rank} R_N = n$ then $\det(R_N R_N^{\mathrm{T}}) \ne 0$ and the matrix $R_N R_N^{\mathrm{T}}$ is nonsingular. If (22) holds and $x_f \in \Re_+^n$ then $u_0^N \in \Re_+^{Nm}$ and

$$x_N = R_N u_0^N = R_N R_N^{\mathrm{T}}[R_N R_N^{\mathrm{T}}]^{-1}x_f = x_f. \tag{24}$$

4. Minimum energy control

Consider positive system (1) with $h = q$ in (1a) and a performance index

$$I(u) = \sum_{i=0}^{N-1} u_i^{\mathrm{T}} Q u_i, \tag{25}$$

where $Q \in \Re^{m\times m}$ is a symmetric positive definite weighting matrix such that

$$Q^{-1} \in \Re_+^{m\times m} \tag{26}$$

and N is the number of steps, in which system (1) is transferred to the state x_f.

Control sequence $u_i \in \Re_+^m$, $i = 0,1,...,N-1$, that minimizes the performance index (25) is called minimal one. The problem of minimum energy control was first solved in [7].

The minimum energy control problem for positive system (1) with $h = q$ can be stated as follows. Given the matrices $A_k \in \Re_+^{n\times n}$ and $B_j \in \Re_+^{n\times m}$ $(k, j = 0,1,...,h)$, the number N of steps, the final state $x_f \in \Re_+^n$ and a weighting matrix Q such that (26) holds. Find a control sequence $u_i \in \Re_+^m$, $i = 0,1,...,N-1$, that transfers system (1) from zero initial conditions to the desired final state $x_f \in \Re_+^n$ and minimizes performance index (25).

Define the matrix

$$W = R_N \overline{Q}_N R_N^{\mathrm{T}} \in \Re_+^{n\times n}, \tag{27}$$

where R_N is the reachability matrix of form (13) and

$$\overline{Q}_N = \mathrm{diag}[Q^{-1},...,Q^{-1}] \in \Re_+^{Nm\times Nm}. \tag{28}$$

From (27) it follows that the matrix W is non-singular if and only if the matrix R_N has full row rank, i.e. the necessary condition of reachability of positive system (1) holds.

Define the sequence of inputs $\hat{u}_0,\ \hat{u}_1,\ ...,\ \hat{u}_{N-1}$ by

$$\hat{u}_0^N = \begin{bmatrix} \hat{u}_0 \\ \hat{u}_1 \\ \vdots \\ \hat{u}_{N-1} \end{bmatrix} = \overline{Q}_N R_N^{\mathrm{T}} W^{-1} x_f. \tag{29}$$

From (29) it follows that $\hat{u}_0^N \in \Re_+^{Nm}$ for any $x_f \in \Re_+^n$ if and only if

$$\overline{Q}_N R_N^{\mathrm{T}} W^{-1} \in \Re_+^{Nm\times n}. \tag{30}$$

Theorem 7. Let the following assumptions hold:

- positive system (1) is reachable in N steps,
- condition (30) is satisfied,
- $\overline{u}_i \in \Re_+^m$, $i = 0,1,...,N-1$, is any sequence of inputs which transfer system (1) from zero initial conditions (2) to the desired final state $x_f \in \Re_+^n$.

Then the sequence of inputs $\hat{u}_0,\ \hat{u}_1,\ ...,\ \hat{u}_{N-1}$ defined by (29) also transferring system (1) from zero initial conditions to the state $x_f \in \Re_+^n$, minimizes performance index (25) and

$$I(\hat{u}) \le I(\overline{u}). \tag{31}$$

Moreover, the minimal value of (25) is given by

$$I(\hat{u}) = x_f^{\mathrm{T}} W^{-1} x_f. \tag{32}$$

Proof. If positive system (1) is reachable in N steps and (30) holds, then $\hat{u}_i \in \Re_+^m$, $i = 0,1,...,N-1$.

From (15) for $u_0^N = \hat{u}_0^N$ and (29) it follows that

$$x_N = R_N \hat{u}_0^N = R_N \overline{Q}_N R_N^{\mathrm{T}} W^{-1} x_f = x_f, \tag{33}$$

because $R_N \overline{Q}_N R_N^{\mathrm{T}} W^{-1} = I_n$. Hence, sequence of inputs (29) provides $x_N = x_f$.

Since both $\overline{u}_0,\ \overline{u}_1,\ ...,\ \overline{u}_{N-1}$ and $\hat{u}_0,\ \hat{u}_1,\ ...,\ \hat{u}_{N-1}$ transfer system (1) from zero initial conditions to $x_f \in \Re_+^n$ then $x_f = R_N \overline{u}_0^N = R_N \hat{u}_0^N$ and

$$R_N(\hat{u}_0^N - \overline{u}_0^N) = 0. \tag{34}$$

From (29) it follows that $R_N^{\mathrm{T}} W^{-1} x_f = \overline{Q}_N^{-1} \hat{u}_0^N$. Hence,

$$(\hat{u}_0^N - \overline{u}_0^N)^{\mathrm{T}} R_N^{\mathrm{T}} W^{-1} x_f = (\hat{u}_0^N - \overline{u}_0^N)^{\mathrm{T}} \hat{Q}_N \hat{u}_0^N = 0, \tag{35}$$

where

$$\hat{Q}_N = \overline{Q}_N^{-1} = \mathrm{diag}[Q,...,Q] \in \Re^{Nm\times Nm}. \tag{36}$$

Using (35) it is easy to show that

$$(\overline{u}_0^N)^{\mathrm{T}} \hat{Q}_N \overline{u}_0^N = (\hat{u}_0^N)^{\mathrm{T}} \hat{Q}_N \hat{u}_0^N + (\overline{u}_0^N - \hat{u}_0^N)^{\mathrm{T}} \hat{Q}_N (\overline{u}_0^N - \hat{u}_0^N). \tag{37}$$

The last term in (37) is always non-negative. Hence, inequality (31) is true.

Substitution (29) into (25) yields

$$\begin{aligned} I(\hat{u}) &= \sum_{i=0}^{N-1} \hat{u}_i^{\mathrm{T}} Q \hat{u}_i = (\hat{u}_0^N)^{\mathrm{T}} \hat{Q}_N \hat{u}_0^N \\ &= (\overline{Q}_N R_N^{\mathrm{T}} W^{-1} x_f)^{\mathrm{T}} \hat{Q}_N (\overline{Q}_N R_N^{\mathrm{T}} W^{-1} x_f) \\ &= x_f^{\mathrm{T}} W^{-1} R_N \overline{Q}_N R_N^{\mathrm{T}} W^{-1} x_f = x_f^{\mathrm{T}} W^{-1} x_f, \end{aligned}$$

since $\hat{Q}_N \overline{Q}_N = I_{Nm}$ and $W^{-1} R_N \overline{Q}_N R_N^{\mathrm{T}} = I_n$. ■

Optimal control which minimizes performance index (25) depends on the weighting matrix Q. From comparison (23) and (29) it follows that control sequence (23) minimizes performance index (25) with $Q = I_m$. This means that u_0^N computed from (23) is minimum energy control with a performance index $I(u) = \sum_{i=0}^{N-1} u_i^{\mathrm{T}} u_i$.

Theorem 8. Let the weighting matrix has the form $Q = aI_m$, $a > 0$. Then $\hat{u}_0^N = u_0^N$, where $\hat{u}_0^N$ and u_0^N are defined by (29) and (23), respectively. In such a case the optimal value of the performance index can be computed from the formula

$$I(\hat{u}) = a x_f^{\mathrm{T}} [R_N R_N^{\mathrm{T}}]^{-1} x_f. \tag{38}$$

Proof. If $Q = aI_m$, then from (28) and (27) it follows that

$$\overline{Q}_N = a^{-1} I_{Nm}, \quad W = a^{-1} R_N R_N^{\mathrm{T}}. \tag{39}$$

Hence,

$$\hat{u}_0^N = \overline{Q}_N R_N^{\mathrm{T}} W^{-1} x_f = a^{-1} R_N^{\mathrm{T}} a (R_N R_N^{\mathrm{T}})^{-1} x_f = R_N^{\mathrm{T}} (R_N R_N^{\mathrm{T}})^{-1} x_f = u_0^N . \quad (40)$$

Substitution of the second formula of (39) into (32) gives (38).

5. Example

Consider positive system (1) with $h = q = 2$ and the matrices

$$A_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0.5 & 0 & 0 \end{bmatrix}, \quad (41a)$$

$$B_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}. \quad (41b)$$

Find the optimal control that transfers this system from zero initial conditions to the final state $x_f = [1 \quad 2 \quad 4]^{\mathrm{T}}$ in three steps and minimizes performance index (25) with

$$Q = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

The necessary condition for reachability in three steps is satisfied because the reachability matrix

$$R_3 = [\Psi(2), \Psi(1), \Psi(0)] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (42)$$

has a full row rank equal to 3.

It is easy to check that the conditions of Theorem 5 are satisfied and the system is reachable in three steps.

The optimal control sequence computed from (29) has the form

$$\hat{u}_0 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \quad \hat{u}_1 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \quad \hat{u}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \quad (43)$$

According to (32), the minimal value of performance index (25) is $I(\hat{u}) = 18.5$.

The control sequence, which also transfers system (1) with the matrices (41) from zero initial conditions to the final state $x_f = [1 \quad 2 \quad 4]^{\mathrm{T}}$, can be computed from (23). This control is of the form

$$u_0 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \quad u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}. \quad (44)$$

The optimal value of (25) for control sequence (44) is equal to $I(u) = 37 > I(\hat{u}) = 18.5$.

6. Concluding remarks

Notion of the internally positive linear discrete-time systems with multiple delays in state and control is introduced. The necessary and sufficient conditions for the internal positivity, reachability and minimum energy control are given.

The results of this study presented here can easily be extended for positive discrete-time systems with different numbers of delays in state and control and for multidimensional systems with delays.

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